FUNCTIONAL CALCULI FOR THE LAPLACE OPERATOR IN $L^p(\mathbb{R})$

Werner J. Ricker*

The Laplace operator $L = -d^2/dx^2$ in $L^p(\mathbb{R})$, $1 < p < \infty$, with domain

$$\mathcal{D}(L) = \{f \in L^p(\mathbb{R});\ f' \in AC(\mathbb{R}),\ f'' \in L^p(\mathbb{R})\}$$

is a closed, densely defined operator with spectrum $\sigma(L) = [0,\infty)$; here $AC(\mathbb{R})$ is the space of functions on the real line $\mathbb{R}$ which are absolutely continuous on bounded intervals. It is known that $-L$ is the infinitesimal generator of a strongly continuous C_0-semigroup of contractions, namely the heat semigroup given by

$$(T_t f)(u) = \frac{1}{2}\,(\pi t)^{-1/2} \int_{-\infty}^{\infty} f(u-w)\, e^{-w^2/4t}\, dw\,, \qquad f \in L^p(\mathbb{R})\,,$$

for each $t > 0$, and that L satisfies the resolvent estimates

$$\|(L - \lambda I)^{-1}\| \le 1/|\lambda| \sin^2\left(\frac{1}{2}\arg(\lambda)\right)\,, \qquad \lambda \in \rho(L)\,. \tag{1}$$

For $0 < \alpha < \pi$, define the open cone $S_\alpha = \{z \in \mathbb{C}\setminus\{0\};\ |\arg(z)| < \alpha\}$. A closed operator T in a Banach space X is said to be of type ω [10], where $0 \le \omega < \pi$, if $\sigma(T) \subseteq \bar{S}_\omega$ (the bar denotes closure and, by definition, $\bar{S}_0 = [0,\infty)$) and, for $0 < \epsilon < (\pi-\omega)$, there is a positive constant c_ϵ such that

$$\|(T - \lambda I)^{-1}\| \le c_\epsilon/|\lambda|\,, \qquad \lambda \notin \bar{S}_{\omega+\epsilon}\,.$$

It follows from (1) that if $0 < \epsilon < \pi$, then

$$\|(L - \lambda I)^{-1}\| \le 1/|\lambda| \sin^2\left(\frac{1}{2}\epsilon\right)\,, \qquad \lambda \notin \bar{S}_\epsilon\,,$$

and hence L is of type $\omega = 0$. In particular, $-L$ then generates an analytic semigroup in the sector $\bar{S}_{\pi/2}$, [10; Theorem 3.3.1].

In the Hilbert space setting it is often the case that operators of type ω admit an $H^\infty(S_\mu)$ functional calculus for every $\omega < \mu < \pi$. For example, this is so for positive

* The support of the Centre for Mathematical Analysis (Canberra) is gratefully acknowledged. The author wishes to thank Professors M. Cowling and A. McIntosh for valuable discussions and suggestions, some of which were relevant to this work.

self-adjoint operators, normal operators with spectrum in a cone and maximal accretive operators. Criteria characterizing those operators of type ω for which this occurs are given in the recent paper [6]; see also [11]. The situation in Banach spaces, even reflexive ones, is less clear and more complex; some positive results in this setting can be found in [2].

Concerning the particular case of the Laplace operator L in $L^p(\mathbb{R})$, $1 < p < \infty$, it can be shown that L admits an $H^\infty(S_\epsilon)$ functional calculus for every $0 < \epsilon < \pi$. Indeed, if m is a bounded measurable function in $\mathbb{R}^+ = [0,\infty)$ such that $m\circ\gamma : \mathbb{R} \to \mathbb{C}$ is a p-multiplier (where $\gamma(z) = z^2$, $z \in \mathbb{C}$), then we can define a continuous linear operator $m(L)$ by

$$m(L) = (m\circ\gamma)(D) . \tag{2}$$

Here $D = -id/dx$ is the closed, densely defined operator of differentiation in $L^p(\mathbb{R})$ with domain

$$\mathcal{D}(D) = \{f \in L^p(\mathbb{R}) \; ; \; f \in AC(\mathbb{R}) \; , \; f' \in L^p(\mathbb{R})\}$$

