DISCREPANCY RESULTS FOR NORMAL NUMBERS

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1. INTRODUCTION

We say that a number x is *normal to base* r ($r \in \mathbb{Z}^+$) if the sequence $(r^n x)_{n=1}^{\infty}$ is uniformly distributed modulo unity. Weyl [16] established that for each base r almost all (Lebesgue) numbers are normal. The nature of the set of numbers normal to all bases in some collection A and non-normal to every base of a collection B was first investigated by W. Schmidt [14,15]. As observed by Schmidt, such numbers can arise if and only if there is no relation of the form $r^n = s^m$ for $r \in A, s \in B$ and $m, n \in \mathbb{Z}^+$. In this event we say A, B are *multiplicatively independent*. We assume this restriction without further comment. Later refinements and some simpler proofs have been given in [1,2,12]. In [2] it was shown that for an appropriately chosen Riesz product measure μ on $[0,1]$, the numbers on $[0,1]$ which are normal to each base of A and normal to no base of B constitute a set of full μ measure. It is seen further in [3] that this set has Hausdorff dimension unity.

Our present purpose is to derive some corresponding discrepancy results pertaining to the rate of convergence to uniformity of the sequences involved. For simplicity we restrict our attention to the case considered in [1] in which B is a singleton set $\{s\}$ with $s > 3$, but the arguments go through with appropriate modifications in the rather more general situation of [2]. The proof employed is based on estimates of Fourier-Stieltjes coefficients available from earlier studies together with some elementary probability theory.

To describe the rate of convergence towards uniformity of a sequence $(x_n)_1^{\infty}$ of numbers on $[0,1]$ we recall the notion of discrepancy. For $E \subset [0,1]$, we define the *counting function*

$$A(E;n) \equiv \mathrm{card}\{k \mid \quad 0 < k \leq n, \quad x_k \in E\} \quad .$$

The *discrepancy* $D_n(x_1, ..., x_n)$ is given by

$$D_n(x_1, ... x_n) \equiv \sup_{0 \le \alpha < \beta \le 1} G_n(\alpha, \beta) \quad ,$$

where

$$G_n(\alpha, \beta) \equiv \left| \frac{A([\alpha, \beta); n)}{n} \quad - (\beta - \alpha) \right|$$

(*c.f.* [10, p.88]). The sequence (x_n) is uniformly distributed modulo unity if $D_n \to 0$ as $n \to \infty$ (*c.f.* [10, p. 89, Thm 1,1]), and when this occurs the asymptotic behaviour of D_n describes the rate of convergence to uniformity of the sequence (x_n).

The literature appears to relate exclusively to discrepancy results for numbers normal to a given base, the case considered in Corollary 1. The early literature deals with a fixed interval, that is, the discrepancy results relate to G_n rather than D_n. Notable amongst this work, surveyed and extended by Koksma [7], is Khintchine's result [5] $G_n = O((n^{-1} loglogn)^{\frac{1}{2}})$. Koksma derives a number of results, such as the following.

Theorem. *Let ψ be a positive function with $\psi(n) \to \infty$ and ϕ a positive non-decreasing function of the positive integer $n \ge n_0$ with*

$$\sum_{n=n_0}^{\infty} \frac{1}{n\phi(n)} \quad < \quad \infty \quad ,$$

$$\phi(n+1) \quad \le \quad (1 + \frac{K}{n})\phi(n) \qquad (n \ge n_0) \quad .$$

Then for almost all (Lebesgue) reals and almost all intervals $[\alpha, \beta)$,

$$G_n \quad = \quad o(n^{-\frac{1}{4}}(\phi(n))^{\frac{1}{2}}) \quad ,$$

$$\liminf_{n \to \infty} G_n/(n^{-\frac{1}{2}}\psi(n)) \quad = \quad 0 \quad .$$

More recent work (see for example [8,9,11,13] concerns the construction of a real number x and base r for which $(r^n x)$ has certain discrepancies (for example $O(n^{-\frac{2}{3}}(\log n)^{\frac{4}{3}})$ in [9]). Accordingly even the limited conclusion of Corollary 1 is not without interest.

