HANKEL OPERATORS ON THE PALEY-WIENER SPACE IN DISK

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1. INTRODUCTION

In [12], Rochberg has studied the Toeplitz and Hankel operators on the Paley-Wiener space in one dimension, and has got the characterizations for the Schatten-von Neumann class S_p criteria. In the end of [12], Rochberg proposed what are analogs of the results in several dimensions. In [11], Peng has studied the case of cube $I^d = \{\xi \in \mathbb{R}^d : -\pi < \xi_j < \pi\,,\, j = 1, \ldots, d\}$. In this paper, we study the case of disk $D = \{\xi \in \mathbb{R}^2 : |\xi| < 1\}$.

Let D denote the unit disk in $\mathbb{R}^2$, and let χ_D denote the characteristic function of D. The Paley-Wiener space on the unit disk, $PW(D)$, is defined to be the image of $L^2(D)$ under inverse Fourier transformations F^{-1}, i.e.

$$PW(D) = \{F^{-1}(\chi_D f) : f \in L^2(D)\}\,. \tag{1.1}$$

Let P_1, P_2 denote the projections defined by $(P_1 g)\hat{} = \chi_D \hat{g}$ and $(P_2 g)\hat{} = \chi_{2D} \hat{g}$, separately.

The Toeplitz operator on $PW(D)$ with symbol b is defined by

$$T_b(f) = P_1(bf)\,, \quad \text{for } f \in PW(D)\,. \tag{1.2}$$

And the Hankel operator on $PW(D)$ with symbol b is defined by

$$H_b(f) = P_1(b\overline{f})\,, \quad \text{for } f \in PW(D)\,. \tag{1.3}$$

Because $PW(D)$ is preserved when taking complex conjugates, these two operators on $PW(D)$ are unitary equivalent. But as they have properties similar to those of classical Hankel operators (see below), we prefer the name Hankel operators in both cases.

Note that $T_b = T_{P_2 b}$, so we assume that $\operatorname{supp} \hat{b} \subset 2D$ throughout this paper.

Taking Fourier transform, we get

$$\widehat{T_b(f)}(\xi) = \int_{\mathbb{R}^2} \hat{b}(\xi - \eta)\chi_D(\xi)\chi_D(\eta)\hat{f}(\eta)\, d\eta\,. \tag{1.4}$$

This turns out to be a paracommutator. But as in the case of cube, it can not be dealt with in the framework of Janson and Peetre [4].

Our idea is the same as that in [11], that is to give a decomposition of D, then to define a kind of the Besov spaces $B_p^{s,q}(D)$ so that they characterize the Schatten-von Neumann class S_p of T_b.

As is well known, the disk multiplier is bounded only on $L^2(\mathbb{R}^2)$. It is quite different from the cube multiplier. Our results on the Schatten-von Neumann class criteria of Hankel operators on $PW(D)$ are also different from either classical case or the case of cube. In fact we get the necessary and sufficient condition of $T_b \in S_p$ only for $1 \leq p \leq 2$, that is $T_b \in S_p$ if and only if $b \in B_p^{\frac{3}{2p},p}(2D)$. For $2 < p \leq \infty$, we get only the necessary condition. (See below Theorems 3.1 and 4.1.)

Note that

$$\widehat{\chi_D}(x) = a\frac{e^{i|x|}}{|x|^{3/2}} + b\frac{e^{-i|x|}}{|x|^{3/2}} + O(|x|^{-5/2})\,, \quad |x| \to \infty\,, \tag{1.5}$$

it is interesting to point out the index $\frac{3}{2}$ is different from either that of classical case or of the case of cube, but is same to the degree of principal part of $\widehat{\chi_D}(x)$.

The sufficient conditions of $2 < p \leq \infty$ are still open.

In §2, we give a decomposition of D, define a kind of Besov spaces of Paley-Wiener type $B_p^{s,q}(D)$, and discuss their elementary functional properties. In §3, we prove the sufficient conditions for $1 \leq p \leq 2$. In §4, we prove the necessary conditions for $1 \leq p \leq \infty$.

