REPRESENTATIONS OF INFINITE DIMENSIONAL GROUPS AND APPLICATIONS

A.L. Carey

This talk reviews some recent work on representations of infinite dimensional groups which I have done jointly with Simon Ruijsenaars, Angas Hurst, Jill Wright and Keith Hannabuss. The main references are [1-5,8]. The point of view adopted here as a result of this work is the following: if $\mathcal{G}$ is a group whose representations one is interested in, then inject $\mathcal{G}$ into Aut $\mathcal{A}$, where $\mathcal{A}$ is a C*-algebra whose representation theory is reasonably well understood. Given an irreducible or factorial representation of $\mathcal{A}$ then, if it is true that $g.\pi$ and π are equivalent for all g in $\mathcal{G}$, the Hilbert space of π carries a projective representation $g \to \rho(g)$ of $\mathcal{G}$ where $\rho(g)$ is a unitary for each g in $\mathcal{G}$ such that

$$(1) \qquad \rho(g_1)\rho(g_2) = \sigma(g_1,g_2)\rho(g_1g_2)$$

with $\sigma(g_1,g_2)$ a unitary in $\pi(\mathcal{A})'$. Now σ is a 2-cocycle which may in general be difficult to compute. However for the groups we consider here (loop or gauge groups or the diffeomorphism group of the circle) extra information enables this cohomological problem to be overcome. For these cases we choose $\mathcal{A}$ to be the C*-algebra of the canonical anticommutation relations (variously known as the fermion algebra or the infinite dimensional Clifford algebra) over the complex Hilbert space H where H is either $L^2(\mathbb{R},\mathbb{C}^N)$ or $L^2(S^1,\mathbb{C}^N)$. This algebra is generated by $\{a(f),a(g)^* | f,g\in H\}$ subject to the relations.

$$(2) \qquad a(f)a(g)^* + a(g)^*a(f) = \langle f,g\rangle I; \quad a(f)a(g) + a(g)a(f) = 0,$$

where I denotes the identity in $\mathcal{a}$.

The representations of $\mathcal{a}$ which we will consider are called quasifree [6] and are determined by an operator A on H with $0 \leq A \leq 1$ via the following formula for the state ω_A:

$$(3) \qquad \omega_A(a(f_1)^*a(f_2)^*\ldots a(f_n)^*a(g_m)a(g_{m-1})\ldots a(g_1)) = \delta_{nm}\det(g_i, Af_j)$$

The essential fact about these representations which we need is that they are factorial.

Now introduce the groups $\Omega U(N)$ and $\Lambda U(N)$ consisting of functions from S^1 (resp. $\mathbb{R}$) to U(N) such that φ (resp. $\varphi-1$) is in $W^{1,2}$. (I will not discuss the diffeomorphism group of the circle, or equivalently, the Virasoro algebra in detail although a treatment analogous to that described here exists in [7,8,9]). These groups act as automorphisms of $\mathcal{a}$ via their action as multiplication operators on H:

$$(4) \qquad a(f) \to a(\varphi f),\ f \in H,\ \varphi \in \Omega U(N)\ (\text{resp. } \Lambda U(N)).$$

Now if π_A denotes the representation of $\mathcal{a}$ corresponding to the state ω_A then for the A which I consider below one can show that $\varphi.\pi_A$ and π_A are equivalent for all $\varphi \in \Omega U(N)$ (resp. $\Lambda U(N)$). Thus we have projective representations ρ_A of each of these groups on the Hilbert space of π_A and I denote the two cocycle by σ_A. One may also show that for those A considered below the automorphisms (4) are weakly inner so that in fact $\sigma_A(\varphi_1,\varphi_2) \in \pi_A(\mathcal{a})' \cap \pi_A(\mathcal{a})'' = \mathbb{C}$.

Now restrict attention to the subgroups of $\Omega U(N)$ and $\Lambda U(N)$ obtained by considering functions into a maximal torus of U(N). If $\varphi_j = \exp(if_j)$ (j=1,2) are two such functions we can consider the relation implied by (1) for the Lie algebra elements f_j. Denoting the Lie algebra representation by $f \to J_A(f)$ we find for the operators A which are considered below:

$$(5) \qquad \left[J_A(f_1), J_A(f_2)\right] = \frac{1}{2\pi i}\int dx \ \mathrm{Tr}\left[\frac{df_1(x)}{dx} \ f_2(x)\right]$$

One should recognise this as the Lie algebra of an infinite dimensional Heisenberg group. (Notice the somewhat surprising fact that the right hand side is independent of A. In fact one may show that σ_A, as a cocycle on the full group, is independent of A for those A listed below).

