A QUALITATIVE UNCERTAINTY PRINCIPLE FOR LOCALLY COMPACT ABELIAN GROUPS

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1. INTRODUCTION

It has long been known that if a function $f \in L^2(\mathbf{R}^n)$ and the supports of f and its Fourier transform $\hat{f}$ are contained in bounded rectangles, then $f = 0$ almost everywhere. In 1974, Benedicks [2] strengthened this result by showing that the supports of f and $\hat{f}$ having finite measure is sufficient to imply that $f = 0$ almost everywhere. Amrein and Berthier [1] reached the same conclusion in 1977 using Hilbert space methods. This result may be thought of as a qualitative uncertainty principle since it limits the "concentration" of the Fourier transform pair $(f, \hat{f})$. Little is known, however, of analogous behaviour for functions on locally compact abelian (LCA) groups.

Let G be an LCA group with dual group Γ. Equip G with a Haar measure m_G. If $\gamma \in \Gamma$ and $f \in L^1(G)$, the Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(\gamma) = \int_G f(x)\, \overline{\gamma(x)}\, dm_G(x).$$

We choose a Haar measure m_Γ on Γ for which the Plancherel identity is valid.

If G is compact, then Γ is discrete and we insist $m_G(G) = 1$. With this convention, $m_\Gamma(\{0\}) = 1$.

If f, g are measurable functions on G, we define their convolution $f*g$ by

$$f*g(x) = \int_G f(xy^{-1})g(y)\,dm_G(y)$$

whenever the integral exists. With this notation,

$$(f*g)^\wedge(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma) \qquad (\gamma \in \Gamma).$$

For $f \in L^2(G)$, let $A_f = \{x \in G;\ |f(x)| > 0\}$ and $B_f = \{\gamma \in \Gamma;\ |\hat{f}(\gamma)| > 0\}$. In 1973, Matolcsi and Szücs [5] showed that for all LCA groups G, $m_G(A_f)m_\Gamma(B_f) < 1 \Rightarrow f = 0\ m_G$ - a.e.

We say that an LCA group G satisfies the qualitative uncertainty principle (QUP) if, for each $f \in L^2(G)$,

$$m_G(A_f) < m_G(G),\ m_\Gamma(B_f) < m_\Gamma(\Gamma) \Rightarrow f = 0\ m_G \text{ - a.e.}$$

We show that the satisfaction (or otherwise) of the QUP is determined by the "level of connectedness" of the group G.

Let $\mathcal{H}$ be a Hilbert space with inner product $(\ ,\ )$ and $T \in \mathcal{B}(\mathcal{H})$ (the set of bounded linear operators on $\mathcal{H}$). Let $\{\phi_k\}$ be a complete orthonormal set in $\mathcal{H}$. We define the Hilbert-Schmidt norm $\|T\|_2$ of T by

$$\|T\|_2^2 = \sum_k \|T\phi_k\|^2 = \sum_k \sum_j |(T\phi_k, \phi_j)|^2.$$

We say T is a Hilbert-Schmidt operator if $\|T\|_2 < \infty$. Suppose $t \in L^2(X \times X)$ where X is a measure space with measure dx. Define an operator T on $L^2(X)$ by

$$(T\phi)(y) = \int_X t(x,y)\phi(x)\,dx \qquad (y \in X).$$

Then $T \in \mathcal{B}(L^2(X))$ and

$$(1.1) \qquad \|T\| \le \|T\|_2 = \|t\|_{L^2(X\times X)} < \infty,$$

where $\|T\|$ denotes the usual operator norm of T.

Now let E and F be orthogonal projections on a Hilbert space $\mathcal{H}$. Let $E \cap F$ denote the unique orthogonal projection onto $\mathcal{H}_1$, the intersection of the ranges of E and F. $\mathcal{H}$ then decomposes as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^{\perp}$ where $\mathcal{H}_1^{\perp} = \{\psi \in \mathcal{H};\ (\psi,\phi) = 0 \text{ for all } \phi \in \mathcal{H}_1\}$. Choose complete orthonormal bases $\{\phi_k\}$ for $\mathcal{H}_1$ and $\{\psi_j\}$ for $\mathcal{H}_1^{\perp}$. Then $\{\phi_k\} \cup \{\psi_j\}$ form a complete orthonormal basis for $\mathcal{H}$. Also, for each k and j, $(E \cap F)\phi_k = \phi_k$ and $(E \cap F)\psi_j = 0$. So

$$\begin{aligned}\|E \cap F\|_2^2 &= \sum_k \|(E \cap F)\phi_k\|^2 + \sum_j \|(E \cap F)\psi_j\|^2 \\ &= \sum_k \|\phi_k\|^2 \\ &= \dim \mathcal{H}_1 .\end{aligned}$$

