A CLASS OF ERGODIC MEASURES

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1. INTRODUCTION

Let $\{\ell(i)\}$ be a sequence of positive integers, let $X = \prod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$ and Γ denote the weak product $\prod_{i=1}^{\infty} \mathbb{Z}_{\ell(i)}$. Then Γ acts on X by $(\gamma x)_i = \gamma_i + x_i$, i.e. as the group of finite coordinate changes.

It is our aim to "understand" measures on X which are ergodic for the action of Γ. Of course, this is a quixotic mission, equivalent to classifying all flows or all von Neumann algebras. At a more realistic level we may hope to understand the relationship of various constructions of classical measures (infinite tensor products, or Riesz products for example) to the general case. It is the purpose of this note to sketch some results in this direction, the proofs of which will appear elsewhere.

2. FORMALISM

It will turn out that the inductive limit structure of Γ allows us to write any sufficiently regular measure which is quasi invariant under Γ in the form

$$d\mu(x) = \lim_{n\to\infty} g_1(x) g_2(x) \ldots g_n(x)\, d\lambda(x)$$

where the function, $g_k(x)$, depends on only the coordinates x_j, for $j > k$, and where λ is Haar measure on X.

To this end let μ be a Γ-quasi-invariant probability measure on X. Consider the cocycle $h(\gamma,x) = \frac{d\mu\circ\gamma}{d\mu}(x)$ (μ a.e.) Letting μ_F be the "tail" measure $\frac{1}{|\Gamma_F|}\sum_{\gamma\in\Gamma_F}\mu\circ\gamma$, we see that μ_F is a probability measure equivalent to μ. Here, for a finite subset F of $\mathbb{N}$, $\Gamma_F = \{\gamma \in \Gamma : \gamma_i = 0,\ i \notin F\}$. A simple calculation reveals that $\frac{1}{|\Gamma_F|}\sum_{\gamma\in\Gamma_F} h(\gamma,x)$ is a version of $\frac{d\mu_F}{d\mu}(x)$. We define a function G_F by

$$G_F(x) = \frac{1}{\sum_{\gamma\in\Gamma_F} g(\gamma,x)}$$

This is a version of $\frac{d\mu}{d\mu_F}$, and we have

$$h(\gamma,x) = \frac{G_F(\gamma x)}{G_F(x)},\ \text{provided}\ \gamma \in \Gamma_F \ \ldots \ (*)$$

The G_F are a <u>compatible family</u> in the sense that if $\gamma \in F_1 \subseteq F_2$, then $\frac{G_{F_1}(\gamma x)}{G_{F_1}(x)} = \frac{G_{F_2}(\gamma x)}{G_{F_2}(x)}$. Conversely, given a compatible family of G_F, we may define a cocycle by (*).

Suppose we have a compatible family and a sequence $F_0 \subseteq F_1 \subseteq F_2 \ldots$ whose union is $\mathbb{N}$, we let $g_i(x) = \frac{G_{F_i}(x)}{G_{F_{i-1}}(x)}$. Then $g_i(x)$ depends only on the coordinates x_k where $k \notin F_{i-1}$, and if F is a finite set and i is the least integer such that $F \subseteq F_i$, we have

$$G_F(x) = g_1(x)\ldots g_i(x) \ \ldots \qquad (**)$$

Thus, we have presented $h(\gamma,x)$ as $\dfrac{g_1(\gamma x)\ldots g_i(\gamma x)}{g_1(x)\ldots g_i(x)}$. Conversely, given $\{g_1,g_2,\ldots\}$ so that g_i is independent of the x_j with $j \in F_i$, we may define a compatible family $\{G_F\}$ by (**).

Let $G = \{G_F\}$. We say μ is a G-measure if its cocycle h satisfies (*).

3. CONTINUOUS FAMILIES

Define a map $q: X \to \mathbb{T}$ by $q(x) = \sum_{i=1}^{\infty} \dfrac{x_i}{\ell(1)\ldots\ell(i)}$. We say that a function f on X is circle continuous if it factors through q.

THEOREM 1 *Let* ν *be a probability measure on* X, *quasi-invariant under* Γ. *There is a* G-*measure* $\mu \sim \nu$ *such that the functions* G_F *corresponding to* μ *have a circle continuous version.*

PROOF One works with the g_i's. Replace them with continuous functions $\tilde{g}_i$, equal to g_i on successively larger sets (by Lusin's theorem). Then invoke the Borel-Cantelli lemma to say that the function $g = \prod^{\infty} \dfrac{g_i}{\tilde{g}_i}$ belongs to $L^1_+(\nu)$. The measure $\dfrac{g.\nu}{\int g d\nu} = \nu$ is the measure we want. □

THEOREM 2 *Let* $G = \{G_F\}$ *be a circle continuous compatible family. There exists a* G-*measure.*

PROOF Let μ_F be any family of probability measures on X then any weak*-limit of

$$\nu_F = G_F \cdot \left\{ \frac{1}{|\Gamma_F|} \sum_{\gamma \in \Gamma_F} \mu_F \circ \gamma \right\} \text{ is a G-measure.}$$ []

For general G, there will be many G-measures.

