A TRANSMUTATION PROPERTY OF THE GENERALIZED ABEL TRANSFORM ASSOCIATED WITH ROOT SYSTEM A_2

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Let Σ be a root system of type A_2 in a real two dimensional Euclidean space $\mathfrak{a}$. Let W denote the Weyl group of Σ and $\mathcal{D}_W(\mathfrak{a})$ the space of W-invariant C^∞-functions on $\mathfrak{a}$ with compact support. Choose a positive Weyl chamber $\mathfrak{a}^+$. For $f \in \mathcal{D}_W(\mathfrak{a})$ and a complex parameter m with positive real part we define (as in [2]) an integral transform on $\mathfrak{a}^+$ which coincides, for certain values of the parameter m, with the Abel transform on some symmetric spaces of the noncompact type. An important property of the Abel transform is that it intertwines the radial part of the Laplace-Beltrami operator on these symmetric spaces with the ordinary Laplacian on $\mathfrak{a}$. In this note we state the result that the generalized Abel transform as introduced in [2] also satisfies this transmutation property. Detailed proofs will appear elsewhere.

In $\mathbb{R}^3$ we have the standard basis $\{e_1, e_2, e_3\}$ and inner product $\langle\cdot,\cdot\rangle$ for which this basis is orthonormal. Let $\mathfrak{a}$ denote the hyperplane in $\mathbb{R}^3$ orthogonal to the vector $e_1+e_2+e_3$. The inner product on $\mathbb{R}^3$ induces an inner product on $\mathfrak{a}$ which we shall also denote by $\langle\cdot,\cdot\rangle$. We identify the dual of $\mathbb{R}^3$ with $\mathbb{R}^3$ and the dual $\mathfrak{a}^*$ with $\mathfrak{a}$ by means of this inner product.

The root system of type A_2 can be identified with the set $\Sigma = \{\pm(e_1-e_2), \pm(e_1-e_3), \pm(e_2-e_3)\}$ in $\mathfrak{a}$. For Σ we take as basis $\Delta = \{e_1-e_2, e_2-e_3\}$ and we denote by Σ^+ the set of positive roots with respect to Δ. The positive Weyl chamber will be denoted by $\mathfrak{a}^+$. Let W denote the Weyl group of Σ. For $m \in \mathbb{C}$ we define $L(m)$, the so-called radial part of the Laplace-Beltrami operator associated with A_2, by

$$(1) \qquad L(m) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + m \sum_{1\le i<j\le 3} \coth(x_i - x_j)\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right).$$

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Here $L(m)$ is considered as differential operator on $\mathfrak{a}^+$ and we used coordinates (x_1, x_2, x_3) on $\mathfrak{a}$ (i.e. $x_1+x_2+x_3 = 0$). If $m = 1,2,4$ or 8 then $L(m)$ is the radial part of the Laplace-Beltrami operator associated with the symmetric spaces of the noncompact type $SL(3,\mathbb{R})/SO(3)$, $SL(3,\mathbb{C})/SU(3)$, $SU^*(6)/Sp(3)$ and $E_{6(-26)}/F_4$ respectively (see e.g. [3, Ch.II, prop. 3.9]). We will say that m corresponds to a group-case if $m = 1,2,4$ or 8. Note that $L(0)$ is the ordinary Laplacian on $\mathfrak{a}$ which we shall denote by $L_{\mathfrak{a}}$.