Suppose $\phi \in \mathcal{H}_1$. Then $E\phi = F\phi = \phi = (EF)\phi$. In particular, $(EF)\phi_k = \phi_k$ for each k. Therefore,

(1.2) $$\|E \cap F\|_2 \le \|EF\|_2 .$$

If E is a measurable subset of an LCA group G, we denote its characteristic function by χ_E and its complement by E' or $G \backslash E$. If H is a closed subgroup of G, the annihilator $A(H)$ of H, is that closed subgroup of Γ defined by

$$A(H) = \{\gamma \in \Gamma,\ \gamma(h) = 1 \text{ for all } h \in H\}.$$

2. RESULTS

PROPOSITION 1 *Let* G *be an LCA group with non-compact identity component* G_0. *Let* m_G *denote Haar measure on* G *and* C *be a measurable subset of* G *with* $0 < m_G(C) < \infty$. *If* $C_0 \subseteq C$ $(m_G(C_0) > 0)$ *and* $\varepsilon > 0$ *are given, then there exists* $a \in G_0$ *such that*

$$m_G(C) < m_G(C \cup aC_0) < m_G(C) + \varepsilon .$$

Proof Define $h: G_0 \to \mathbb{R}^+ (= \{y \in \mathbb{R};\ y \ge 0\})$ by $h(a) = m_G(C \cup aC_0)$. Then h may also be written as

$$h(a) = \|L(a)\chi_{C_0} - \chi_C\|_2^2 + (L(a)\chi_{C_0}, \chi_C)$$

where we have made use of the continuous unitary representation L of G on $L^2(G)$ given by $(L(a)f)(x) = f(a^{-1}x)$. The strong continuity of the representation L implies the continuity of h on G_0.

Choose δ such that $0 < 2\delta < m_G(C_0)$. By the regularity of Haar measure, there exists a compact set K, $K \subseteq C$, with $m_G(C\backslash K) < \delta$. Let $M = KK^{-1}$, a compact subset of G. $M \cap G_0$ is then either compact in G_0 or empty and since G_0 is not compact, we may choose $a \in G_0\backslash(M \cap G_0)$. With this choice of a, $aK \cap K$ is empty and so,

$$\begin{aligned} aC_0 \cap K &= a((C_0 \cap K) \cup (C_0 \cap K')) \cap K \\ &= (aC_0 \cap aK \cap K) \cup (aC_0 \cap aK' \cap K) \\ &= aC_0 \cap aK' \cap K \\ &\subseteq a(C \cap K'). \end{aligned}$$

Hence

(2.1) $$m_G(aC_0 \cap K) \leq m_G(C \cap K') < \delta.$$

Then, $h(a) = m_G(C \cup aC_0)$

$$\begin{aligned} &= m_G((C \cap K) \cup (C \cap K') \cup (aC_0 \cap K) \cup (aC_0 \cap K')) \\ &\geq m_G((C \cap K) \cup (aC_0 \cap K')) \\ &= m_G(C \cap K) + m_G(aC_0 \cap K') \\ &= m_G(C) - m_G(C \cap K') + m_G(aC_0) - m_G(aC_0 \cap K) \\ &> m_G(C) + m_G(aC_0) - 2\delta \end{aligned}$$

(by (2.1) and the choice of K)

$$> m_G(C) = h(0)$$

(by the choice of δ).

So h is a non-constant continuous function on the connected set G_0. We may then choose $a \in G_0$ with

$$m_G(C) = h(0) < h(a) = m_G(C \cup aC_0) < h(0) + \varepsilon = m_G(C) + \varepsilon. \qquad \square$$

Now let G be an LCA group with dual group Γ and suitably normalized Haar measures m_G, m_Γ on G,Γ respectively. Let $A \subseteq G$ and $B \subseteq \Gamma$ be measurable subsets with $m_G(A)m_\Gamma(B) < \infty$. Define projections E_A and F_B on $L^2(G)$ by

$$(E_A f)(x) = \chi_A(x)f(x)$$

(2.2)

$$(F_B f)(x) = (\chi_B \hat{f})^{\vee}(x) = \chi_B^{\vee} * f(x)$$

where $^{\vee}$ denotes the inverse Fourier transform. With this notation, a non-compact LCA group G satisfies the QUP if, for all such subsets A and B, $(E_A \cap F_B)L^2(G) = \{0\}$.

THEOREM 1 *If* G *is an LCA group with non-compact identity component, then* G *satisfies the* QUP.

