A REMARK ON THE RELATIVE ENTROPY

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INTRODUCTION

The present article is a report of our joint works [4] and [5] with Mr. H. Yoshida on Pimsner-Popa's relative entropy $H(M|N)$ for a pair $M \supset N$ of finite von Neumann algebras. The notion of the relative entropy appeared first in Connes-Stormer's work [1] as a technical tool for finite dimensional algebras. In [6], M. Pimsner and S. Popa extended this notion for finite von Neumann algebras and made clear the relationship between $H(M|N)$ and Jones index $[M:N]$ for a pair $M \supset N$ of finite factors [3].

The aim of this article is to give complete formulas on $H(M|M^{\alpha})$ for an arbitrary action α of a locally compact group G on a finite von Neumann algebra M, applying Pimsner-Popa's deep results and our complementary general results. When M is a factor of type II_1, $H(M|M^{\alpha})$ is computed by using some conjugacy invariants of actions which are defined in a modified way of Jones' one [2].

§1 SOME GENERAL RESULTS.

Before entering in description, we fix some notations used hereafter. For a von Neumann algebra M, M^+ = {all positive elements of M} and $Z(M)$ = the center of M. For a set S, $|S|$ = the cardinal number of S.

Throughout the article, M denotes a finite von Neumann algebra on a separable Hilbert space with a faithful normal normalized trace τ. Let N be a von Neumann subalgebra of M. Then, a function h on M^+ is defined by

$$h(x) = \tau\eta E(x) - \tau\eta(x) \quad \text{for} \quad x \in M^+.$$

Here, E is a τ-preserving conditional expectation of M onto N and η is a continuous function for $t \geq 0$ such that $\eta(0) = 0$ and $\eta(t) = -t\cdot \log t$ if $t > 0$. Set

$$S(M) = \{\Delta = (x_i)_{i\in I};\ x_i \in M^+ \text{ and } \sum_{i\in I} x_i \leq 1 \text{ where } |I| < +\infty\}$$

$$H_\Delta(M|N) = \sum_{i\in I} h(x_i) \quad \text{for} \quad \Delta = (x_i)_{i\in I} \quad \text{in} \quad S(M).$$

Pimsner-Popa's relative entropy $H(M|N)$ is now given by

$$H(M|N) = \sup\{H_\Delta(M|N);\ \Delta \in S(M)\}.$$

Corresponding to an abelian von Neumann algebra $Z(M) \cap Z(N)$, there exists a standard probability measure space (Γ,μ) such that

$$(M,\tau) \underset{\theta}{\cong} \int_\Gamma^\oplus (M(\gamma))d\mu(\gamma), \quad N \underset{\theta}{\cong} \int_\Gamma^\oplus N(\gamma)d\mu(\Gamma),$$

$$Z(M) \cap Z(N) \underset{\theta}{\cong} \{\text{diagonalizable operators}\} \cong L^\infty(\Gamma,\mu).$$

Then, for μ-almost all $\gamma \in \Gamma$, the relative entropy $H(M(\gamma)|N(\gamma))$ is defined associated with the trace τ^γ.

THEOREM 1.1. *The function* $\Gamma \ni \gamma \to H(M(\gamma)|N(\gamma)) \in [0,\infty]$, *is* μ-*measurable and*

$$H(M|N) = \int_\Gamma H(M(\gamma)|N(\gamma))d\mu(\gamma).$$

The component algebras $M(\gamma)$ and $N(\gamma)$ satisfy that $Z(M(\gamma)) \cap Z(N(\gamma)) = \mathbb{C}$ for μ-almost all $\gamma \in \Gamma$. Thus, what we should do next is to evaluate the restive entropy $H(M|N)$ for such a pair $M \supset N$ that $Z(M) \cap Z(N) = \mathbb{C}$. Unfortunately, we can not succeed in it in general, but, under some stronger conditions, we may give some formulas on $H(M|N)$ in the next theorems. Here, we also note that the formula $H(M|N) = H(M|L) + H(L|N)$ is not assured in general for an intermediate subalgebra L with $N \subset L \subset M$.

