This talk reviews some recent work on representations of infinite dimensional groups which I have done jointly with Simon Ruijsenaars, Angas Hurst, Jill Wright and Keith Hannabuss. The main references are [1-5,8]. The point of view adopted here as a result of this work is the following: if $g$ is a group whose representations one is interested in, then inject $g$ into $\text{Aut} \, a$, where $a$ is a $C^*$-algebra whose representation theory is reasonably well understood. Given an irreducible or factorial representation of $a$ then, if it is true that $g.\pi$ and $\pi$ are equivalent for all $g$ in $g$, the Hilbert space of $\pi$ carries a projective representation $g \rightarrow \rho(g)$ of $g$ where $\rho(g)$ is a unitary for each $g$ in $g$ such that

$$\rho(g_1)\rho(g_2) = \sigma(g_1, g_2)\rho(g_1 g_2)$$

with $\sigma(g_1, g_2)$ a unitary in $\pi(a)'$. Now $\sigma$ is a 2-cocycle which may in general be difficult to compute. However for the groups we consider here (loop or gauge groups or the diffeomorphism group of the circle) extra information enables this cohomological problem to be overcome. For these cases we choose $a$ to be the $C^*$-algebra of the canonical anticommutation relations (variously known as the fermion algebra or the infinite dimensional Clifford algebra) over the complex Hilbert space $H$ where $H$ is either $L^2(\mathbb{R}, \mathbb{C}^N)$ or $L^2(S^1, \mathbb{C}^N)$. This algebra is generated by $\{a(f), a(g)^* | f, g \in H\}$ subject to the relations.

$$a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle I; \quad a(f)a(g) + a(g)a(f) = 0,$$
where $I$ denotes the identity in $a$.

The representations of $a$ which we will consider are called quasifree [6] and are determined by an operator $A$ on $H$ with $0 < A < 1$ via the following formula for the state $\omega_A$:

$$\omega_A(a(f_1)^{*}a(f_2)^{*}...a(f_n)^{*}a(g_m)a(g_{m-1})...a(g_1)) = \delta_{nm}\det(g_i,Af_j)$$

The essential fact about these representations which we need is that they are factorial.

Now introduce the groups $\Omega U(N)$ and $\Lambda U(N)$ consisting of functions from $S^1$ (resp. $\mathbb{R}$) to $U(N)$ such that $\varphi$ (resp. $\varphi^{-1}$) is in $W^{1,2}$. (I will not discuss the diffeomorphism group of the circle, or equivalently, the Virasoro algebra in detail although a treatment analogous to that described here exists in [7,8,9]). These groups act as automorphisms of $a$ via their action as multiplication operators on $H$:

$$a(f) \rightarrow a(\varphi f), f \in H, \varphi \in \Omega U(N) \text{ (resp. } \Lambda U(N)).$$

Now if $\pi_A$ denotes the representation of $a$ corresponding to the state $\omega_A$ then for the $A$ which I consider below one can show that $\varphi, \pi_A$ and $\pi_A$ are equivalent for all $\varphi \in \Omega U(N)$ (resp. $\Lambda U(N)$). Thus we have projective representations $\rho_A$ of each of these groups on the Hilbert space of $\pi_A$ and I denote the two cocycle by $\sigma_A$. One may also show that for those $A$ considered below the automorphisms (4) are weakly inner so that in fact

$$\sigma_A(\varphi_1,\varphi_2) \in \pi_A(a)^{\prime} \cap \pi_A(a)^{\prime\prime} = 1.$$

Now restrict attention to the subgroups of $\Omega U(N)$ and $\Lambda U(N)$ obtained by considering functions into a maximal torus of $U(N)$. If $\varphi_j = \exp(\imath f_j)$ ($j=1,2$) are two such functions we can consider the relation implied by (1) for the Lie algebra elements $f_j$. Denoting the Lie algebra representation by $f \rightarrow J_A(f)$ we find for the operators $A$ which are considered below:
One should recognize this as the Lie algebra of an infinite dimensional Heisenberg group. (Notice the somewhat surprising fact that the right hand side is independent of $A$. In fact one may show that $\sigma_A$ as a cocycle on the full group, is independent of $A$ for those $A$ listed below).