Physicists refer to the operators $J_A(f)$ as smeared boson fields. The fact that starting with a representation of the fermion algebra $\mathcal{a}$, one obtains a representation of this Heisenberg or boson algebra is the easy part of what is known as the boson-fermion correspondence in the physics literature. The hard part arises through the so-called vertex operator construction, which from the point of view of this talk amounts to the fact that there exist special group elements $\gamma_{r,\varepsilon}$ (here r is either an S^1 or $\mathbb{R}$ variable) and constants c_ε such that for g in H with Fourier transform of compact support.

$$(6) \qquad \int g(r)c_\varepsilon \rho_A(\gamma_{r,\varepsilon})dr \text{ converges to } \pi_A(a(g)^*)$$

as ε goes to zero either strongly or weakly (this depends on A) on a dense domain. (Actually even when we cannot prove strong convergence we can show that products of the operators on the left hand side converge to the corresponding products of the fermion algebra operators.) In other words, starting from the representation of the Heisenberg algebra one may reconstruct the fermion algebra elements in the sense of the limiting operation (6).

One is not restricted however to this result alone since one may also 'twist' the Heisenberg algebra representation by an automorphism $\varkappa$ i.e. replace ρ_A by $\rho_A \circ \varkappa$ in (6). Then, while the resulting operators still converge, one generally obtains 'interacting' fermions i.e. representations of the fermion algebra which are not quasifree and in some cases interacting 'fields' which are not even fermions (i.e. do not satisfy (2)).

It is worth emphasizing at this point that detailed information about the representations of $\Omega U(N)$ or $\Lambda U(N)$ follows only because we have been able in some cases to establish in (6), strong convergence on a dense domain. While there is a considerable physical and mathematical literature on the boson-fermion correspondence the question of convergence is largely unconsidered. In those cases where convergence is discussed only far weaker results are obtained.

To describe the representations which arise is the object of the remainder of this talk. First I want to explain how to obtain the representations π_A.

The Fermion Fock space over H is the Hilbert space obtained by completing the exterior algebra $\wedge H$ over H in the obvious Hilbert space topology. We define an action of $a(g)^*$ by

$$a(g)^* \, g_1 \wedge g_2 \wedge \ldots \wedge g_n = g \wedge g_1 \wedge g_2 \wedge \ldots \wedge g_n$$

for g_j in H ($j=1,2,\ldots,n$). Then $a(g)$ may be identified with the Hilbert space adjoint of $a(g)^*$ and the relations (2) hold. When A is a projection, say P, then the representation π_P is constructed in terms of this action by:

$$\pi_P(a(g)) = a((1-P)g) + a(CPg)^*$$

where C is a conjugation on H commuting with P (C is essential because $a(g)$ depends conjugate linearly on g).

When A is not a projection we let $K = H \oplus H$, form the fermion algebra over K, denoted $\mathfrak{a}(K)$, and define a projection K by

$$P(A) = \begin{pmatrix} A & A^{1/2}(1-A)^{1/2} \\ A^{1/2}(1-A)^{1/2} & 1-A \end{pmatrix}$$

Then the representation π_A is the restriction of the representation $\pi_{P(A)}$ of $\mathcal{a}(K)$ to the subalgebra $\mathcal{a}(H\oplus(0))$. The action of φ in $\Omega U(N)$ or $\Lambda U(N)$ on $\mathcal{a}(K)$ is given by

$$a(h_1\oplus h_2) \to a(\varphi h_1 \oplus h_2).$$

In this context one may also consider representations of $\Omega U(N)\oplus\Omega U(N)$ or $\Lambda U(N)\oplus\Lambda U(N)$ via the obvious diagonal 2x2 matrix action on K.

To distinguish the various operators A which have been considered I will introduce the following notation.

- denote by P_- the projection on $L^2(\mathbb{R},\mathbb{C}^N)$ (resp. $L^2 S^1,\mathbb{C}^N)$) onto functions which are boundary values of functions holomorphic in the lower half plane in $\mathbb{C}$ (resp. exterior of the unit disc).