THEOREM 1.2. *Suppose that the expectation* E *of* M *onto* N *satisfies* (*) $E(x) = \tau(x)$ *for* $x \in Z(M)$. *Then, we get the following.*

i) If $H(M|N) < +\infty$, *then* $Z(M)$ *is atomic.*

ii) When $Z(M)$ *is atomic, we denote all atoms of* $Z(M)$ *by* $\{p_i\}_{i\in I}$ *and the subalgebra* $\sum_{i\in I} p_i N p_i$ *of* M *by* L. *Then, we obtain* $H(M|N) = H(M|L) + H(L|N)$ *where*

$$H(M|L) = \sum_{i\in I} \tau(p_i) H(M_{p_i} | N_{p_i}) \text{ and } H(L|N) = \sum_{i\in I} \eta\tau(p_i).$$

THEOREM 1.3 *Suppose that* M *is a factor of finite type. Then, we have the following.*

i) If $H(M|N) < +\infty$, *then* $N' \cap M$ *is atomic, especially,* $Z(N)$ *is atomic.*

ii) When $Z(N)$ *is atomic, we denote all atoms of* $Z(N)$ *by* $\{q_j\}_{j\in J}$ *and* $\sum_{j\in J} q_j M q_j$ *by* L. *Then, we obtain* $H(M|N) = H(M|L) + H(L|M)$, *where*

$$H(M|L) = \sum_{j\in J} \eta\tau(q_j) \quad \textit{and} \quad H(L|N) = \sum_{j\in J} \tau(q_j) H(M_{q_j} | N_{q_j}).$$

§2 THE RELATIVE ENTROPY OF FIXED POINT ALGEBRAS

Let α be an action of a locally compact group G on a finite von Neumann algebra M. We denote the fixed point algebra of M under the action α by M^{α}, or by M^{G} if there is no need of mention of α. In this section, we shall give complete formulas on $H(M|M^{\alpha})$.

The action α of G on M induces an action of G on $Z(M)$ and $Z(M)^{G} = Z(M) \cap Z(M^{G})$. Corresponding to the abelian von Neumann subalgebra $Z(M)^{G}$, there exists a standard probability measure space (Γ,μ) such that

$$(M,\tau) \underset{\theta}{\cong} \int_{\Gamma}^{\oplus} (M(\gamma),\tau^{\gamma})d\mu(\gamma) \quad \text{and} \quad Z(M)^{G} \cong L^{\infty}(\Gamma,\mu).$$

Moreover, for μ-almost all $\gamma \in \Gamma$, there exists an action α^{γ} of G on the component algebra $M(\gamma)$ satisfying that

$$\alpha \underset{\mathrm{Ad}\theta}{\cong} \int_{\Gamma}^{\oplus} \alpha^{\gamma}\, d\mu(\gamma) \quad \text{and} \quad M^{G} \underset{\theta}{\cong} \int_{\Gamma}^{\oplus} M(\gamma)^{G}\, d\mu(\gamma).$$

Hence, we have the following immediate consequence of Theorem 1.1.

PROPOSITION 2.1. *In the above situation, we have*

$$H(M|M^{G}) = \int_{\Gamma} H(M(\gamma)|M(\gamma)^{G})\, d\mu(\gamma)$$

Here, we note that almost all actions α^{γ} of G on $M(\gamma)$ are centrally ergodic, namely, $Z(M(\gamma))^{G} = \mathbb{C}$. Therefore, we shall consider the case that an action is centrally ergodic.

LEMMA 2.2. *If an action* α *of* G *on* M *is centrally ergodic, the assumption (*) in Theorem 1.2 is satisfied for the pair* $M \supset M^{G}$.

PROPOSITION 2.3. *Suppose that an action* α *of* G *on* M *is centrally ergodic. Then, we get the following.*

i) If $H(M|M^{G}) < +\infty$, *then* $Z(M)$ *is atomic.*

ii) When $Z(M)$ *is atomic, we denote by* $\{p_i\}_{i\in I}$ *the family of all atoms of* $Z(M)$ *and by* H *the stabilizer at* p *for a fixed projection* p *among* p_i*'s. Then, we have*

$$H(M|M^G) = \sum_{i\in I} \eta\tau(p_i) + H(M_p|M_p^H).$$

Now, the rest to do is to compute the relative entropy $H(M|M^G)$ in the case that M is a factor of type II_1.

LEMMA 2.4. *Let* α *be an outer action of* G *on a factor* M *of finite type. Then, we get* $H(M|M^G) = \log|G|$.

This lemma is easily generalized as follows by applying Proposition 2.1 and 2.3.

COROLLARY 2.5. *Let* M *be a finite von Neumann algebra with a faithful normal normalized trace* τ *and* α *be a* τ*-preserving properly outer action of* G *on* M. *Then, we get* $H(M|M^G) = \log|G|$.