Physicists refer to the operators $J_A(f)$ as smeared boson fields. The fact that starting with a representation of the fermion algebra $\mathcal{A}$, one obtains a representation of this Heisenberg or boson algebra is the easy part of what is known as the boson-fermion correspondence in the physics literature. The hard part arises through the so-called vertex operator construction, which from the point of view of this talk amounts to the fact that there exist special group elements $\gamma_{r,\epsilon}$ (here $r$ is either an $S^1$ or $\mathbb{R}$ variable) and constants $c_\epsilon$ such that for $g$ in $H$ with Fourier transform of compact support.

One is not restricted however to this result alone since one may also 'twist' the Heisenberg algebra representation by an automorphism $\kappa$ i.e. replace $\rho_A$ by $\rho_{A^*\kappa}\ln(5)$. Then, while the resulting operators still converge, one generally obtains 'interacting' fermions i.e. representations
of the fermion algebra which are not quasifree and in some cases inter-
acting 'fields' which are not even fermions (i.e. do not satisfy (2)).

It is worth emphasizing at this point that detailed information about
the representations of $\hat{U}(N)$ or $\tilde{U}(N)$ follows only because we have been
able in some cases to establish in (6), strong convergence on a dense
domain. While there is a considerable physical and mathematical literature
on the boson-fermion correspondence the question of convergence is largely
unconsidered. In those cases where convergence is discussed only far weaker
results are obtained.

To describe the representations which arise is the object of the
remainder of this talk. First I want to explain how to obtain the repre-
sentations $\pi_A$.

The Fermion Fock space over $H$ is the Hilbert space obtained by
completing the exterior algebra $\wedge H$ over $H$ in the obvious Hilbert space
topology. We define an action of $a(g)^*$ by

$$a(g)^* g_1 \wedge g_2 \wedge \cdots g_n = g^A g_1 \wedge g_2 \wedge \cdots g_n$$

for $g_j$ in $H$ ($j=1,2,\ldots ,n$). Then $a(g)$ may be identified with the Hilbert
space adjoint of $a(g)^*$ and the relations (2) hold. When $A$ is a projection,
say $P$, then the representation $\pi_P$ is constructed in terms of this action by:

$$\pi_P(a(g)) = a((1-P)g) + a(CPg)^*$$

where $C$ is a conjugation on $H$ commuting with $P$ ($C$ is essential because
$a(g)$ depends conjugate linearly on $g$).

When $A$ is not a projection we let $K = H \otimes H$, form the fermion algebra
over $K$, denoted $\hat{a}(K)$, and define a projection $K$ by

$$P(A) = \begin{pmatrix} A & A^{1/2}(1-A)^{1/2} \\ (A^{1/2}(1-A)^{1/2})^T & 1-A \end{pmatrix}$$
Then the representation $\pi_A$ is the restriction of the representation $\pi_{P(A)}$ of $a(K)$ to the subalgebra $a(H \oplus \mathbb{R})$. The action of $\varphi$ in $\Omega U(N)$ or $\Delta U(N)$ on $a(K)$ is given by

$$a(h_1 \oplus h_2) \rightarrow a(\varphi h_1 \oplus h_2).$$

In this context one may also consider representations of $\Omega U(N) \oplus \Omega U(N)$ or $\Delta U(N) \oplus \Delta U(N)$ via the obvious diagonal $2 \times 2$ matrix action on $K$.

To distinguish the various operators $A$ which have been considered I will introduce the following notation.