and, for any p-multiplier $\psi : \mathbb{R} \to \mathbb{C}$, $\psi(D)$ is the bounded operator in $L^p(\mathbb{R})$ specified by

$$(\psi(D)f)\hat{} = \psi\hat{f} \, , \qquad f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R}) \, ,$$

where $\hat{\cdot}$ denotes the Fourier transform. Fix $0 < \epsilon < \pi$. If $\psi \in H^\infty(S_\epsilon)$, then $\psi \circ \gamma \in H^\infty(-S_{\epsilon/2} \cup S_{\epsilon/2})$ and an application of the Cauchy integral formula shows that

$$|(\psi \circ \gamma)'(x)| \leq ||\psi||_\infty/|x| \sin(\tfrac{1}{2}\epsilon) \, , \qquad x \in \mathbb{R}\backslash\{0\} \, .$$

It follows [9; p.96 Theorem 3] that the restriction to $\mathbb{R}$ of $\psi \circ \gamma$, again denoted by $\psi \circ \gamma$, is a p-multiplier and so the operator $\psi(L) = (\psi \circ \gamma)(D)$ is defined. Furthermore, the multiplier theorem just indicated can also be used to show that

$$||\psi(L)|| \leq \alpha_p ||\psi||_\infty/\sin(\tfrac{1}{2}\epsilon), \qquad \psi \in H^\infty(S_\epsilon) \, ,$$

where α_p depends only on p and so $\psi \mapsto \psi(L)$ is a continuous homomorphism of $H^\infty(S_\epsilon)$ into the space of bounded linear operators on $L^p(\mathbb{R})$ equipped with the uniform operator topology. In addition, the range of the $H^{\infty}(S_{\epsilon})$ functional calculus includes the resolvent operators $(L - \lambda I)^{-1}$ whenever $\lambda \notin \bar{S}_{\epsilon}$.

The formula (2) also provides another functional calculus for L. Indeed, if $BV(\mathbb{R}^+)$ denotes the algebra of functions $f : [0,\infty) \to \mathbb{C}$ such that $f \circ \gamma$ is of bounded variation on $\mathbb{R}$ (equipped with the usual variation norm), then it follows from [1; pp.208-209], for example, that the map

$$f \mapsto f(L) = (f \circ \gamma)(D) \ , \qquad f \in BV(\mathbb{R}^+) \ ,$$

is a continuous homomorphism. Again the resolvent operators of L are included in this functional calculus since, if $\lambda \in \rho(L)$, the function $x \mapsto (x - \lambda)^{-1}$, $x \in \mathbb{R}^+$, is an element of $BV(\mathbb{R}^+)$. We remark that this functional calculus can be specified via an integral formula of the type

$$f(L) = (f \circ \gamma)(D) = \int_{-\infty}^{\infty} f(\gamma(\lambda))\, dE(\lambda) \ , \qquad f \in BV(\mathbb{R}^+) \ ,$$

where $E : \mathbb{R} \to L(L^p(\mathbb{R}))$ is the spectral family given by $E(\lambda) = \chi_{(-\infty,\lambda]}(D)$, $\lambda \in \mathbb{R}$, and the integral exists as a strong operator limit of certain Riemann-Stieltjes sums; see [7; Chapter 2] for the terminology and properties of the integral. Here $L(L^p(\mathbb{R}))$ is the space of all continuous linear operators from $L^p(\mathbb{R})$ into itself.

At this stage it is natural to ask whether L admits a functional calculus based on some richer family of functions. Indeed, this is the case for $p = 2$. Suppose that $J \subseteq [0,\infty)$ is an interval. Then $\chi_J \circ \gamma \in BV(\mathbb{R})$ is the characteristic function of the set $\{t^{1/2};\, t \in J\} \cup \{-t^{1/2};\, t \in J\}$ which, with obvious notation, is the union of the two intervals $J^{1/2}$ and $-J^{1/2}$. Accordingly, $\chi_J \circ \gamma = \chi_{J^{1/2}} + \chi_{-J^{1/2}} - \chi_J(0)\chi_{\{0\}}$ and so the operator $\chi_J(L)$ defined via (2) is just $\chi_{J^{1/2}}(D) + \chi_{-J^{1/2}}(D)$; it is a projection commuting with L (in the sense of (11.2) below). Furthermore, the family of projections $\{\chi_J(L);\, J$ an interval in $\mathbb{R}^+\}$ is uniformly bounded in $L^p(\mathbb{R})$, [9; p.100]. For the case $p = 2$ this family of projections can be extended so that a projection is assigned to each Borel subset of $[0,\infty)$ and the so extended family forms the resolution of the identity for the self-adjoint operator L in $L^2(\mathbb{R})$. There is then available an extensive functional calculus, namely that based on all bounded Borel functions on $[0,\infty)$. However, if $p \neq 2$, then it turns out that