4. UNIQUE ERGODICITY

We suppose throughout this section that the G_F are compatible, continuous and normalized in the sense that $\frac{1}{|\Gamma_F|} \sum G_F(\gamma x) = 1$. (We may always assume this without loss of generality.)

PROPOSITION *(i) If there's a unique G-measure then it is ergodic.*

(ii) If every G-measure is ergodic, then there's just one.

We isolate this property of <u>unique ergodicity</u> of G and wish to study certain conditions on the family G which ensure that it happens. If it does, the measure may be constructed analogously to the classical measures mentioned above:

THEOREM 3 *The following conditions are equivalent*

(i) *There's a unique G-measure.*

(ii) *The F-net in* $C(X)$ *defined by*

$$A_F(f)(x) = \frac{1}{|\Gamma_F|} \sum_{\gamma \in \Gamma_F} G_F(\gamma x) f(\gamma x)$$

converges uniformly to a constant

(iii) $A_F(f)$ *converges pointwise to a constant.*

In this case, the unique measure is ergodic and may be realized as

$$\text{weak}^*\text{-}\lim_{n\to\infty} \prod_{k=1}^{n} g_k(x)\,d\lambda(x),$$ where λ denotes the Haar measure on X.

5. SUFFICIENT CONDITIONS FOR UNIQUENESS

We would like now to specify some conditions on the functions G_F (or g_i) which guarantee uniqueness of μ. We'd like to include the usual examples - infinite products, and Riesz products, as well as the g-measures studied by Keane, Ledrappier and Walters.

These examples are seen as G-measures as follows

(a) INFINITE TENSOR PRODUCTS. We are given, for each k, a probability measure α^k on $\mathbb{Z}_{\ell(k)}$. Set $g_k(x) = \alpha^k(x_k)$

(b) RIESZ PRODUCTS. Take $g_k(x) = 1 + a_k \cos(2\pi q_k(x))$ where

$$q_k(x) = q(x_k, x_{k+1}, \ldots) = \sum_{i=k}^{\infty} \frac{x_i}{\ell(k)\ldots\ell(i)} .$$

(c) g-MEASURES. Suppose that $\ell(n) = \ell$(constant) for all n and take $g_k(x) = g(x_k, x_{k+1}, \ldots)$, where g is a sufficiently nice (e.g. Lipschitz) function on X. The latter have been studied by Keane et al for their invariance under the shift - not under finite coordinate changes.

These classes of examples are all included in Theorem 4 below. We have found that one can obtain uniqueness by imposing two conditions, the first a weak kind of equicontinuity condition and the second a type of mixing condition. In order to state these, notice that we may assume without loss of generality that g_k is q_{k+1} continuous, i.e. that for all $\gamma \in \mathbb{Z}_{\ell(k)}$ there exists $g'_k(\gamma,\bullet)$ a continuous function on $\mathbb{T}$ such that $g_k(\gamma,x_{k+1},x_{k+2}\ldots) = g'_k(\gamma,q_{k+1}(x_{k+1},\ldots))$.

We assume furthermore that our functions satisfy

(E) $\forall\varepsilon > 0 \ \exists n \in \mathbb{N}$ such that $\forall k, \forall\gamma \in \Gamma^{I_{k+n}}$

$$\sup_{x\in X} \left|1 - \frac{G_{I_k}(\gamma x)}{G_{I_k}(x)}\right| < \varepsilon , \quad \text{and}$$

(M) $\liminf \{g'_n(\gamma,x) : \gamma \in \mathbb{Z}(\ell(n))\} > 0$ for all $x \in \mathbb{T}$.

THEOREM 4 *In the presence of* (E) *and* (M), *there is a unique* G-*measure.*

PROOF One is able to reduce the problem to one on $\mathbb{T}$ and use the Arzelà-Ascoli theorem. Notice that (E) is a condition on the "tail". □

The uniqueness fails with no conditions on the G_n's. We have been working on weakening (M).

For example, the theorem remains valid if M is replaced by M_2 below. To formulate our condition, let ℓ be a positive integer, let $\delta > 0$ and take

$$A_\delta(\ell,t) = \{\gamma \in \mathbb{Z}_\ell : \liminf \{g'_n(\gamma,t) : \ell(n) = \ell\} \ge \delta\}$$

and $\quad A(\ell,t) = \bigcup_{\delta>0} A_\delta(\ell,t)$. Condition (M_2) reads as follows.

For all $s,t \in \mathbb{T}$ and for all ℓ such that $\ell(i) = \ell$ infinitely often

$$A(\ell,t) \cap A(\ell,s) \neq \phi .$$

Further weakening is possible, but the description of the conditions becomes more complicated. Details will appear elsewhere.

Similar results may be formulated and proved for the case where $\ell(i)$ is unbounded.

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