- let $A(\beta)$ denote the operator on $L^2(S^1,\mathbb{C}^N)$ (resp. $L^2(\mathbb{R},\mathbb{C}^N)$) which is given by multiplication by the function

$$k \to e^{-\beta k}/(1+e^{-\beta k}), \quad k\in\mathbb{Z} \text{ (resp. } k\in\mathbb{R}) \ (\beta \geq 0)$$

on the Fourier transform,

- let $A(m)$ denote the operator on $L^2(\mathbb{R},\mathbb{C}^N)$ given by multiplication on the Fourier transform by the function

$$p \to (1-p/(p^2+m^2)^{1/2})/2, \quad (m \geq 0).$$

These operators arise respectively as

(i) the spectral projection of the massless Dirac hamiltonian corresponding to the negative part of the spectrum.

(ii) from the K.M.S. states on the Fermion algebra for the one parameter group of automorphisms generated by the massless Dirac operator and finally

(iii) $P(A(m))$ is the spectral projection of the massive Dirac hamiltonian corresponding to the interval $(-\infty,-m]$.

In the following table I summarise the results of the analysis described above for the representations of the groups $\Omega U(N)$ and $\Lambda U(N)$. The table should be read as follows. The first column denotes the Hilbert space H, the second the choice of A, the third the properties of the resulting representations of $\Omega U(N)$ or $\Lambda U(N)$, the fourth the consequences of considering the limiting operation (6) both in the absence and presence of a 'twist'.

$L^2(S^1,\mathbb{C}^N)$	P_-	This gives the basic representation of the Kac-Moody algebra $A_{N-1}^{(1)}$	The limiting procedure (6) gives free fermions in the P_- representation.
$L^2(\mathbb{R},\mathbb{C}^N)$	P_-	This is equivalent via the Cayley transform to the previous case.	
$L^2(S^1,\mathbb{C}^N)$	$A(\beta)$	This gives a K.M.S. state on the C*-algebra generated by operators representing $\Omega U(N)$. The representation $\rho_{A(\beta)}$ is quasi-equivalent to that given by P_-	One recovers fermions in the $\pi_{A(\beta)}$ representation but in such a way as to obtain identities between theta functions and between Jacobi elliptic functions. These are reminiscent of the identities obtained using the Kac-Moody character formula. In fact it seems that the analytic continuation to imaginary time given by the K.M.S. state is precisely that arising in the Kac-Moody character formula

			suggesting a direct representation theoretic interpretation of it.
$L^2(S^1,\mathbb{C}) \oplus L^2(S_1,\mathbb{C})$	$P(A(\beta))$	Form, for a suitable $\varkappa$, the representation $\rho_{P(A(\beta))}\circ\varkappa$ of $\Omega U(N)\oplus\Omega U(N)$	One constructs interacting fields corresponding to the Luttinger model of statistical mechanics. The correlation functions of the model may be rigorously calculated.
$L^2(\mathbb{R},\mathbb{C}^N)$	$A(\beta)$	$\rho_{A(\beta)}$ is a type III_1 factor representation of $\Lambda U(N)$.	This returns fermions in the $\pi_{A(\beta)}$ representation.
$L^2(\mathbb{R},\mathbb{C}^N)$	$A(m)$	$\rho_{A(m)}$ is a type III_1 factor representation of $\Lambda U(N)$.	This returns fermions in the $\pi_{A(m)}$ representation.
$L^2(\mathbb{R},\mathbb{C}) \oplus L^2(\mathbb{R},\mathbb{C})$	$P(A(m))$	The bosons may be interpreted as sine-Gordon fields at the critical value of the coupling constant where the theory is free.	This returns fermions in the $\pi_{P(A(m))}$ representation.
$L^2(\mathbb{R},\mathbb{C}) \oplus L^2(\mathbb{R},\mathbb{C})$	$P(A(m))$	Form, for a suitable $\varkappa$, the representation $\rho_{P(A(m))}\circ\varkappa$ of $\Lambda U(N)\oplus\Lambda U(N)$	When $m=0$ one obtains interacting massless Thirring fields. When m is positive we conjecture that one obtains massive Thirring fields.

Some comments in the right hand column represent work in progress, in particular those involving $\rho_{A(\beta)}$ and $\rho_{P(A(m))}\circ\varkappa$ for m positive. Note that the first two cases of the above table are considered, using a slightly different viewpoint by Segal and Pressley [8] from which I have borrowed some notation. There is also a large literature on these first two cases from a Lie algebra viewpoint and this may be traced from [8].

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A.L. Carey

Department of Pure Mathematics

University of Adelaide

Adelaide, South Australia