NORMAL SUBRELATIONS OF ERGODIC EQUIVALENCE RELATIONS

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§0 INTRODUCTION

In this paper we introduce the notion of a "normal" subrelation of an ergodic equivalence relation, and study some of its consequences. Details will appear elsewhere [2].

Throughout, we deal with a countable non-singular equivalence relation S on a standard non-atomic probability space (X,B,μ), as in [1] or [4]. Thus $S \subseteq X \times X$ is a Borel set, for each $x \in X$, $S(x) = \{y \in X : (y,x) \in S\}$ is countable, and for each $E \in B$ with $\mu(E) = 0$, $\mu(S(E)) = 0$ where $S(E) = \cup\{S(x) : x \in E\}$. For our purposes, there is no loss of generality in assuming that μ is a probability measure, and that S is ergodic i.e. for $E \in B$, $\mu(S(E)) \in \{0,1\}$; we will hence forth make these assumptions. In addition, we assume for simplicity that the measure μ is finite and invariant, as in [1] i.e. that S is of type II_1.

In recent years, much has been learned about the structure of such relations, up to isomorphism; here S_1 on (X_1,B_1,μ_1) is isomorphic to S_2 on (X_2,B_2,μ_2) if there is a bimeasurable map $\phi : X_1' \to X_2'$ between conull subsets of X_1 and X_2 for which $\mu_2 \circ \phi$ has the same null sets as μ_1, and for which $(x,y) \in S_1$ if and only if $(\phi(x),\phi(y)) \in S_2$. The reader is referred to [1,3,7] for a glimpse of what is known. Our interest however is not so much in the relations themselves, but in the possible subrelations R of a given ergodic equivalence relation S

on (X,B,μ); alternatively, if R and S are given equivalence relations, we are interested in the various embeddings of R in S.

§1 THE INDEX COCYCLE

Let S be an ergodic, countable, non-singular II_1-equivalence relation on (X,B,μ) as above, and let $R \subseteq S$ be a (Borel) subrelation; we do not assume R is ergodic. A modification of the selection argument of [1] guarantees there is a set $J = \{1,2,\ldots,N\} \subseteq \mathbb{N}$ (where we allow $N = \infty$), and a Borel map $\Psi = (y,x) \in S \to (\psi_x(y),x) \in J \times X$ such that $\Psi(y,x) = \Psi(y',x)$ if and only if $(y,y') \in R$; effectively, $\{\psi_x(y) : y \in S(x)\}$ are a set of labels for the R-equivalence classes within $S(x)$ - the number of such classes (N) is essentially independent of x by the ergodicity of S. N is called the <u>index</u> of R in S. We now define a map $\sigma_\Psi : S \to \Sigma(J)$, the group of all permutations on the set J, by

$$\sigma_\Psi(x,y)(\psi_y(z)) = \psi_x(z)$$

for $(x,y) \in S$, $(y,z) \in S$.

THEOREM 1.1 [(2]) *In the situation described above*

a) σ_Ψ *is a 1-cocycle, or* $\sigma_\Psi \in Z^1(S,\Sigma(J))$ *i.e. for* $(x,y) \in S$, $(y,z) \in S$, *we have*

$$\sigma_\Psi(x,y)\sigma_\Psi(y,z) = \sigma_\Psi(x,z);$$

b) *If* $\Phi : S \to J \times X$ *is another "labelling map", then there is a Borel map* $x \in X \to v_x \in \Sigma(J)$ *such that for* $(x,y) \in S$,

$$\sigma_\Phi(x,y) = v_x\sigma_\Psi(x,y)v_y^{-1};$$

thus the cohomology class σ *of* σ_Ψ *depends only on the pair* $R \subseteq S$.

c) *If* $R, R' \subseteq S$ *are ergodic and define cohomology classes* $\sigma, \sigma' \in H^1(S, \Sigma(J))$, *then there is an automorphism* α *of* S *such that* $\alpha^{(2)}(R) = R'$ *if and only if there is an automorphism* β *of* S *such that* $\sigma \circ \beta^{(2)} = \sigma'$. *(As in [4],* $\alpha^{(2)}$ *is the restriction of* $\alpha \times \alpha$ *to* S*).*

Thus the isomorphism classes of ergodic subrelations R of S are controlled completely by (a subset of) the cohomology $H^1(S, \Sigma(J))$. We will refer to the cocycle σ (or its class) corresponding to a subrelation R of S as the <u>index cocycle</u> of the pair; where appropriate it will be denoted σ_R.

