SINGULAR INTEGRALS ON BMO

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Let f be a locally integrable function on $\mathbb{R}^n$. We say f has bounded mean oscillation, $f \in BMO$, if

$$(1) \qquad \sup_B \inf_{c \in \mathbb{R}} \frac{1}{|B|} \int_B |f(y)-c|\,dy < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Identifying functions which differ by an additive constant a.e. makes BMO a Banach space with norm $\| \ \|_{BMO}$ equal to the left hand side of (1). Note that L^∞ is a proper subset of BMO, since $\log|x| \in BMO$.

Let K be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that $Tf(x) = \lim_{\epsilon \downarrow 0} \int_{\{|y|>\epsilon\}} K(y)f(x-y)dy$ is a bounded operator on L^2. We say K satisfies condition H_r, $1 \le r < \infty$, if there is a non-decreasing function s on (0,1) such that $\sum_{j=1}^{\infty} s(2^{-j}) < +\infty$ and

$$\left[\int_{\{x:R<|x|<2R\}} |K(x-y)-K(x)|^r\,dx\right]^{1/r} \le s\left(\frac{|y|}{R}\right) R^{-n/r'}, \quad \text{for } |y|<R/2.$$

Define H_∞ by the obvious modification.

If $f \in L^\infty$ is supported on a set of finite measure and $K \in H_1$, then Tf exists a.e. (i.e., the limit exists and is finite), $Tf \in BMO$, and $\|Tf\|_{BMO} \le C\|f\|_{BMO}$ [2]. On the other hand, if f is merely bounded, then without a suitable modification Tf may fail to exist on a set of positive measure. For example, if $f(x) = \chi_E(x)$ is the characteristic function of $E = \{x \in \mathbb{R}^n: x_i > 0,\ i=1,\dots,n\}$, then the Riesz transforms of f, defined by the kernels $K_j(x) = \dfrac{x_j}{|x|^{n+1}}$, j=1,...,n, are infinite a.e.

Let $I(x)$ be a constant function on $\mathbb{R}^n$. We say $K \in H_r^+$, $1 \le r \le \infty$, if $K \in H_r$, $TI = 0$, and $\sum_{j=1}^{\infty} js(2^{-j}) < +\infty$.

THEOREM: Suppose $K \in H_r^+$, $1 < r \le \infty$, and $f \in BMO$. Either Tf fails to exist almost everywhere or $Tf \in BMO$ and

$$\|Tf\|_{BMO} \le C\|f\|_{BMO}.$$

The constant C is independent of f.

Given $x \in \mathbb{R}^n$ and $\delta > 0$, set $B(x,\delta) = \{y \in \mathbb{R}^n\colon |x-y| \le \delta\}$. For $B = B(x,\delta)$, let $f_B = \frac{1}{|B|}\int_B f(y)dy$. The proof of the theorem is based on the following lemma. (See [4].)

LEMMA: Let $1 \le p < \infty$. There is a constant C depending on n and p so that for $f \in BMO$, $B = B(x,\delta)$, and $k \ge 1$,

$$\left[\int_{B(x,2^k\delta)} |f(y)-f_B|^p dy\right]^{1/p} \le Ck(2^k\delta)^{n/p}\|f\|_{BMO}.$$

We now sketch a proof of the theorem. (See [6,4].) Suppose $E = \{x \in \mathbb{R}^n\colon Tf(x) \text{ exists}\}$ has positive measure. Let x_0 be a point of density of E and $\delta > 0$. Set $B = B(x_0,\delta)$ and $\tilde{B} = B(x_0,4\delta)$. Write $f(x) = f_B + \left[f(x)-f_B\right]\chi_{\tilde{B}}(x) + \left[f(x)-f_B\right]\chi_{\mathbb{R}^n\setminus\tilde{B}}(x) = f_B+g_B(x)+h_B(x)$. Since f_B is constant, $Tf_B = 0$. By the lemma, $g_B \in L^2$ and

$$\text{(2)} \qquad \int_B |Tg_B(y)|dy \le |B|^{1/2}\|Tg_B\|_2 \le C_1|B|^{1/2}\|g_B\|_2 \le C_2|B|\,\|f\|_{BMO}.$$