Proof Suppose $f_0 \in (E_A \cap F_B)L^2(G)$ and $f_0 \not\equiv 0$.

Let $A_0 = \{x \in G;\ |f_0(x)| > 0\}$ $(m_G(A_0) > 0)$. Choose $N \in \mathbb{Z}^+ (= \{n \in \mathbb{Z};\ n > 0\})$ with $2m_G(A_0)m_\Gamma(B) < N$. We define a sequence of measurable sets $\{A_i;\ 1 \le i \le N\}$, $A_i \subseteq A_{i+1}$, by applying Proposition 1 with $\varepsilon = 1/(2m_\Gamma(B))$, $C = A_{i-1}$, $C_0 = A_0$. For each i, choose $a_i \in G_0$ with

$$m_G(A_{i-1}) < m_G(A_{i-1} \cup a_iA_0) < m_G(A_{i-1}) + 1/(2m_\Gamma(B))$$

and set $A_i = A_{i-1} \cup a_iA_0$. A simple calculation shows

$$(E_{A_i}F_B f)(x) = \int_G \chi_{A_i}(x)\chi_B^{\vee}(xy^{-1})f(y)\,dm_G(y) \tag{2.3}$$

and so, by (1.1),

$$\|E_{A_i}F_B\|_2^2 = \int_G\int_G |\chi_{A_i}(x)\chi_B^{\vee}(xy^{-1})|^2 dm_G(y)\,dm_G(x) \tag{2.4}$$
$$= m_G(A_i)m_\Gamma(B).$$

Therefore, by (1,2) and (2.4),

$$\dim(E_{A_N} \cap F_B)L^2(G) \le m_G(A_N)m_\Gamma(B) \tag{2.5}$$
$$< [m_G(A_0) + \frac{N}{2m_\Gamma(B)}]m_\Gamma(B)$$

$$< N$$

(by our choice of N). Define $f_i = L(a_i)f_0$, so that $(f_i)\hat{}(\gamma) = \overline{\gamma(a_i)}\hat{f}_0(\gamma)$ $(\gamma \in \Gamma)$. Then $F_B f_i = f_i$ $(0 \le i \le N)$. Since $A_m = A_0 \cup a_1A_0 \cup \ldots \cup a_mA_0$ and $f_i = 0$ m_G- a.e. on $(a_iA_0)'$, we see that $E_{A_m} f_i = f_i$ for $0 \le i \le m$. Further $E_{A_m \backslash A_{m-1}} f_i = 0$ for $0 \le i \le m-1$ and $E_{A_m \backslash A_{m-1}} f_m \ne 0$. Therefore, f_m is not a linear combination of $f_0, \ldots, f_{m-1}$ and so $\{f_0, \ldots, f_N\}$ is a set of $N + 1$ linearly independent functions in $(E_{A_N} \cap F_B)L^2(G)$, thus contradicting (2.5). We conclude that $(E_A \cap F_B)L^2(G) = \{0\}$. □

A simple argument extends this result to functions in $L^p(G)$, $1 \le p \le \infty$.

COROLLARY 1 *Let* G, Γ, A, B *be as above and* $1 \le p \le \infty$. *If* $f \in L^p(G)$, $f(x) = 0$ m_G-a.e. *on* A' *and* $\hat{f}(\gamma) = 0$ m_Γ-a.e. *on* B', *then* $f = 0$ m_G-a.e.

Proof If $f \in L^p(G)$, $1 \le p \le \infty$ and $f(x) = 0$ m_G-a.e. on A', then $f \in L^1(G)$ since

$$\begin{aligned}\|f\|_1 &= \int_G |f(x)|\chi_A(x)\,dm_G(x)\\ &\le \|f\|_p\|\chi_A\|_{p'} < \infty\end{aligned}$$

(by Hölder's inequality with $p' = p/(p-1)$ for $1 < p < \infty$, $1' = \infty$ and $\infty' = 1$). So $\hat{f} \in L^\infty(\Gamma)$ and $f \in L^2(G)$ since

$$\begin{aligned}\|f\|_2^2 = \|\hat{f}\|_2^2 &= \int_\Gamma |\hat{f}(\gamma)|^2\chi_B(\gamma)\,dm_\Gamma(\gamma)\\ &\le \|\hat{f}\|_\infty^2 m_\Gamma(B) < \infty.\end{aligned}$$

Applying Theorem 1 we see that $f = 0$ m_G - a.e. □

Restricting attention for the moment to non-compact groups, we might ask whether the conditions given in Theorem 1 for the QUP to be satisfied are necessary, i.e., does there exist a non-compact group G with compact identity component G_0 that satisfies the QUP?

