TRIGONOMETRIC SUMS AND POLYNOMIAL ZEROS

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1. INTRODUCTION

This is a preliminary report on work in progress on an ARGS project concerned with positive trigonometric sums and their applications.

Consider the cosine series

\[ G_m(\theta) = \sum_{j=1}^{\infty} j^{-m} \cos j\theta, \quad m \in \mathbb{N}, \]

and its partial sums

\[ G^n_m(\theta) = \sum_{j=1}^{n} j^{-m} \cos j\theta. \]

We establish the following

\begin{enumerate}
  \item \( G_m(\theta) \) is decreasing on \((0,\pi)\),
  \item the unique zero of \( G_m(\theta) \) lying in \((0,\pi)\) increases with \( m \),
  \item \( G^n_m(\theta) \) is decreasing on \((0,\pi)\) for \( m \geq 2 \),
  \item the unique zero of \( G^n_m(\theta) \) lying in \((0,\pi)\) increases with \( m \geq 2 \) for fixed \( n \).
\end{enumerate}

Apart from the obvious connection with the Riemann zeta function, such series arise in the context of a quadrature-based method for solving boundary integral equations currently being developed by I.H. Sloan and W.L. Wendland [3]: the zeros of \( G_m(\theta) \) in \((0,2\pi)\) correspond to the quadrature points, and a consequence of (ii) is the stability of some forms of the method.
The special values \( m = 1, 2, 4, \ldots \) give an idea of the general behaviour of \( G_m(\theta) \):

\[
G_1(\theta) = -\frac{1}{2} \log(2(1 - \cos \theta)),
\]

\[
G_2(\theta) = \frac{\theta}{4} - \frac{\pi \theta}{2} + \frac{\pi}{6},
\]

\[
G_4(\theta) = -\frac{\theta}{48} + \frac{\pi \theta}{12} - \frac{\pi^2 \theta^2}{12} + \frac{\pi^4 \theta^4}{90},
\]

\[
G_{2m}(\theta) = \cos \theta;
\]

note that up to a constant \( G_{2m}(\theta) \) are the Bernoulli polynomials.

2. PROOF OF THEOREM

(i) For \( m = 1 \) we see immediately from the explicit formula that

\( G_1(\theta) \) is decreasing on \((0, \pi)\). For \( m > 1 \), the series may validly be differentiated termwise \([2, 196, 199.4]\) so that we reduce to proving

\[
H_{\beta}(\theta) = \sum_{j=1}^{\infty} j^{-\beta} \sin j\theta \text{ positive on } (0, \pi), \quad \beta > 1, \quad \beta \in \mathbb{N}.
\]

In fact that result is valid for all positive real \( \beta \) and Dick Askey showed us how to prove it using the correct kernel: write

\[
j^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-jt} \, dt
\]

so that

\[
H_{\beta}(\theta) = \frac{1}{\Gamma(\beta)} \sum_{j=1}^{\infty} \sin j\theta \int_0^{\infty} t^{\beta-1} e^{-jt} \, dt
\]

\[
= \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} \sum_{j=1}^{\infty} \sin j\theta (e^{-t})^j \, dt
\]

\[
= \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} \frac{e^{-t} \sin \theta}{1 - 2e^{-t} \cos \theta + e^{-2t}} \, dt
\]

\[
> 0 \quad \text{for } \theta \in (0, \pi).
\]

(ii) Denote by \( z(m) \) the unique zero of \( G_m(\theta) \) lying in \((0, \pi)\).

Notice that \( z(1) = \frac{\pi}{3} \) and that for \( m > 1 \) we have

\[
G_m(\frac{\pi}{3}) = \frac{1}{2}(1 - 2^{1-m})(1 - 3^{1-m})z(m) > 0 \quad \text{and} \quad G_m(\frac{\pi}{2}) = -2^{-m}(1 - 2^{1-m})z(m) < 0.
\]
Thus $z(m) \in \left[ \frac{\pi}{3}, \frac{\pi}{2} \right]$, and it is enough to show that \((G_{m+1} - G_m)(\theta)\) is positive on \(\left[ \frac{\pi}{3}, \frac{\pi}{2} \right] \), \(m \in \mathbb{N} \), for then \(G_{m+1}(z(m)) > G_m(z(m)) = 0\) which implies \(z(m + 1) > z(m)\) by (i).