2. BESOV SPACES $B_p^{s,q}(D)$

Let $S_D = \{f \in S(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset \overline{D}\}$, $S'_D = \{f \in S'(\mathbb{R}^2) : \operatorname{supp} \hat{f} \subset \overline{D}\}$, and let I^σ denote a kind of fractional integration operators defined by

$$(I^\sigma f)\hat{}(\xi) = (1 - |\xi|)^\sigma \hat{f}(\xi)\,, \quad \text{for } \sigma \in \mathbb{R},\ f \in S'_D\,.$$

Definition (2.1). For $1 \le p \le \infty$, $S \in \mathbb{R}$,

$$H_p^s(D) = \left\{ f \in S'_D : \|f\|_{H_p^s(D)} = \|I^s f\|_{L^p} < \infty \right\} .$$

It is obvious that I^σ maps $H_p^s(D)$ isomorphically onto $H_p^{s-\sigma}(D)$, and that $H_2^0(D) = PW(D)$.

To define a kind of Besov spaces of Paley-Wiener type on D, we give a decomposition of D as follows.

Let $Q_{j,k_j} = \{re^{i\theta} \in D : 4^{j-1} \le 1 - r \le 4^j , (k_j - 1)2^j\pi \le \theta \le k_j 2^j \pi\}$, for $j = -1, -2, \ldots$, $k_j \in \{1, 2, \ldots, 2^{-j+1}\}$, $Q_{0,k_0} = \{re^{i\theta} \in D : 0 \le r \le \frac{3}{4} , (k_0 - 1)\pi \le \theta \le k_0 \pi\}$, $k_0 = 1, 2$, thus

$$D = \bigcup_{\substack{j \in \mathbb{Z}_- \\ k_j \in \{1,\ldots,2^{-j+1}\}}} Q_{j,k_j} , \quad \text{where } \mathbb{Z}_- = \{0, -1, -2, \ldots\} . \tag{2.1}$$

Each Q_{j,k_j} has its height $3 \times 4^{j-1}$ which is comparable to the distance from the boundary, and has its length $r2^j\pi$ which is comparable to the square root of the distance from the boundary.

Definition (2.2). Let $\Phi(D)$ be the collection of all test function systems $\{\varphi_{j,k_j}\}$ such that

(i) $\operatorname{supp} \hat{\varphi}_{j,k_j} \subset \overline{Q}_{j,k_j} = \{re^{i\theta} \in D : \frac{3}{4} \times 4^{j-1} \le 1 - r \le \frac{5}{4} \times 4^j , (k_j - \frac{3}{2})2^j\pi \le \theta \le (k_j + \frac{1}{2})2^j\pi\}$,

(ii) $\hat{\varphi}_{j,k_j} \ge 0$, $\hat{\varphi}_{j,k_j}(\xi) \ge C > 0$ for $\xi \in Q_{j,k_j}$, $\hat{\varphi}_{j,k} \in C_0^\infty$,

(iii) $C_1 \le \sum \hat{\varphi}_{j,k_j}(\xi) \le C_2$ for $\xi \in D$.

Moreover, we can also require that $\sum \hat{\varphi}_{j,k_j}(\xi) \equiv 1$ for $\xi \in D$.

Definition (2.3). Let $s \in \mathbb{R}$, $0 < p, q \le \infty$, $\{\varphi_{j,k_j}\} \in \Phi(D)$.

$$B_p^{s,q}(D) = \left\{ f \in S'_D : \|f\|_{B_p^{s,q}(D)} = \left[\sum_{\substack{j \in \mathbb{Z}_- \\ k_j \in \{1,\ldots,2^{-j+1}\}}} (4^{sj} \|f * \varphi_{j,k_j}\|_p)^q \right]^{\frac{1}{q}} < \infty \right\} .$$

The following Theorem contains some of the elementary functional properties of $B_p^{s,q}(D)$.

THEOREM (2.1).

(i) $B_p^{s,q}(D)$ *is a quasi-Banach space if* $s \in \mathbb{R}$, $0 < p,q \leq \infty$ *(Banach space if* $1 \leq p,q \leq \infty$*), and the quasi-norms* $\|f\|^{\varphi}_{B_p^{s,q}(D)}$ *with* $\varphi \in \Phi(D)$ *are equivalent.*

(ii) $B_2^{s,2}(D) = H_2^s(D)$.