We may recover $R \subseteq S$ (up to a transitive relation) from σ_R as follows. Given $\kappa \in Z^1(S, \Sigma(J))$, we define the <u>skew product</u> $\hat{R} = S \times J$ as the equivalence relation on $X \times J$ given by

$$((x,i)(y,j)) \in \hat{R} \text{ if and only if } (x,y) \in S \text{ and } \kappa(x,y)(j) = i.$$

We also define $\hat{S}$ on $X \times J$ by

$$((x,i),(y,j)) \in \hat{S} \text{ if and only if } (x,y) \in S.$$

THEOREM 1.2 ([2]) *If* $R \subseteq S$ *is an ergodic subrelation and* $\kappa = \sigma_R$, *then there is an isomorphism of* $\hat{S}$ *with* $S \times T(J)$ *which carries* $\hat{R}$ *to* $R \times T(J)$, *where* $T(J)$ *denotes the transitive relation* $J \times J$ *on* J. *Conversely, if* $\kappa \in Z^1(S, \Sigma(K))$ *is arbitrary, then the index cocycle* $\hat{\sigma}$ *for* $\hat{R} \subseteq \hat{S}$ *can be taken to be given by* $\hat{\sigma}((x,i),(y,j)) = \kappa(x,y)$. *Finally, if* $\kappa \in Z^1(S, \Sigma(J))$ *is ergodic (i.e.* $S \times_\kappa J$ *is ergodic), the index cocycle for* $R = \{(x,y) \in S : \kappa(x,y)(1) = 1\}$ *is cohomologous to* κ.

Thus there is a complete correspondence between ergodic subrelations $R \subset S$ (up to automorphisms of S) and the ergodic cocycle in $Z^1(S,\Sigma(J))$.

§2 NORMALITY OF SUBRELATIONS

<u>Definition 2.1</u> A subrelation $R \subset S$ on (X,μ) is said to be <u>normal</u> if the restriction of σ_R to R cobounds i.e. if there is a measurable map $x \in X \to v_x \in \Sigma(J)$ such that $\sigma_R(x,y) = v_x v_y^{-1}$ a.e. for $(x,y) \in R$.

THEOREM 2.2 [2] *Let* $R \subset S$ *be equivalence relations on* (X,μ), *and suppose* R *is ergodic. Then the following are equivalent:*

i) R *is normal in* S.

ii) there are endomorphisms $\phi_j \in \mathrm{End}(R)$, $j \in J$, *such that*

$$S(x) = \bigcup_J R(\phi_j(x)) \quad \textit{and} \quad R(\phi_j(x)) \cap R(\phi_k(x) = \phi \quad \text{for} \quad j \neq k;$$

iii) there is a discrete group G *and a homomorphism* $\theta : S \to G$ *such that*

a) $\ker \theta = \{(x,y) \in S : \theta(x,y) = e\} = R;$

b) *if* $g \in G$ and $x \in X$ *are given, there exists* $y \in X$ *with* $(y,x) \in S$ *and* $\theta(y,x) = g$,

c) *for any other discrete group* H *and homomorphism* $\kappa : S \to H$ *with* $\ker\kappa \supseteq R$, *there is a homomorphism* $\kappa' : G \to H$ *with* $\kappa = \kappa' \circ \theta$.

iv) The extension $S \times_\sigma J$ *of* S *by the index cocycle of* R *is normal in the sense of* [6].

The theorem may be generalized to nonergodic subrelations R of S at the expense of replacing the group G of iii) by a discrete measured ergodic groupoid $\mathcal{G}$. The group G or groupoid $\mathcal{G}$ is termed the <u>quotient</u> of S by R, and it is routine to see that any discrete group or groupoid can occur as a quotient; we should also note that the "quotient relations" of [4] are a special case of our construction.

It is easy to see where the group G of 2.2.iii) comes from: for $j,k \in J$ and $x \in X$, define $j *_x k = \ell$ to mean $R(\phi_j(\phi_k(x)) = R(\phi_\ell(x))$ where $\{\phi_j\}$ are "choice functions as in 2.2.ii). Since $j *_x k = j *_y k$ for $(x,y) \in R$ (as $\phi_j \in \mathrm{End}(R)$ for all j) and since R is ergodic $j *_x k$ is (essentially) independent of x, and provides a group law on J. The group G is of course the same as that provided by Zimmer's theory of normal extensions.