2. RESULTS

In the remainder of the paper $D_n(r,x)$ refers to the choice $x_n = r^n x$.

PROPOSITION 1. *Let r be a positive integer exceeding unity and ν a probability measure on $[0,1]$. Define*

$$S_n(x) \equiv \frac{1}{n}\sum_{k=1}^{n} exp(2\pi i r^k x) \quad x \in [0,1], \quad n > 0,$$

$$I_n(l) \equiv \int_0^1 |S_n(lx)|^2 d\nu(x) \quad , \quad l \in \mathbb{Z}\backslash\{0\}.$$

Suppose there exist positive constants b (independent of l) and C such that

(2.1) $$I_n(l) \leq C/n^b \quad , \forall n > 0 \quad .$$

Then

(i) normality to base r occurs a.e. (ν) on $[0,1]$;

(ii) for any $d > \frac{3}{2}b$ and each $\epsilon > 0$,

$$\nu(\{x \in [0,1] | D_n(r,x) = O(n^{-b/2}(\log n)^d)\}) > 1 - \epsilon.$$

Proof. From (2.1) we have that $\sum I_n(l)/n < \infty$ for each integer l. By a theorem of Davenport, Erdös and LeVeque [4], $S_n(lx) \to 0$ a.e. (ν) as $n \to \infty$ for all $l \in \mathbb{Z}\backslash\{0\}$. The first part derives from Weyl's celebrated criterion that a necessary and sufficient condition for (x_n) to be uniformly distributed (mod 1) is that for each $l \in \mathbb{Z}\backslash\{0\}$

$$\frac{1}{N}\sum_{n=1}^{N} exp(2\pi i l x_n) \to 0 \quad \text{as} \quad N \to \infty.$$

For part (ii), we observe that (2.1) implies that for each $\delta > 0$

(2.2) $$\sum_{n=2}^{\infty} n^b I_N(l)/(n(\log n)^{\frac{1}{2}(1+\delta)}) < \infty \quad .$$

Put

$$T_n(x) \equiv S_n(x) n^{\frac{1}{2}b} / (\log n)^{\frac{1}{2}(1+\delta)} \quad (n \geq 1) \quad , \tag{2.3}$$

$$J_n(l) \equiv \int_0^1 |T_n(lx)|^2 d\nu(x) \quad .$$

Then (2.2) implies that $\sum J_n(l)/n < \infty$ for each $l \in \mathbb{Z} \backslash \{0\}$, so that the Davenport, Erdös and LeVeque result gives $T_n(lx) \to 0$ a.e. (ν) as $n \to \infty$. Each $T_n(lx)$ is everywhere finite, and by Egorov's theorem

$$T_N(lx) \to 0 \quad (\nu) \quad \text{almost} \quad \text{uniformly.}$$

Hence for each $l \in \mathbb{Z} \backslash \{0\}$ and $\epsilon > 0$, there exists an $A > 0$ and a ν-measurable set $F(\epsilon)$ such that

$$|S_n(lx)| < A (\log n)^{\frac{1}{2}(1+\delta)} / n^{\frac{1}{2}b} \quad (\forall n \in \mathbb{Z}^+, x \in F(\epsilon))$$

and

$$\nu(F(\epsilon)) > 1 - \epsilon.$$

In fact a simple modification of the proof of Egorov's theorem (see, for example [6]) shows that this can be done simultaneously for all $l \in \mathbb{Z} \backslash \{0\}$, that is, the constant A may be chosen independently of l. By a theorem of Erdös and Turán (see [10, p.114, relation (2.42)]), the discrepancy D_n of the sequence $(r^n x)$ satisfies an inequality

$$D_n(r, x) \leq C \left\{ \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} |S_n(hx)| \right\}$$

for any positive integer m, where $C > 0$ is an absolute constant. Therefore

$$D_n(r, x) \leq C \left\{ \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} A (\log n)^{\frac{1}{2}(1+\delta)} / n^{\frac{1}{2}b} \right\} \quad (\forall x \in F(\epsilon)) \quad (n > 1) \quad .$$