It follows that Tg_B exists a.e. so that Tf exists at almost every point such that

Since x_0 is a point of density of E and Tg_B exists a.e., there is a point $y_0 \in B(x_0,\delta)$ such that $Th_B(y_0) = Tf(y_0) - Tg_B(y_0)$ exists. Suppose $x \in B$. Set $A_j = \{z \in \mathbb{R}^n : 2^j\delta < |x_0 - z| \le 2^{j+1}\delta\}$. By the lemma, since $K \in H_r^+$ and $|x-y_0| \le 2\delta$,

$$(3)\quad |Th_B(x)-Th_B(y_0)| \le \int |K(x-z)-K(y_0-z)|\ |h_B(z)|dz$$

$$= \sum_{j=2}^{\infty} \int_{A_j} |K(x-z)-K(y_0-z)|\ |f(z)-f_B|dz$$

$$\le \sum_{j=2}^{\infty} \left[\int_{A_j} |K(x-z)-K(y_0-z)|^r dz\right]^{1/r} \left[\int_{B(x_0,2^{j+1}\delta)} |f(z)-f_B|^{r'} dz\right]^{1/r'}$$

$$\le C \sum_{j=2}^{\infty} s\left[\frac{|x-y_0|}{2^j\delta}\right] (2^j\delta)^{-n/r'}\ j(2^{j+1}\delta)^{n/r'}\ \|f\|_{BMO}$$

$$\le C' \sum_{j=1}^{\infty} s(2^{-j})\ j\|f\|_{BMO} = C''\|f\|_{BMO}.$$

As a consequence of (3), Th_B exists a.e. in B, which implies Tf exists a.e. in B. By considering only $B(x_0,\delta)$ with δ a positive integer, it follows that Tf exists a.e. in $\mathbb{R}^n$.

To show $\|Tf\|_{BMO} \le C\|f\|_{BMO}$, fix $B = B(x,\delta)$ and choose y_0 as before. By (2) and (3),

$$\frac{1}{|B|}\int_B |Tf(y) - Th_B(y_0)|dy \le \frac{1}{|B|}\int_B |Tg_B(y)|dy + \frac{1}{|B|}\int_B |Th_B(y)-Th_B(y_0)|dy$$

$$\le C\ \|f\|_{BMO}.$$

Since B was arbitrary, we see that $Tf \in BMO$ and $\|Tf\|_{BMO} \le C\|f\|_{BMO}$.

Let $\sum_{n-1} = \{x \in \mathbb{R}^n\colon |x|=1\}$ and ρ be a rotation of $\sum_{n-1}$ with $|\rho| = \sup_{x\in\Sigma_{n-1}} |x-\rho x|$. Suppose $K(x) = \frac{\Omega(x)}{|x|^n}$, where Ω is homogeneous of degree 0 and $\int_{\Sigma_{n-1}} \Omega(x)d\sigma(x) = 0$. Let ω_r be the L^r modulus of continuity of Ω on $\sum_{n-1}$, $\omega_r(\delta) = \sup_{|\rho|\le\delta} \left[\int_{\Sigma_{n-1}} |\Omega(x)-\Omega(\rho x)|^r d\sigma(x)\right]^{1/r}$. (For $r=\infty$, use the L^∞ norm). Then $K \in H_r^+$ if $\int_0^1 \frac{\omega_r(\delta)\ell n\delta}{\delta} d\delta <+ \infty$. (This is a slightly stronger condition than the L^r–Dini condition, which implies $K \in H_r$.) In particular, if $\Omega \in Lip(\alpha)$, $\alpha > 0$,

$$|\Omega(x)-\Omega(y)| \le C\,|x-y|^\alpha,$$

then $\Omega \in H_\infty^+$. Thus, the Riesz transforms satisfy the theorem.

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