$$\{\chi_J(L);\ J \text{ a finite disjoint union of intervals in } \mathcal{R}^+\} \tag{3}$$

is not a uniformly bounded set of continuous operators in $L^p(\mathcal{R})$. Accordingly, the family of projections (3) cannot be enlarged to form a spectral measure in $L^p(\mathcal{R})$, [4; XVII Lemma 3.3. and Corollary 3.10]. Using this observation it is possible to establish (see the Appendix) that L is not an (unbounded) scalar-type spectral operator in the classical sense of N. Dunford [4] when $p \neq 2$.

Nevertheless, we wish now to indicate that for the case $p \neq 2$ something positive can still be said. There is available a functional calculus for L based on a certain algebra of bounded Borel functions on $\mathcal{R}^+$ (but not all) which has many features in common with the L^1-space corresponding to a classical spectral measure.

Denote by $\mathcal{A}^{(p)}(\mathcal{R}^+)$ the Boolean algebra consisting of those Borel sets $E \subseteq \mathcal{R}^+$ for which $\chi_E \circ \gamma$ is a p-multiplier and, for each such set E, let $P(E) = \chi_E(L)$ be defined by (2). Then it is possible (due to some recent work of I. Kluvánek [5]) to associate with P an L^1-type space via an "integration procedure" such that the integration mapping $f \mapsto \int_0^\infty f dP$ is a continuous algebra homomorphism. We proceed to outline this procedure.

The assignment $E \mapsto P(E)$, $E \in \mathcal{A}^{(p)}(\mathcal{R}^+)$, is finitely additive, multiplicative and $P(\mathcal{R}^+) = I$. If $\text{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$ denotes the vector space of all $\mathcal{A}^{(p)}(\mathcal{R}^+)$-simple functions, then P has an unique additive and multiplicative extension to $\text{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$ defined in an obvious way; its value at an element $f \in \text{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$ is denoted by $\int_0^\infty f dP$. The set function P turns out to be closable (see [8]) in Kluvánek's sense, meaning that

$$\lim_{n \to \infty} \left\| \sum_{j=1}^{n} \int_0^\infty f_j dP \right\| = 0$$

whenever $f_j \in \text{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$, $j = 1,2,\ldots$, are functions satisfying

$$\sum_{j=1}^{\infty} \left\| \int_0^\infty f_j dP \right\| < \infty \tag{4}$$

and $\sum_{j=1}^{\infty} f_j(w) = 0$ for every $w \in \mathcal{R}^+$ such that

$$\sum_{j=1}^{\infty} |f_j(w)| < \infty . \tag{5}$$

A function $f : \mathcal{R}^+ \to \mathcal{C}$ is said to be P-integrable [5] if, and only if, there exist functions $f_j \in \mathrm{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$, $j = 1,2,...$, satisfying (4), such that

$$f(w) = \sum_{j=1}^{\infty} f_j(w) \tag{6}$$

holds for every $w \in \mathcal{R}^+$ for which the inequality (5) holds. The closability of P guarantees that the operator $\sum_{j=1}^{\infty} \int_0^{\infty} f_j \mathrm{d}P$, denoted by $\int_0^{\infty} f \mathrm{d}P$, is well-defined. Indeed, suppose that $\{g_j\} \subseteq \mathrm{sim}(\mathcal{A}^{(p)}(\mathcal{R}^+))$ is another sequence such that $\sum_{j=1}^{\infty} \|\int_0^{\infty} g_j \mathrm{d}P\| < \infty$ and $f(w) = \sum_{j=1}^{\infty} g_j(w)$ for every $w \in \mathcal{R}^+$ for which $\sum_{j=1}^{\infty} |g_j(w)| < \infty$. Then the sequence $\{h_j\}$ defined by $h_{2k-1} = f_k$ and $h_{2k} = -g_k$, $k = 1,2,...$, satisfies