(iii) $S_D \subset B_p^{s,q}(D) \subset S'_D$.

(iv) If $p,q < \infty$, S_D *is dense in* $B_p^{s,q}(D)$.

(v) $\forall \sigma \in \mathbb{R}$, I^σ *maps* $B_p^{s,q}(D)$ *isomorphically onto* $B_p^{s-\sigma,q}(D)$.

(vi) $\left(B_p^{s,q}(D)\right)' = B_{p'}^{-s,q'}(D)$, *for* $S \in \mathbb{R}$, $1 \leq p,q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

(vii) $\left(B_{p_0}^{s_0,q_0}(D), B_{p_1}^{s_1,q_1}(D)\right)_{[\theta]} = B_{p^*}^{s^*,q^*}(D)$, *for* $s_0, s_1 \in \mathbb{R}$, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $0 < \theta < 1$, $s^* = (1-\theta)s_0 + \theta s_1$, $\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$.

Proof. All of the conclusions can be proved similarly to the ones of classical case. (See, e.g. Peetre [7], Triebel [15], also cf. Peng [11] for (vi) and (vii)). □

We can also define $B_p^{s,q}(2D)$ similarly according to the decomposition of $2D$:

$$2D = \sum_{\substack{j \in \mathbb{Z}_- \\ k_j \in \{1,2,\ldots,2^{-j+1}\}}} Q'_{j,k_j} ,$$

where $Q'_{j,k_j} = \{re^{i\theta} \in 2D : 2 \times 4^{j-1} \leq 2 - r \leq 2 \times 4^j , (k_j - 1)2^j\pi \leq \theta \leq k_j 2^j \pi\}$ for $j = -1, -2, \ldots$, $k_j \in \{1, 2, \ldots, 2^{-j+1}\}$, $Q'_{0,k_0} = \{re^{i\theta} \in 2D : 0 \leq r \leq \frac{3}{2}, (k_0 - 1)\pi \leq \theta \leq k_0\pi\}$. And $B_q^{s,q}(2D)$ have the properties in Theorem (2.1).

3. SUFFICIENT CONDITIONS FOR $1 \leq p \leq 2$.

We adopt the notation of Janson and Peetre [4] for $\|k(\xi,\eta)\|_{S_p(U \times V)}$. Extending the definition of T_b, we consider $T_b^{s,t}$ defined by

$$\widehat{T_b^{s,t} f}(\xi) = \int_{\mathbb{R}^2} \hat{b}(\xi - \eta)(1 - |\xi|)^s (1 - |\eta|)^t \chi_D(\xi)\chi_D(\eta)\hat{f}(\eta)\, d\eta \,, \tag{3.1}$$

for $s, t \in \mathbb{R}$.

THEOREM (3.1). *Suppose that* $1 \leq p \leq 2$, $b \in B_p^{\frac{3}{2p},p}(2D)$. *Then* $T_b \in S_p$ *and*

$$\|T_b\|_{S_p} \leq C \|b\|_{B_p^{\frac{3}{2p},p}(2D)} . \tag{3.2}$$

We need two lemmas.

LEMMA (3.1). *For $b \in S'_D$, $T_b \in S_2$ if and only if $b \in B_2^{\frac{3}{4},2}(2D)$ and that*

$$\|T_b\|_{S_2} \simeq \|b\|_{B_2^{\frac{3}{4},2}(2D)} . \tag{3.3}$$

Proof. According to Janson and Peetre [4], we have

$$\begin{aligned}
\|T_b\|_{S_2}^2 &= \iint \left|\hat{b}(\xi-\eta)\chi_D(\xi)\chi_D(\eta)\right|^2 \mathrm{d}\xi\,\mathrm{d}\eta \\
&= \int_{2D} \left|\hat{b}(\xi)\right|^2 \left(2\arcsin\tfrac{1}{2}\sqrt{4-|\xi|^2} - \tfrac{|\xi|}{2}\sqrt{4-|\xi|^2}\right) \mathrm{d}\xi \\
&\simeq \int_{2D} \left|\hat{b}(\xi)\right|^2 (4-|\xi|)^{\frac{3}{2}}\,\mathrm{d}\xi \\
&= \|b\|^2_{H_2^{\frac{3}{4}}(2D)} \\
&\simeq \|b\|^2_{B_2^{\frac{3}{4},2}(2D)} .
\end{aligned}$$