THEOREM 2 *Let* G *be a non-compact LCA group with compact identity component* G_0. *Then the* QUP *is violated.*

Proof The quotient group G/G_0 is totally disconnected and therefore has a compact open subgroup K. let $\pi: G \to G/G_0$ be the natural homomorphism. π is continuous and open and there exists a compact open subset C of G such that $\pi(C) = K$. (See [7], App. B6, A7.) Then $G_1 = \pi^{-1}(K) = CG_0$ is a compact open subgroup of G. Let $m_G(G_1) = \alpha > 0$. If μ_{G_1} is the restriction of m_G to G_1 then $m_{G_1} = \alpha^{-1}\mu_{G_1}$ is a Haar measure on G_1 for which $m_{G_1}(G_1) = 1$. Let $f = \chi_{G_1}$. Then

$$\|f\|_2^2 = \int_G |\chi_{G_1}(x)|^2 dm_G(x) = m_G(G_1) = \alpha$$

Also,

$$\begin{aligned}\hat{f}(\gamma) &= \int_G \chi_{G_1}(x)\overline{\gamma(x)}\,dm_G(x)\\ &= \int_{G_1} \overline{\gamma(x)}\,\alpha\, dm_{G_1}(x)\\ &= \alpha\chi_{A(G_1)}(\gamma).\end{aligned}$$

Then,

$$\begin{aligned}\|\hat{f}\|_2^2 &= \int_\Gamma |\alpha\chi_{A(G_1)}(\gamma)|^2 dm_\Gamma(\gamma)\\ &= \alpha^2 m_\Gamma(A(G_1)).\end{aligned}$$

By Plancherel's theorem, $\alpha = \alpha^2 m_\Gamma(A(G_1))$, so $m_\Gamma(A(G_1)) = \alpha^{-1}$. For this function f, $A_f = G_1$, $B_f = A(G_1)$ and so $m_G(A_f)m_\Gamma(B_f) = \alpha\alpha^{-1} = 1$, hence the QUP is violated. ❑

A compact group G satisfies the QUP if, for each $f \in L^2(G)$,

$$m_G(A_f) < 1,\ m_\Gamma(B_f) < \infty \Rightarrow f = 0 \ m_G\text{-a.e.}$$

Once again, the connectedness of the group G determines whether or not the QUP is satisfied.

THEOREM 3 *If* G *is a compact abelian group, the* QUP *is satisfied iff* G *is connected.*

Proof First note that those functions $f \in L^2(G)$ satisfying $m_\Gamma(B_f) < \infty$ are the trigonometric polynomials, and it is well known that if G is connected, the only such function f satisfying $m_G(A_f) < 1$ is $f \equiv 0$.

Now suppose G has identity component $G_0 \subsetneq G$. There exists an open subgroup H with $G_0 \subseteq H \subsetneq G$. Since H is open, it is also closed and $0 < m_G(H) < 1$. Let $f = \chi_H$. Then as before, $\|f\|_2^2 = m_G(H)$, $A_f = H$ and $\hat{f}(\gamma) = m_G(H)\chi_{A(H)}(\gamma)$. So $B_f = A(H)$ and the Plancherel theorem then implies $m_\Gamma(B_f) = m_G(H)^{-1} < \infty$, so the QUP is violated.□

A pair of projections P,Q on a Hilbert space $\mathcal{H}$ have numerical range defined by

$$\text{num ran}(P,Q) = \{(x,y) \in [0,1] \times [0,1] \; ; \; x = \|Pf\|^2, \; y = \|Qf\|^2 \text{ for some } f \in \mathcal{H}, \; \|f\| = 1\}.$$

For general Hilbert spaces $\mathcal{H}$ and projections P,Q, num ran(P,Q) was studied extensively by Lenard in [4]. If G is an LCA group satisfying the QUP and F_B, E_A are projections on $L^2(G)$ defined by (2.2), a closer examination of the operator $F_B E_A$ allows a fairly complete description of num ran(E_A,F_B) - the only ambiguity being the delicate question of whether the points (0,1) and (1,0) lie in num ran(E_A,F_B). The form of the numerical range (in particular, the fact that it is bounded away from (1,1)) furnishes a more quantitative uncertainty principle that that established here.The case $G = \mathbb{R}$, $A = [-1,1]$, $B = [-1,1]$ is treated in [3] and [4].

The same analysis gives some familiar results related to the space of bandlimited functions, i.e. those functions in the range of F_B.

The techniques used in this paper (in particular, the proof of Theorem 1) may be applied to give a QUP on a fairly broad class of locally compact (not necessarily abelian) groups, similar to the type considered in [6].

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