$$\sum_{j=1}^{\infty} \| \int_0^{\infty} h_j \mathrm{d}P\| = \sum_{j=1}^{\infty} \| \int_0^{\infty} f_j \mathrm{d}P\| + \sum_{j=1}^{\infty} \| \int_0^{\infty} g_j \mathrm{d}P\| < \infty$$

and $\sum_{j=1}^{\infty} h_j(w) = 0$ for every $w \in \mathcal{R}^+$ such that $\sum_{j=1}^{\infty} |h_j(w)| < \infty$. Since

$$\sum_{k=1}^{2n} \int_0^{\infty} h_k \mathrm{d}P = \sum_{j=1}^{n} \int_0^{\infty} f_j \mathrm{d}P - \sum_{j=1}^{n} \int_0^{\infty} g_j \mathrm{d}P ,$$

for each $n = 1,2,...$, the closability of P ensures that $\sum_{j=1}^{\infty} \int_0^{\infty} f_j \mathrm{d}P = \sum_{j=1}^{\infty} \int_0^{\infty} g_j \mathrm{d}P$.

The space of all P-integrable functions is denoted by $L(P)$. It turns out that $L(P) \subseteq L^{\infty}(\mathcal{R}^+)$ and $\|f\|_{\infty} \leq \| \int_0^{\infty} f \mathrm{d}P\|$, for every $f \in L(P)$. In addition, if $f,g \in L(P)$, then also $fg \in L(P)$ and $\int_0^{\infty} fg \mathrm{d}P = (\int_0^{\infty} f \mathrm{d}P)(\int_0^{\infty} g \mathrm{d}P)$, that is, $L(P)$ is an algebra of functions. Concerning the spectrum, it is the case that

$$\sigma(\int_0^{\infty} f \mathrm{d}P) = \bigcap_{U \in \mathcal{N}} \overline{\{f(w);\, w \in \mathcal{R}^+ \backslash U\}} , \tag{7}$$

for each $f \in L(P)$, where $\mathcal{N}$ is the collection of all null sets in $[0,\infty)$ with respect to Lebesgue measure. These statements constitute a special case of Proposition 2 in [5].

Since the functional $f \mapsto \| \int_0^\infty f dP\|$ is a seminorm on $L(P)$ it is possible to form the associated normed space in the usual way; this space is denoted by $L^1(P)$. Then $L^1(P)$ is actually complete and the integration mapping

$$f \mapsto \int_0^\infty f dP, \qquad f \in L^1(P) , \tag{8}$$

induces an isomorphism of the (semisimple) Banach algebra $L^1(P)$ onto the uniformly closed algebra generated by $\{P(E); E \in \mathcal{A}^{(p)}(\mathcal{R}^+)\}$, [5]; denote this algebra by $<P>$.

Concerning the space $L(P)$ it is known to contain every function of bounded variation on $[0,\infty)$ which vanishes at infinity and whose continuous singular component is zero [8]. In particular, $x \mapsto (x - \lambda)^{-1}$, $x \in \mathcal{R}^+$, is P-integrable whenever $\lambda \in \rho(L)$ and hence, $<P>$ contains all the resolvent operators of L. Of course, $L(P)$ also contains many functions which are not of bounded variation. We remark that if $f \in L(P)$, then also the functions $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ and $\bar{f}$ (complex conjugation) are P-integrable [8], although $|f|$ need not be.