Now \(G_m(0) = \zeta(m)\) and \(G_m(\pi) = -(1 - 2^{1-m})\zeta(m)\) both decrease (to 1 and \(-1\) respectively), whereas \(G_{m,3}(\pi)\) and \(G_{m,2}(\pi)\) increase with \(m\).

In particular, \((G_{m+1} - G_m)(\theta)\) has an even number, at least 2, of zeros in \((0,\pi)\). It is easily verified that \((G_2 - G_1)(\theta)\) and \((G_3 - G_2)(\theta)\) have exactly 2 zeros in \((0,\pi)\); we proceed inductively.

Since \((G_{m+3} - G_{m+2})(\theta) = -(G_{m+1} - G_m)(\theta)\), \((G_{m+3} - G_{m+2})(\theta)\) has precisely 2 points of inflexion in \((0,\pi)\), and since it is negative and concave up at 0 and at \(\pi\), \((G_{m+3} - G_{m+2})(\theta)\) cannot have more than two zeros in \((0,\pi)\).

Hence \((G_{m+1} - G_m)(\theta), m \in \mathbb{N}\), has exactly two zeros in \((0,\pi)\):

one in \(0,\frac{\pi}{3}\) and the other in \(\frac{\pi}{2}, \pi\); in particular \((G_{m+1} - G_m)(\theta)\) is positive on \(\left[ \frac{\pi}{3}, \frac{\pi}{2} \right] \).

(iii) For the partial sums it does not seem possible to mimic the elegant use of the gamma-function kernel. However the classical Jackson-Gronwall result on the positivity of the partial sums of \(H_1(\theta)\) gives all the information required (and that result has been given many pretty proofs over the years).

(iv) \(z_n(m)\) increases with \(m, m \in \mathbb{N}, m \geq 2\).

Note first that the assertion is trivial for \(n = 1\) since \(G_1(\theta) = \cos \theta\) and \(z_1(m) = \frac{\pi}{2}\), so we suppose \(n \geq 2\). Then
$z_n(m) \in \left[ \frac{\pi}{4'2} \right]$ since $G_m^n(\pi) < 0$ ($G_m^n(\pi)$ is an alternating sum of terms decreasing in absolute value, the first of which is negative) and since $G_m^n(\pi) > 0$ (to see this, pair the $j$th term with the $(j - 4)$th, $j = 3, 4, 5 \mod 8, \ j \geq 11$). As before it suffices to prove

$$(G_{m+1}^n - G_m^n) > 0 \text{ on } \left[ \frac{\pi}{4'2} \right], \text{ that is, to prove } \sum_{j=2}^{n} \frac{j - 1}{m + 1} \cos j \theta < 0,$$

$\theta \in \left[ \frac{\pi}{4'2} \right], \ n, m \geq 2$. Summing by parts we see that it is enough to prove

$$C_n(\theta) = \sum_{j=2}^{n} \frac{j - 1}{j^2} \cos j \theta < 0, \ \theta \in \left[ \frac{\pi}{4'2} \right], \ n \geq 2.$$

Since $\cos 2\theta, \cos 3\theta$ and $(\cos 2\theta + \cos 4\theta)$ are negative throughout $\left[ \frac{\pi}{4'2} \right]$ we have $C_n(\theta) < 0$ on $\left[ \frac{\pi}{4'2} \right]$ for $n = 2, 3, 4$. For $n \geq 5$ we sum twice by parts to see that

$$2 \sin^2 \frac{\theta}{2} C_n(\theta) = \frac{1}{4} \sin^2 \frac{\theta}{2} - \frac{5}{18} \sin^2 \phi - \frac{1}{144} \sin^2 \frac{3\theta}{2} \sin^2 \frac{\phi}{2}$$

$$+ \sum_{j=3}^{n-2} \left( \frac{j - 1}{j^2} - \frac{2}{(j + 1)^2} + \frac{1 + 1}{(j + 2)^2} \right) \sin^2 \frac{(j + 1)\theta}{2}$$