Now, we shall concentrate our attention to the structure of an action α of a locally compact group G on a factor M of type II_1 such that $H(M|M^\alpha) < +\infty$. We denote by $K(\alpha)$, or often abbreviated by K, a subgroup $\{g \in G;\ \alpha_g$ is an inner automorphism of $M\}$ of G. We note that K is a normal subgroup of G but not necessarily closed in general.

Suppose that $H(M|M^G) < +\infty$. Then, we get (a), (b), (c), and (d) which will be described below.

(a) K *is a closed normal subgroup of* G *such that* G/K *is a finite group.*

Then, by choosing a suitable Borel section, there exist a Borel multiplier μ of K and a Borel μ-representation V of K such that $\alpha_k = \mathrm{Ad}V_k$ and $V_h V_k = \mu(h,k)V_{hk}$ $(V_e = 1)$. Moreover, there exists a Borel $\mathbb{T}$-valued function λ of $G \times K$ satisfying that $\alpha_g(V_k) = \lambda(g,k)V_{gkg^{-1}}$ $(g \in G,\ k \in K)$. Since $H(M|M^K) < +\infty$, $(M^K)' \cap M \supset V(K)''$ must be atomic by (i) of Theorem 1.3. Therefore, V is decomposed as a direct sum of multiples of finite dimensional irreducible μ-representations of K. Here, we denote by X the set of all unitary equivalence classes of finite dimensional irreducible μ-representations of K and we define the action $\hat{\alpha}$ of G on X by $\hat{\alpha}_g(\pi_k) = \lambda(g,k)\pi_{gkg^{-1}}$ $(g \in G,\ k \in K)$ for $[\pi] \in X$. We denote by Ω the G-orbit space of X. For each orbit $\omega \in \Omega$, set $d_\omega = \dim\chi$ $(\chi \in \omega)$ and $|\omega|$ = the number of $\chi \in \omega$. Denote by $\{f_\chi\}_{\chi \in X}$ the family of central minimal projections of $V(K)''$ corresponding to the canonical central decomposition of V and set $e_\omega = \sum_{\chi \in \omega} f_\chi$ for $\omega \in \Omega$. Then, we get the following.

(b) *Each* f_χ *is an atom of* $Z(M^K)$ *if* $f_\chi \neq 0$, *and* $\sum_{\chi \in X} f_\chi = 1$.

(c) *Each* e_ω *is an atom of* $Z(M^G)$ *if* $e_\omega \neq 0$, *and* $\sum_{\omega \in \Omega} e_\omega = 1$.

(d) $(M^G)' \cap M = (M^K)' \cap M = V(K)''$.

We note that (λ,μ) is a representative of characteristic invariant of actions and $(\tau(e_\omega))_{\omega \in \Omega}$ is a representative of inner invariant of actions in a modified way of Jones' sense [2]. Under these investigations, we get the following.

THEOREM 2.6. *Let* M *be a factor of type* II_1 *with the canonical trace* τ *and* α *be an action of a locally compact group* G *on* M. *If* $H(M|M^G) < +\infty$, *we have*

$$H(M|M^G) = H(M|M^K) + H(M^K|M^G)$$
$$= \log|G/K| + \sum_{\omega\in\Omega} \tau(e_\omega)\log(d_\omega^2|\omega|/\tau(e_\omega)).$$

Finally, we remark on the case that G is a finite group. Let α be an action of a finite group G on a factor M of type II_1. We name α a *Jones action* if $\tau(e_\omega) = d_\omega^2|\omega|/|K(\alpha)|$. For each characteristic invariant $[\lambda,\mu] \in \Lambda(G,K)$, Jones constructed a model action $s_{G,K}^{(\lambda,\mu)}$ of G on the hyperfinite factor R of type II_1 in [2]. We note that, when $M = R$, α is a Jones action if and only if α is conjugate to $s_{G,K}^{(\lambda,\mu)}$ for $K(\alpha) = K$ and $\Lambda(\alpha) = [\lambda,\mu]$. The next is an immediate consequence of Theorem 2.6.

COROLLARY 2.7. *Let* α *be an action of a finite group* G *on a factor* M *of type* II_1. *Then* $0 \leq H(M|M^\alpha) \leq \log|G|$. *Moreover,* $H(M|M^\alpha) = \log|G|$ *if and only if the action* α *is a Jones action.*

By this corollary, when $M = R$, we see that there is one and only one action α up to conjugacy in each stable conjugacy class (characterized by each characteristic invariant) such that $H(R|R^\alpha)$ attains the maximum value $\log|G|$, which is nothing but Jones' model action. Corollary 2.7 is easily generalized in the case that M is not necessarily a factor by applying the formulas in Proposition 2.1 and 2.3. For the details, see [4].

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