So, the integration mapping (8) provides a functional calculus for L based on the Banach algebra $L^1(P)$ which includes the resolvent operators of L and has associated with it the spectral mapping theorem (7). In addition, any operator $T \in <P>$, necessarily of the form $\int_0^\infty f dP$ for some P-integrable function f, can be approximated by linear combinations of disjoint values of P, a feature in common with the case when P is the resolution of the identity of a scalar-type spectral operator (in the sense of N. Dunford [4]). The formulae (7) and (8) are obvious analogues of similar formulae known to be valid for scalar-type operators. So, even though L is not a scalar-type spectral operator in the classical sense (for $p \neq 2$), it is still natural to inquire whether L exhibits further similarities (if suitably interpreted) with scalar-type operators? This is indeed the case. It turns out [8] that if $\lambda^{(n)}$, $n = 1,2,...$, denotes the function $w \mapsto w\, \chi_{[0,n]}(w)$, $w \in \mathcal{R}^+$, then each $\lambda^{(n)}$ is P-integrable and

$$\mathcal{D}(L) = \{f \in L^p(\mathbb{R});\ \lim_{n\to\infty} \left(\int_0^\infty \lambda^{(n)}\, dP\right) f \text{ exists in } L^p(\mathbb{R})\} \tag{9}$$

with

$$Lf = \lim_{n\to\infty} \left(\int_0^\infty \lambda^{(n)} dP\right) f\,, \qquad f \in \mathcal{D}(L)\,; \tag{10}$$

see [4; p.2238] for the case of scalar-type spectral operators. Furthermore, a bounded operator T in $L^p(\mathbb{R})$ commutes with L (in the sense of (11.2) below) if and only if it commutes with each projection $P(E)$, $E \in \mathcal{A}^{(p)}(\mathbb{R}^+)$, [8]; see [4; XVIII Corollary 2.4] for the case of spectral operators. In addition (see [8]), the Boolean algebra of projections $\{P(E);\ E \in \mathcal{A}^{(p)}(\mathbb{R}^+)\}$ satisfies all the properties of being a classical resolution of the identity for L, in the sense of Definition 2.1 of [4; Ch.XVIII], except the boundedness requirement. Namely,

(11.1) $\mathcal{D}(L) \supseteq \{P(E)f;\ f \in L^p(\mathbb{R})\}$ whenever $E \in \mathcal{A}^{(p)}(\mathbb{R}^+)$ is a bounded set ,

(11.2) $P(E)(\mathcal{D}(L)) \subseteq \mathcal{D}(L)$ and $LP(E)f = P(E)Lf$, $f \in \mathcal{D}(L)$, for every $E \in \mathcal{A}^{(p)}(\mathbb{R}^+)$ and

(11.3) if $E \in \mathcal{A}^{(p)}(\mathbb{R}^+)$, then $\sigma(L_{P(E)}) \subseteq \bar{E}$, where $L_{P(E)}$ denotes the restriction of L to the closed subspace $\{P(E)f;\ f \in L^p(\mathbb{R})\}$.

In conclusion we wish to make some remarks concerning the connection between the various functional calculi. The function $z \mapsto z^i$ belongs to $H^\infty(S_\epsilon)$ for every $0 < \epsilon < \pi$ but its restriction to $[0,\infty)$ is surely not of bounded variation. On the other hand, the characteristic function of any interval $J \subsetneq \mathbb{R}^+$ (other than $\mathbb{R}^+$) belongs to $BV(\mathbb{R}^+)$ but it is not the restriction to $\mathbb{R}^+$ of any element of $H^\infty(S_\epsilon)$ for any $\epsilon > 0$. The function ψ_s: $x \mapsto e^{isx^{1/2}}$, $x \in \mathbb{R}^+$, is known to belong to $L(P)$, [8], for every $s \in \mathbb{R}$, but it is not in $BV(\mathbb{R}^+)$ if $s \neq 0$. If ψ_s were the restriction to $\mathbb{R}^+$ of a holomorphic function in S_ϵ, then this would have to be the function $z \mapsto e^{isz^{1/2}}$, $z \in S_\epsilon$, which is not bounded when $s < 0$. So, there exist functions in $L(P)$ which are not the restriction to $\mathbb{R}^+$ of any element of $H^\infty(S_\epsilon)$ for any $0 < \epsilon < \pi$. Concerning the converse however, it turns out that if $0 < \epsilon < \pi$, then $H^\infty(S_\epsilon)$ is contained in $L(P)$ in the sense that the restriction to $\mathbb{R}^+$ of any element from $H^{\infty}(S_{\epsilon})$ belongs to $L(P)$, [8].

APPENDIX. In this section we establish the following result (mentioned earlier).