$$+ \left( \frac{n - 2}{(n - 1)^2} - \frac{n - 1}{n^2} \right) \sin^2 \frac{n\theta}{2}$$

$$+ \frac{n - 1}{n^2} \sin(2n + 1) \frac{\phi}{2} \sin \frac{\phi}{2}$$

$$\leq \frac{1}{4} \sin^2 \frac{\theta}{2} - \frac{5}{18} \sin^2 \phi - \frac{1}{144} \sin^2 \frac{3\theta}{2} + \frac{5}{144}$$

$$+ \frac{n - 1}{n^2} \sin(2n + 1) \frac{\phi}{2} \sin \frac{\phi}{2}$$

$$= f(\sin^2 \frac{\theta}{2}) + \frac{n - 1}{n^2} \sin(2n + 1) \frac{\phi}{2} \sin \frac{\phi}{2}$$

where

$$f(x) = \frac{1}{4} x - \frac{5}{18} x - \frac{1}{144} x^2 + \frac{5}{144}.$$
where \( f(t) = \frac{1}{144}(5 - 133t + 184t^2 - 16t^3) \). Because \( f \) is concave up we have 
\[
\left( \sin^2 \frac{\theta}{2} \right) \leq \max \left\{ \sin^2 \frac{\theta_1}{2}, \sin^2 \frac{\theta_2}{2} \right\} \quad \text{on } [\theta_1, \theta_2],
\]
and 
\[
C_n(\theta) < 0 \quad \text{on } [\theta_1, \theta_2] \quad \text{whenever } \quad F(\theta_1, \theta_2, n) = \max \left\{ \sin^2 \frac{\theta_1}{2}, \sin^2 \frac{\theta_2}{2} \right\}.
\]
\[
\left( \sin^2 \frac{\theta}{2} \right) + \frac{n - 1}{n} \sin \frac{\theta}{2} < 0. \quad \text{Also, since } \quad \sin^2 \frac{\theta}{2} < 0 \quad \text{on } \left[ \frac{\pi}{4}, \frac{\pi}{2} \right]
\]
we have \( C_n(\theta) < 0 \) on any subinterval where \( \sin(2n + 1)\frac{\theta}{2} \leq 0 \).

For \( n \geq 9 \) we have \( F(\frac{\pi}{4}, 2, 9) \leq F(\frac{\pi}{4}, 2, 9) < 0 \); for \( 5 \leq n \leq 8 \) it is necessary to subdivide the interval:

for \( n = 8 \) we have \( F(\frac{6\pi}{4}, 17, 8) < 0, \quad F(\frac{8\pi}{17}, 8) < 0 \) and \( \sin \frac{17\pi}{2} \leq 0 \)

on \( \left[ \frac{6\pi}{17}, \frac{8\pi}{17} \right] \),

for \( n = 7 \) we have \( F(\frac{4\pi}{15}, 7) < 0 \) and \( \sin \frac{15\pi}{2} \leq 0 \)

on \( \left[ \frac{4\pi}{15}, \frac{6\pi}{15} \right] \),

for \( n = 6 \) we have \( F(\frac{2\pi}{3}, 6) < 0, \quad F(\frac{3\pi}{2}, 6) < 0 \) and

for \( n = 5 \) we have \( F(\frac{4\pi}{11}, 5) < 0 \) and \( \sin \frac{11\pi}{2} \leq 0 \) on \( \left[ \frac{4\pi}{11}, \frac{4\pi}{11} \right] \).

\[\square\]

3. REMARKS

Statement (i) of the theorem is valid for arbitrary real numbers \( a \geq 1 \), as the proof shows. We will discuss the extension of the remainder of the theorem to non-integral \( m \) on another occasion, [1].

For \( a < 2 \) no even partial sum is decreasing; nevertheless it seems
that these partial sums still have a unique zero in \((0, \pi)\). If \(a \geq \frac{\theta}{8}\),
this can be proved using Vietoris’ methods (see [1], [4]).

REFERENCES


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