- denote by $P_-$ the projection on $L^2(\mathbb{R}, \mathbb{C}^N)$ (resp. $L^2 S^1, \mathbb{C}^N$) onto functions which are boundary values of functions holomorphic in the lower half plane in $\mathbb{C}$ (resp. exterior of the unit disc).

- let $A(\beta)$ denote the operator on $L^2(S^1, \mathbb{C}^N)$ (resp. $L^2(\mathbb{R}, \mathbb{C}^N)$) which is given by multiplication by the function

$$k \rightarrow e^{-\beta k} / (1 + e^{-\beta k}), \quad k \in \mathbb{Z} \text{ (resp. } k \in \mathbb{R}) \quad (\beta > 0)$$

on the Fourier transform,

- let $A(m)$ denote the operator on $L^2(\mathbb{R}, \mathbb{C}^N)$ given by multiplication on the Fourier transform by the function

$$p \rightarrow (1 - p/(p^2 + m^2)^{1/2})/2, \quad (m \geq 0).$$

These operators arise respectively as

(i) the spectral projection of the massless Dirac hamiltonian corresponding to the negative part of the spectrum.

(ii) from the K.M.S. states on the Fermion algebra for the one parameter group of automorphisms generated by the massless Dirac operator and finally

(iii) $P(A(m))$ is the spectral projection of the massive Dirac
In the following table I summarise the results of the analysis described above for the representations of the groups $\mathbb{Q}U(N)$ and $\Delta U(N)$. The table should be read as follows. The first column denotes the Hilbert space $H$, the second the choice of $A$, the third the properties of the resulting representations of $\mathbb{Q}U(N)$ or $\Delta U(N)$, the fourth the consequences of considering the limiting operation (6) both in the absence and presence of a 'twist'.

<table>
<thead>
<tr>
<th>$L^2(S^1, \mathbb{C}^N)$</th>
<th>$\mathbb{P}_-$</th>
<th>This gives the basic representation of the Kac-Moody algebra $A^{(1)}_{N-1}$</th>
<th>The limiting procedure (6) gives free fermions in the $\mathbb{P}_-$ representation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L^2(\mathbb{R}, \mathbb{C}^N)$</td>
<td>$\mathbb{P}_-$</td>
<td>This is equivalent via the Cayley transform to the previous case.</td>
<td></td>
</tr>
</tbody>
</table>
| $L^2(S^1, \mathbb{C}^N)$ | $A(\beta)$ | This gives a K.M.S. state on the C*-algebra generated by operators representing $\mathbb{Q}U(N)$. The representation $\rho_{A(\beta)}$ is quasi-equivalent to that given by $\mathbb{P}_-$ that generated by operators such a way as to obtain identities between theta functions and between Jacobi elliptic functions. These are reminiscent of the identities obtained using the Kac-Moody character formula. In fact it seems that the analytic continuation to imaginary time given by the K.M.S. state is precisely that arising in the Kac-Moody character formula. | One recovers fermions in the $\tau_{A(\beta)}$ representation but in such a way as to obtain identities between theta functions.
suggesting a direct representation theoretic interpretation of it.

One constructs interacting fields corresponding to the Luttinger model of statistical mechanics. The correlation functions of the model may be rigorously calculated.

This returns fermions in the $\pi_{A(\beta)}$ representation.

This returns fermions in the $\pi_{A(m)}$ representation.

This returns fermions in the $\pi_{P(A;m)}$ representation.

When $m=0$ one obtains interacting massless Thirring fields. When $m$ is positive we conjecture that one obtains massive Thirring fields.

Some comments in the right hand column represent work in progress, in
particular those involving $\rho_{A(\beta)}$ and $\rho_{P(A(m))}^\chi$ for $m$ positive. Note that the first two cases of the above table are considered, using a slightly different viewpoint by Segal and Pressley [8] from which I have borrowed some notation. There is also a large literature on these first two cases from a Lie algebra viewpoint and this may be traced from [8].

REFERENCES


9. S.N.M. Ruijsenaars, unpublished notes.

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