THEOREM. *Let $1 < p < \infty$ with $p \neq 2$. Then the operator L is not an (unbounded) scalar-type spectral operator in $L^p(\mathcal{R})$.*

It suffices to consider the case $p \in (1,2)$. This follows from the fact that the dual operator of an (unbounded) scalar-type operator in a reflexive Banach space is also a scalar-type operator and the fact that the dual operator of L (when L is considered in $L^p(\mathcal{R})$) is just L in $L^q(\mathcal{R})$ where $p^{-1} + q^{-1} = 1$. So, from now on it is assumed that $p \in (1,2)$. In this case the Fourier transform maps $L^p(\mathcal{R})$ into $L^q(\mathcal{R})$. Then

$$\mathcal{D}(L) = \{f \in L^p(\mathcal{R});\ \xi^2\hat{f}(\xi) = \hat{g}(\xi) \text{ for some } g \in L^p(\mathcal{R})\} \tag{12}$$

and, for each $f \in \mathcal{D}(L)$, it is the case that $Lf = g$ where $g \in L^p(\mathcal{R})$ satisfies $\xi^2\hat{f}(\xi)=\hat{g}(\xi)$.

For the definition and basic properties of an (unbounded) scalar-type spectral operator T in a Banach space X we refer to [4; Chapter XVIII]. In particular, such an operator T is necessarily closed, densely defined and has a unique resolution of the identity (i.e. a spectral measure), say $Q : \mathcal{B} \to L(X)$, which is σ-additive for the strong operator topology and such that

$$\mathcal{D}(T) = \{x \in X;\ \lim_{n\to\infty} \left(\int_{\mathcal{C}} \lambda^{(n)} dQ\right)x \text{ exists in } X\}$$

with

$$Tx = \lim_{n\to\infty} \left(\int_{\mathcal{C}} \lambda^{(n)} dQ\right)x\ . \qquad x \in \mathcal{D}(T)\ .$$

Here $L(X)$ is the space of all continuous linear operators of X into itself, $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathcal{C}$ and, for each $n = 1,2,...$, $\lambda^{(n)}$ is the bounded measurable function $w \mapsto w\chi_n(w)$, $w \in \mathcal{C}$, where χ_n is the characteristic function of the set $\{z \in \mathcal{C}; |z| \leq n\}$. In particular, each function $\lambda^{(n)}$ is Q-integrable (in the sense of [4; Ch.XVII, §2]) and so $\int_{\mathcal{C}} \lambda^{(n)} dQ$ is an element of $L(X)$. The support of the spectral measure Q is precisely $\sigma(T)$, [3; Theorem 17], and the residual spectrum of T is necessarily empty [3;

Theorem 21].

If $J \subseteq \mathcal{R}^+$ is an interval, then the identity $\chi_J \circ \gamma = \chi_{J^{1/2}} + \chi_{-J^{1/2}} - \chi_J(0)\chi_{\{0\}}$, together with the fact that intervals in $\mathcal{R}$ are p-multiplier sets, shows that $J \in \mathcal{A}^{(p)}(\mathcal{R}^+)$ and $P(J) = \chi_{J^{1/2}}(D) + \chi_{-J^{1/2}}(D)$. Using the formulation (12) of $\mathcal{D}(L)$ and the corresponding definition of L in terms of the Fourier transform it is not difficult to establish that $P(J)(\mathcal{D}(L)) \subseteq \mathcal{D}(L)$ and

$$P(J)Lf = L\,P(J)f\,, \qquad f \in \mathcal{D}(L)\,. \tag{13}$$

To establish the Theorem we proceed by contradiction. So,suppose that L is a scalar-type operator with resolution of the identity Q. Then Q is supported by $\sigma(L) = \mathcal{R}^+$ and $\{Q(E);\ E \subseteq \mathcal{R}^+,\ E$ a Borel set$\}$ is uniformly bounded in $L(L^p(\mathcal{R}))$, [4; XVII Lemma 3.3 and Corollary 3.10]. Suppose that

$$Q(J) = P(J)\,, \qquad J \subseteq \mathcal{R}^+, J \text{ an interval}\,. \tag{14}$$

Then by finite additivity of P and Q, (14) would be valid for every set $J \subseteq \mathcal{R}^+$ which is the union of finitely many disjoint intervals in $\mathcal{R}^+$ and hence, the family of operators (3) would be uniformly bounded in $L(L^p(\mathcal{R}))$. This provides the desired contradiction. So, it remains to establish (14) for which some auxiliary results are needed.