□

LEMMA (3.2). *If $b \in B_1^{s+t+\frac{3}{2},1}(2D)$, $s,t > -\frac{1}{2}$. Then $T_b^{3,t} \in S_1$ and that*

$$\|T_b^{s,t}\|_{S_1} \le C\|b\|_{B_1^{s+t+\frac{3}{2},1}(2D)} . \tag{3.4}$$

Proof. Let $\{\varphi_{j,k_j}\} \in \Phi(2D)$ such that $\sum_{\substack{j\in\mathbb{Z}_- \\ k_j\in\{1,2,\dots,2^{-j}+1\}}} \hat{\varphi}_{j,k_j}(\xi) = 1$ on $2D$. Then

$$\begin{aligned}
\|T_b^{s,t}\|_{S_1} &\le \sum_{\substack{j\in\mathbb{Z}_- \\ k_j\in\{1,2,\dots,2^{-j}+1\}}} \left\|\hat{b}(\xi-\eta)\hat{\varphi}_{j,k_j}(\xi-\eta)(1-|\xi|)^s(1-|\eta|)^t\right\|_{S_1(D\times D)} \\
&= \sum_{\substack{j\in\mathbb{Z}_- \\ k_j\in\{1,2,\dots,2^{-j}+1\}}} I_{j,k_j} .
\end{aligned}$$

If $j < -10$, let us estimate I_{j,k_j} as follows. Note that

$$\begin{aligned}
&\operatorname{supp}\hat{\varphi}_{j,k_j} \subset \overline{Q}'_{j,k_j} \\
&\quad = \left\{re^{i\theta} \in 2D : \frac{3}{2}\times 4^{j-1} \le 2-\gamma \le \frac{5}{2}\times 4^j ,\ (k_j - \frac{3}{2})2^j\pi \le \theta \le (k_j+\tfrac{1}{2})2^j\pi\right\} ,
\end{aligned}$$

if $\xi \notin \{r_1e^{i\theta_1} : 1-r_1 \le 4^{j+2} , (k_j-4)2^j\pi \le \theta_1 \le (k_j+3)2^j\pi\}$ or $\eta \notin \{r_2e^{i\theta_2} : 1-r_2 \le 4^{j+2} , (k_j-4)2^j\pi+\pi \le \theta_2 \le (k_j+3)2^j\pi+\pi\}$ then $\xi-\eta \notin \overline{Q}'_{j,k_j}$.

Thus we have, by Lemmas 3.1 and 3.3 of Janson and Peetre [4],

$$I_{j,k_j} \le C\|b * \varphi_{j,k_j}\|_1 \sum_{l_1=-\infty}^{j+2} \sum_{l_2=-\infty}^{j+2} \left\|(1-|\xi|)^s(1-|\eta|)^t\right\|_{S_1(Q_{l_1,k_j}\times Q_{l_2,k'_j})}$$

(where $Q_{l,k_j} = \{re^{i\theta} : 4^{l-1} \le 1-r \le 4^l , (k_j-4)2^j\pi \le \theta \le (k_j+3)2^j\pi\}$, $k'_j = 2^{-j}+k_j$)

$$\leq C\|b*\varphi_{j,k_j}\|_1 \sum_{l_1=-\infty}^{j+2}\sum_{l_2=-\infty}^{j+2} 4^{l_1 s}\cdot 4^{l_2 t}\cdot (4^{l_1}\cdot 2^j)^{1/2}\cdot (4^{l_2}\cdot 2^j)^{1/2}$$
$$= C4^{j(s+t+\frac{3}{2})}\|b*\varphi_{j,k_j}\|_1 \,.$$

If $j \geq -10$,

$$\begin{aligned} I_{j,k_j} &\leq C\|b*\varphi_{j,k_j}\|_1 \sum_{l_1=-\infty}^{0}\sum_{l_2=-\infty}^{0} 4^{l_1 s}\cdot 4^{l_2 t}\cdot (2\pi\cdot 4^{l_1})^{1/2}\cdot (2\pi\cdot 4^{l_2})^{1/2} \\ &\leq C\|b*\varphi_{j,k_j}\|_1 \\ &\leq C4^{j(s+t+\frac{3}{2})}\|b*\varphi_{j,k_j}\|_1 \,. \end{aligned}$$

This completes the proof. □

The proof of Theorem (3.1). Theorem (2.1)-(vii), Lemma (3.1) and Lemma (3.2) give the proof of Theorem (3.1) by complex interpolation. □

4. NECESSARY CONDITIONS FOR $1 \leq p \leq \infty$.

The necessary conditions can be treated in more generality.