Choose

$$m = \left[n^{\frac{1}{2}b} (\log n)^{\frac{1}{2}(1+\delta)} / A \right] + 1,$$

where [.] denotes the integer part function. Then

$$D_n(r, x) \leq C_1 A n^{-\frac{1}{2}b} (\log n)^{\frac{1}{2}(3+\delta)} \quad (\forall x \in F(\epsilon)) \quad (n > 1),$$

where C_1 is an absolute constant. Therefore for all d above $\frac{3}{2}$,

$$D_n = O(n^{-\frac{1}{2}b}(\log n)^d) \quad (\forall x \in F(\epsilon)),$$

and (ii) follows. $\square$

We note some applications. First take ν to be Lebesgue measure λ . Then $I_n(l) \equiv 1/n$. We derive

COROLLARY 1 *If $r > 1$ is an integer, then for each $\epsilon > 0$ and $d > \frac{3}{2}$,*

$$\lambda(\{x|x \in [0,1], D_n(r,x) = O(n^{-\frac{1}{2}}(\log n)^d)\}) > 1 - \epsilon.$$

Secondly, for r, s multiplicatively independent integers exceeding unity, denote by η_r the type of $log_s r$ in the sense of [7, p. 121], that is,

$$\eta_r = \liminf_{n \to \infty}\{\gamma | n^\gamma \langle n \log_s r\rangle = 0\} \quad ,$$

where $\langle . \rangle$ is the fractional part function. It is noted in [7] that the value of a type is not less than unity. We have the following result.

COROLLARY 2 *Let $s > 3$ be a fixed integer and A a subset (proper or improper) of the integers r multiplicatively independent of s and satisfying $r > s$. Then there exists a probability measure ν on $[0,1]$ with the following properties:*

(a) the numbers on $[0,1]$ non-normal to base s and normal to each base of A form a set of full ν measure;

(b) for each $\epsilon > 0$ and sequence d_r satisfying

$$0 < d_r < (1 - \log_s 3)/(2\eta_r), \tag{2.4}$$

we have

$$\nu(\{x|x \in [0,1], D_n(r,x) = O(n^{-d_r}) \quad \forall r \in A\}) > 1 - \epsilon.$$

Proof Choose ν to be the Riesz product μ constructed in [1]. Then (a) is immediate. For part (b), if suffices to show that for any $r \in A$ and d_r satisfying (2.4), we have

(2.5) $$\mu(\{x|x \in [0,1], \quad D_n(r,x) = O(n^{-d_r})\}) > 1 - \epsilon/2^r.$$

By the argument of [1], for a given l, r, we have a decomposition

$$I_n(l) = 1/n \quad + \quad H_n(l),$$

and by relation (22) of [1] the quantity H_n has an upper bound

$$H_n(l) \quad \leq \quad t(n)/n \quad + \quad 2.3^{t(n)}\{D_n^1 + 3.s^{-t(n)}\}$$

for each monotone non-decreasing map $t : \mathbf{Z}^+ \to \mathbf{Z}^+$ with $t(n) \leq n$ for all $n \in \mathbf{Z}^+$. By the relation immediately preceding [1,(17)], D_n^1 satisfies

$$D_n^1 \quad = \quad O(n^{\xi - 1/\eta})$$

for each $\xi > 0$. Select t so that

$$t(n) \sim \frac{\log n}{\eta_r \log s} \quad .$$

It is readily verified that (2.1) holds for b satisfying

$$b < \min[1, (1 - \log_s 3)/\eta_r] \quad ,$$

that is (since $\eta_r \geq 1$)

$$b < (1 - \log_s 3)/\eta_r \quad .$$

Relation (2.5) follows from Proposition 1, giving part (b) as desired. □

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