Let T be an (unbounded) scalar-type operator in a Banach space X with Q its resolution of the identity. Let F be a bounded projection in X such that $F(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$ and $FTx = TFx$ for each $x \in \mathcal{D}(T)$. Then necessarily

$$FQ(U) = Q(U)F\,, \qquad U \in \mathcal{B}\,,$$

[4; XVIII Corollary 2.4]. It follows that the range of F, denoted by $F(X)$, is invariant for each operator $Q(U)$, $U \in \mathcal{B}$. Accordingly, the set function $Q_{F(X)} : \mathcal{B} \to L(F(X))$ defined in the obvious way by restriction to $F(X)$ is a spectral measure. This induces the (unbounded) scalar-type operator $\tilde{T}_F$ in $F(X)$ with

$$\mathcal{D}(\tilde{T}_F) = \{z \in F(X);\ \lim_{n\to\infty} \left(\int_{\mathcal{C}} \lambda^{(n)} dQ_{F(X)}\right)z \text{ exists}\}$$

and

$$\tilde{T}_F z = \lim_{n\to\infty} \left(\int_C \lambda^{(n)} dQ_{F(X)}\right) z, \qquad z \in \mathcal{D}(\tilde{T}_F).$$

There is also the operator T_F with $\mathcal{D}(T_F) = \mathcal{D}(T) \cap F(X)$ defined by

$$T_F z = Tz, \qquad z \in \mathcal{D}(T_F).$$

Of course, such an operator can be defined even if T is not a scalar-type operator. Since each $z \in \mathcal{D}(T_F)$ satisfies $z = Fz$ and F commutes with T in the sense indicated above, we have

$$T_F z = Tz = TFz = FTz, \qquad z \in \mathcal{D}(T_F).$$

It is not difficult to check that T_F is a closed, densely defined operator in the Banach space $F(X)$.

Lemma 1. $T_F = \tilde{T}_F$.

Proof. If $z \in \mathcal{D}(\tilde{T}_F) \subseteq F(X)$, then $z = Fz$. It follows from the definition of $Q_{F(X)}$ that $Q(U)z = Q_{F(X)}(U)z$ for all $U \in \mathcal{B}$ and hence, that

$$\left(\int_C f dQ_{F(X)}\right) z = \left(\int_C f dQ\right) z \tag{15}$$

for all $\mathcal{B}$-simple functions f. By a standard approximation argument (15) then holds for all bounded measurable functions f. In particular, $(\int_C \lambda^{(n)} dQ_{F(X)}) z = (\int_C \lambda^{(n)} dQ) z$ for each $n = 1,2,\ldots$, and so

$$\lim_{n\to\infty} \left(\int_C \lambda^{(n)} dQ\right) z = \lim_{n\to\infty} \left(\int_C \lambda^{(n)} dQ_{F(X)}\right) z \tag{16}$$

exists (as $z \in \mathcal{D}(\tilde{T}_F)$), thereby showing that $z \in \mathcal{D}(T_F) = \mathcal{D}(T) \cap F(X)$. Furthermore, it follows from (16) and the definition of T and $\tilde{T}_F$ that $\tilde{T}_F z = Tz = T_F z$. A similar type of arugment shows that $\mathcal{D}(T_F) \subseteq \mathcal{D}(\tilde{T}_F)$ and $\tilde{T}_F z = T_F z$ for each $z \in \mathcal{D}(T_F)$. □

COROLLARY 1.1. *Let J be a closed subset of C such that $\sigma(T_F) \subseteq J$. Then $F = Q(J)F = FQ(J)$.*

Proof. Since $\tilde{T}_F$ is a scalar-type operator in $F(X)$ with resolution of the identity

$Q_{F(X)}$ it follows from an earlier remark that $Q_{F(X)}$ is supported by $\sigma(\hat{T}_F)$. But, Lemma 1 implies that $\sigma(\hat{T}_F) = \sigma(T_F)$, which is contained in J by hypothesis, and so $I|_{F(X)} = Q_{F(X)}(J) = Q(J)|_{F(X)}$. Hence, if $x \in X$, then $Fx \in F(X)$ and it follows that

$$Fx = I|_{F(X)}Fx = Q(J)|_{F(X)}Fx = Q(J)Fx \ . \qquad \square$$

COROLLARY 1.2. *Suppose that T has no eigenvalues. Let J be a closed set in $\mathbb{C}$ such that*