Theorem 2.2 has some surprising consequences for group actions.

COROLLARY 2.3 *Let* H_j *be discrete groups with normal subgroups* N_j, *and suppose* H_j *acts freely on* (X_j,μ_j) *with* μ_j *finite and invariant, and with* N_j *acting ergodically* (j=1,2). *Then, if there is a measure space isomorphism of* (X_1,μ_1) *with* (X_2,μ_2) *carrying* H_1-*orbits to* H_2-*orbits and* N_1-*orbits to* N_2 *orbits, we have* H_1/N_1 *isomorphic with* H_2/N_2.

Note that if R is normal in S and ergodic the quotient group G "acts" as endomorphisms of R; in the notation of Theorem 2.2, we may choose $\phi_g \in \mathrm{End}(R)$, $g \in G$ such that $\theta(\phi_g x, x) = g$ for all $x \in X$ and $g \in G$, and we have $(\phi_g \circ \phi_h(x), \phi_{gh}(x)) \in R$ for all $x \in X$. It is routine to show that we may assume ϕ_g is invertible for each g (although this may fail if R is not ergodic - see [4]). Furthermore, we have

THEOREM 2.4 [2] *Suppose* R *is normal in* S *and ergodic with the quotient group* G *being amenable. Then there is a homomorphism* $g \in G \to \phi_g \in \mathrm{Aut}(R)$ *such that* $\theta(\phi_g x, x) = g$ *for all* $g \in G$.

§3 CLASSIFICATION RESULTS

In this section, S on (X,μ) denotes the hyperfinite relation of type II_1 (c.f. [4]; Our objective is to classify (some of) the subrelations R of S.

THEOREM 3.1 [2] *Let* S *on* (X,μ) *be as above.*

a) There is a bijective correspondence between ergodic (not necessarily normal) subrelations of S *(up to automorphisms of* S*) of finite index* N *and conjugacy classes of transitive subgroups of the symmetric group* Σ_N *on* N *symbols.*

b) There is a bijective correspondence between ergodic normal subrelations R *of* S *(up to automorphisms of* S*) and (isomorphism classes of) countable amenable groups.*

§4 RIGIDITY RESULTS

Throughout this section H will denote a connected, non-compact simple Lie group with finite centre, and $\Gamma \subset H$ will denote a lattice (see [7] for definitions and discussion).

THEOREM 4.1 ([2]) *For* $j = 1,2$, *Let* S_j *on* (X_j,μ_j) *be a* II_1 *ergodic relation with normal ergodic subrelation* R_j. *Suppose that the quotient groups* S_j/R_j *are amenable, that* R_j *is generated by a free action of a lattice* Γ_j *in* H_j, *and that* $\mathbb{R}$*-rank* $H_j \geq 2$. *Then if* S_1 *is isomorphic to* S_2, H_1 *is locally isomorphic with* H_2.

THEOREM 4.2 [2] *Let* $\Gamma \subset H$ *be a lattice and let* S *be the equivalence relation generated by a free* II_1 *ergodic action of* Γ *and suppose* $\mathbb{R}$*-rank* $H \geq 2$. *Then if* $R \subset S$ *is normal and ergodic,* R *has finite index in* S.

THEOREM 4.3 [2] *Let* $\Gamma \subset H$ *be a lattice (with no rank assumption on* H*) and let* S *be the equivalence relation generated by a free ergodic* II_1*-action of* Γ. *Then, if* $R \subset S$ *is amenable and strongly normal,* R *is finite (i.e.* $R(x)$ *is finite for each* x*).*

The same conclusion as in Theorem 4.3 holds if Γ is the fundamental group of a complete Riemannian manifold with sectional curvature k satisfying $k \leq c < 0$ for some constant c. In particular, it holds for free groups.

In Theorem 4.3, R being <u>strongly normal</u> means that $\{g \in [S]: g$ normalizes $R\}$ generates S. Strong normality implies normality in general; the converse is true for ergodic subrelations, but not in general (c.f. [4]).

§5 COMMENTS

Many ergodic equivalence relations (but not all) admit normal, ergodic, amenable subrelations, and hence maximal such ones. While these maximal, normal, ergodic, amenable subrelations are not unique, we know of no example of an equivalence relation S with maximal, normal, ergodic, amenable subrelations which are not conjugate via automorphisms of S.

Finally, we should note there is a strong connection between subrelations and pairs of von Neumann algebras (M,N) with $M \subset N$. This aspect will be discussed in [5].

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