Let $\mathcal{D}_W(\mathfrak{a})$ denote the space of W-invariant C^∞-functions on $\mathfrak{a}$ with compact support. For $f \in \mathcal{D}_W(\mathfrak{a})$ and $m \in \mathbb{C}$, $\mathrm{Re}\, m > 0$ the Abel transform $F_f^{(m)}$ of f is the function on

$$\mathfrak{a}^+ = \{(t_1,t_2,t_3) \in \mathfrak{a} \mid t_1>t_2>t_3\}$$

defined by:

$$F_f^{(m)}(t_1,t_2,t_3) =$$

$$= \frac{\pi^{3m/2}\, 2^{m+4}}{\Gamma(\frac{1}{2}m)^3} \cdot \int_{y_3=-\infty}^{t_3} \mathrm{sh}(y_2-y_3)^{-(m-2)}\, (\mathrm{ch}(y_2-y_3)-\mathrm{ch}(t_2-t_3))^{\frac{1}{2}(m-2)}\, dy_3$$

$$(2) \qquad \int\!\!\int_{x_1>y_2>x_2>y_3>x_3} f(x_1,x_2,x_3) \cdot \prod_{1\le i<j\le 3} \mathrm{sh}(x_i-x_j) \cdot$$

$$\cdot \left\{-\prod_{i=1}^{3} (\mathrm{ch}(2x_i-t_2-t_3)-\mathrm{ch}(y_2-y_3))\right\}^{\frac{1}{2}(m-2)} dx_2\, dx_3 \quad .$$

In the inner integral x_1 is such that $x_1+x_2+x_3= 0$ and in the outer integral y_2 is such that $y_2+y_3= t_2+t_3$. Note that since $y_3<t_3$ we have $y_2-y_3> t_2-t_3> 0$. Also

$$-\prod_{i=1}^{3} (\mathrm{ch}(2x_i-t_2-t_3)-\mathrm{ch}(y_2-y_3)) = -2^3 \prod_{i=1}^{3} \mathrm{sh}(x_i-y_2)\, \mathrm{sh}(x_i-y_3) > 0 \ .$$

In [1] Aomoto obtained $F_f^{(1)}$ and $F_f^{(2)}$ as integral representation for the Abel transform for $SL(3,\mathbb{R})$ and $SL(3,\mathbb{C})$. In [2, section 6] we showed that this is also the case for $SU^*(6)$ and $E_{6(-26)}$ where $m = 4$ and 8 respectively. For other values of the parameter m there is no interpretation of $F_f^{(m)}$ as the Abel transform on some noncompact semisimple Lie group. We also showed in [2] that there exists a differential operator $D(m)$ on $\mathfrak{a}^+$ such that

$$F^{(m)}_{D(m)f} = \text{const.}\ F^{(m-2)}_{f}\ , \text{ on } \mathfrak{a}^+,\ \text{Re } m > 2\ ,$$

and

$$F^{(2)}_{D(2)f} = \text{const.}\ f\ , \text{ on } \mathfrak{a}^+ .$$

In particular the transform $f \longrightarrow F^{(m)}_{f}$ can be inverted on the right by a differential operator if m is even. If moreover m corresponds to a group case then this differential operator is also a left-inverse.

An important property of the Abel transform in the group-cases is the transmutation property with respect to the operator $L(m)$ in (1). Using the explicit expression (2) for the generalized Abel transform one can show that this transmutation property also holds for general m.

Theorem. *For* $f \in \mathcal{D}_W(\mathfrak{a})$ *and* $m \in \mathbb{C}$, Re $m > 0$ *let* $F^{(m)}_{f}$ *be defined by* (2). *Then*

$$F^{(m)}_{L(m)f} = (L_{\mathfrak{a}} - 2m^2)\ F^{(m)}_{f} \quad \text{on } \mathfrak{a}^+ .$$

Here $L(m)$ *is defined by* (1) *and* $L_{\mathfrak{a}} = L(0)$ *is the* ordinary Laplacian on $\mathfrak{a}$.

Note that $L_{\mathfrak{a}}$ is precisely the highest order term in $L(m)$. The number $2m^2$ is equal to $\langle\rho(m),\rho(m)\rangle$ where $\rho(m) = \tfrac{1}{2}m \sum_{\alpha \in \Sigma^+} \alpha = m(e_1-e_3)$. In the group cases the theorem follows from general theory (see e.g. [3, Ch. II, (39)]. The proof of the theorem for general m is a direct calculation (which we shall not present here).

References

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