(i) $\sigma(T_F) \subseteq J$,

(ii) $\sigma(T_{(I-F)}) \subseteq \overline{\sigma(T)\backslash J}$ *and*

(iii) $J \cap \overline{\sigma(T)\backslash J}$ *is a countable set.*

Then $F = Q(J)$.

Proof. Applying Corollary 1.1 to (i) gives $F = Q(J)F = FQ(J)$ and applying Corollary 1.1 to (ii) gives

$$(I - F) = Q(\overline{\sigma(T)\backslash J})(I - F) = (I - F)\, Q(\overline{\sigma(T)\backslash J}) \ .$$

Since T has no residual spectrum (as it is in a scalar-type operator) and no eigenvalues (by hypothesis) it follows that $Q(\{z\}) = 0$ for every $z \in \mathbb{C}$, [3; Theorem 21]. Then (iii) and the σ-additivity of Q imply that

$$Q(\overline{\sigma(T)\backslash J}) = Q(\sigma(T)\backslash J) + Q(J \cap \sigma(T)) \quad Q(\sigma(T)\backslash J) = I - Q(J) \ .$$

The desired conclusion follows. $\square$

We now apply the above results in the setting of $T = L$ and $X = L^p(\mathbb{R})$; it is assumed that L is a scalar-type operator with resolution of the identity Q. Let $J \subseteq \mathbb{R}^+$ be any closed interval and let $F = P(J)$. Then F commutes with L by (13) and condition (iii) of Corollary 1.2 is certainly satisfied. Suppose for the moment that (i) and (ii) also hold. Then Corollary 1.2 would imply that $P(J) = Q(J)$, for every closed interval J in

$\mathcal{R}^+$. Since L has no eigenvalues it follows that necessarily $Q(\{z\}) = 0$ for every $z \in \sigma(L) = \mathcal{R}^+$. But, singleton subsets of $\mathcal{R}^+$ are also P-null and hence (14) would follow. So, to complete the proof of the Theorem it remains to show that

$$\sigma(L_{P(J)}) \subseteq J \text{ and } \sigma(L_{(I-P(J))}) \subseteq \overline{\mathcal{R}^+ \backslash J}, \tag{17}$$

for every closed interval $J \subseteq \mathcal{R}^+$.

Suppose $\lambda \in \mathcal{C}\backslash J$. Then there is $M > 0$ such that $|\xi^2 - \lambda| \geq M$ for all $\xi \in J^{1/2} \cup (-J^{1/2})$ and hence, the function

$$h_\lambda: \xi \mapsto (\chi_{J^{1/2}}(\xi) + \chi_{-J^{1/2}}(\xi))/(\lambda - \xi^2), \qquad \xi \in \mathcal{R},$$

is bounded and measurable. In fact, h_λ is a p-multiplier. Since $P(J) = \chi_{J^{1/2}}(D) + \chi_{-J^{1/2}}(D)$ is a p-multiplier operator it follows that $h_\lambda(D)P(J) = P(J)h_\lambda(D)$ and so the range $\mathcal{R}(P(J))$ of $P(J)$ is invariant for $h_\lambda(D)$. Accordingly, the restriction, $h_\lambda(D)_{P(J)}$, of $h_\lambda(D)$ to $\mathcal{R}(P(J))$ is an element of $L(\mathcal{R}(P(J)))$. Using the formulation (12) of $\mathcal{D}(L)$ and the corresponding definition of L in terms of the Fourier transform it is not difficult to check that $h_\lambda(D)_{P(J)}$ is the resolvent operator of $L_{P(J)}$ at the point λ (in the space $L(\mathcal{R}(P(J)))$, of course). This shows that $\sigma(L_{P(J)}) \subseteq J$. The other inclusion in (17) can be established in a similar way. □

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Centre for Mathematical Analysis
Australian National University
Canberra, A.C.T. 2600
Australia