THEOREM (4.1). *If $1 \leq p \leq \infty$, $b \in S'_{2D}$ such that $T_b^{s,t} \in S_p$, then $b \in B_p^{s+t+\frac{3}{2p},p}(2D)$, and*

$$\|b\|_{B_p^{s+t+\frac{3}{2p},p}(2D)} \leq C\|T_b^{s,t}\|_{S_p} \,. \tag{4.1}$$

Proof. Suppose that $\{\varphi_{j,k}\} \in \Phi(D)$. Let P_{j,k_j} denote the projection defined by $(P_{j,k_j}g)\hat{} = \chi_{\overline{Q}_{j,k_j}}\hat{g}$.

Since $\{P_{j,k_j}\}$ are joint at most 9 times we have

$$\|T_b^{s,t}\|_{S_p}^p \geq C \sum_{\substack{j\in\mathbb{Z}_- \\ k_j\in\{1,2,\ldots,2^{-j}+1\}}} \|P_{j,k_j}T_b^{s,t}P_{j,k'_j}\|_{S_p}^p \,,\ (k'_j = k_j + 2^{-j}) \,. \tag{4.2}$$

Let

$$\hat{\psi}_{j,k_j}(\xi) = 4^{-j(s+t+\frac{3}{2})}\int \hat{\varphi}_{j,k_j}(\xi+\eta)(1-|\xi+\eta|)^s(1-|\eta|)^t\hat{\varphi}_{j,k'_j}(\eta)\,d\eta \,.$$

It is easy to show that

(i) $\operatorname{supp}\hat{\psi}_{j,k_j} \subset \{re^{i\theta} : \frac{3}{2} \times 4^{j-1} \le 2 - r \le \frac{5}{2} \times 4^j,\ (k_j - \frac{5}{2})2^j\pi \le \theta \le (k_j + \frac{5}{2})2^j\pi\}$,

(ii) $\hat{\psi}_{j,k_j} \in C_0^\infty,\ \hat{\psi}_{j,k_j} \ge 0$ and

$$\hat{\psi}_{j,k_j}(\xi) \ge C > 0 \quad \text{for } \xi \in \{re^{i\theta} : 2 \times 4^{j-1} \le 2 - r \le 2 \times 4^j\ ,$$
$$(k_j - \tfrac{3}{2})2^j\pi \le \theta \le (k_j + \tfrac{1}{2})2^j\pi\}\ ,$$

(iii) $C_1 \le \sum_{\substack{j \in \mathbb{Z}_- \\ k_j \in \{1,\ldots,2^{-j+1}\}}} \hat{\psi}_{j,k_j}(\xi) \le C_2$ for $\xi \in 2D$,

therefore $\{\psi_{j,k_j}\}$ can be used to define $B_p^{s,q}(2D)$.

Now we claim that

$$\|P_{j,k_j} T_b^{s,t} P_{j,k_j}\|_{S_p} \ge 4^{j(s+t+\frac{3}{2p})} \|b * \psi_{j,k_j}\|_p\ . \tag{4.3}$$

In fact, for $p = 1$, it is true by Lemma 3 of Timotin [16]. For $p = \infty$,

$$\begin{aligned} &4^{j(s+t)}|b * \psi_{j,k_j}(x)| \\ &= 4^{-\frac{3}{2}j}|\langle \varphi_{j,k_j}\ , T_b^{s,t}\varphi_{j,k'_j}\rangle|\ . \\ &\le C\|P_{j,k_j} T_b^{s,t} P_{j,k'_j}\|_{S_\infty} \end{aligned}$$

So by interpolation, (4.3) holds.

Finally, (4.2) and (4.3) imply (4.1). □

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