RANDOM WALKS ON A DISCRETE SEMIGROUP AND CHEBYSHEV POLYNOMIALS

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1. Suppose that $F_q = Z * \cdots * Z$ (q times) is the free product of q copies of Z (the relative integers). Let $E_n = \{X \in F_q : |X| = n\}$ for all $n = 0, 1, 2, \ldots$, where $|\cdot|$ is the geodesic distance. Let moreover χ_n denote the characteristic function of E_n. It is well known (see e.g. [5]) that

$$\chi_n = (2q-1)^{n/2}\left\{P_n\left(\frac{\chi_1}{2\sqrt{2q-1}}\right) - \frac{1}{2q-1}P_{n-2}\left(\frac{\chi_1}{2\sqrt{2q-1}}\right)\right\} \tag{1}$$

where P_n is the n-th Chebyshev polynomial of the second kind. Equation (1) turns out to be important in the study of representation theory as well as in the study of random walks on F_q (see e.g. [5], [4]).

We will now produce a discrete semigroup such that χ_n is exactly equal (up to dilations) to a Chebyshev polynomial. This will reduce the study of random walks on N (the nonnegative integers), endowed with the hypergroup structure induced by Chebyshev polynomials, to the study of random walks on a discrete semigroup.

2. Let S_1 denote the discrete semigroup generated by two symbols a, b with the unique relation: $ab = e$. In other words

$$S_1 = \langle a, b \mid ab = e\rangle$$

(here e is the identity).

It is easily seen that every $x \in S_1$ can be uniquely represented as a reduced word of the form

$$x = b^r a^s \qquad r \geq 0, s \geq 0\,. \tag{2}$$

If x is as in (2), we will set

$$|x| = r + s$$

$$A(x) = s\ , \quad B(x) = r$$

so that $|x| = B(x) + A(x)$. We could also define semigroups $S_\kappa = S_1 * \cdots * S_1$ (κ times), but let us, for simplicity, study only the case of S_1.

Some remarks are in order:

a) There are idempotents: e.g. $(ba)^2 = ba$, so that the counting measure on S_1 is not translation invariant

b) for the same reason S_1 is not left nor right cancellative

c) the action of S_1 on its Cayley graph is not simple nor transitive

d) it can be proved that S_1 is amenable (while S_κ is not for $\kappa > 1$).

The convolution on S_1 (or on any discrete semigroup) is defined via the formula:

$$f * g(x) = \sum_{uv=x} f(u)g(v)\ .$$

It has the familiar properties of the convolution on groups.

Let now E_n and χ_n have the same meaning as in section 1. Then it is easy to see that

(3) $$\chi_1 * \chi_n = \chi_{n+1} - \chi_{n-1} \qquad n = 1, 2, \ldots$$

so that $\chi_n = P_n(\frac{\chi_1}{2})$. An l^1-function f on S_1 is called radial if $f = \sum_{n=0}^{\infty} C_n \chi_n$. By (3) radial functions form a Banach subalgebra of l^1 generated by χ_1 and $\chi_0 = \delta_e$.

There is another interesting subalgebra of l^1. Fix positive number u, v and set for all $n \geq 0$:

(4) $$\chi_n = \sum_{\kappa=0}^{n} u^\kappa v^{n-\kappa}\, \delta_{b^\kappa a^{n-k}}$$

(throughout this paper δ_x means the Dirac measure at x). If $u = v = 1$, then $\xi_n = \chi_n$.

We get from (4)

(5) $$\xi_1 * \xi_n = \xi_{n+1} - uv\xi_{n-1} \qquad n = 1, 2, \ldots\ .$$

An l^1-function f on S_1 is called pseudoradial if $f = \sum_{n=0}^{\infty} c_n \xi_n$. By (5) pseudoradial functions form a Banach subalgebra of l^1. We also get by (5):

$$\xi_n = (uv)^{n/2} P_n\left(\frac{\xi_1}{2\sqrt{uv}}\right). \tag{6}$$

3. Let now μ denote a probability measure on S_1. Let $X_1, X_2, \ldots, X_n, \ldots$ denote independent, identically distributed random variables with values in S_1 and common distribution μ. The process

$$Y_n = X_1 X_2 \cdots X_n \tag{7}$$

will be called (right) random walk on S_1. Left random walks can be defined as well, but it is not hard to see that the study of left random walks can be reduced to the study of right random walks. We will be interested in three processes associated with the process (7):

$$|Y_n|\,,\ A(Y_n)\,,\ B(Y_n)\,.$$

These processes "arise" from the subadditive processes

$$|X_r \cdots X_s|\,,\ A(X_r \cdots X_s)\,,\ B(X_r \cdots X_s)$$

which are stationary subadditive in the sense of Kingman [3], provided that $\mathcal{E}(|X_1|) < \infty$ (here and in the following $\mathcal{E}$ denotes the expectation).

It is possible to show that if μ is pseudoradial, then $|Y_n|$ is a Markov process.

LEMMA. *For every* $n \geq 1$

$$B(Y_n) - A(Y_n) = \sum_{j=1}^{n} \{B(X_j) - A(X_j)\}\,.$$

As a consequence of this lemma we obtain that

$$|Y_n| = \sum_{j=1}^{n} \{B(X_j) - A(X_j)\} + \text{perturbation}\,.$$

In several cases it is possible to control the perturbation. Here are two theorems which give the idea of what can be proved for $|Y_n|$.

THEOREM 1. *If $\mathcal{E}(|X_1|) < \infty$ then*

$$\frac{|Y_n|}{n} \to \left|\mathcal{E}\left(B(X_1) - A(X_1)\right)\right| \quad \textit{almost surely}\,.$$

THEOREM 2. *If $\mathcal{E}(|X_1|^2) < \infty$, $\kappa = \left|\mathcal{E}\left(B(X_1) - A(X_1)\right)\right| \neq 0$ and $\sigma = \sigma\left(B(X_1) - A(X_1)\right) \neq 0$, then*

$$\frac{|Y_n| - n\kappa}{\sigma\sqrt{n}} \to N(0,1) \quad \textit{vaguely}\,,$$

where $N(0,1)$ is the normal density with mean 0 and variance 1.

Remark. If $\kappa = 0$, theorem 2 is no longer true. The limit density can in fact be different from the normal one. This happens e.g. if μ is radial (see [1]).

4. We will outline now the relation of the processes $|Y_n|$ with the random walks on hypergroups.

Let us define a hypergroup structure on N in the following way. Fix $u, v > 0$ and set

$$z = \frac{u+v}{2\sqrt{uv}}\,, \quad Q_n(x) = \frac{P_n(\frac{x}{2\sqrt{uv}})}{P_n(z)}\,.$$

Then from the multiplication theorem for Chebyshev polynomials we get:

$$Q_m Q_n = \sum_{\kappa=|m-n|}^{m+n} \frac{P_\kappa(z)}{P_m(z)P_n(z)} Q_\kappa \tag{8}$$

when κ ranges over the integers such that $\kappa - |m-n|$ is even (compare with (6)). Making the formal substitution $\delta_j = Q_j$ ($j = m, n, \kappa$) in (8) we define a generalized convolution for Dirac measures:

$$\delta_m \times \delta_n = \sum_{\kappa=|m-n|}^{m+n} \frac{P_\kappa(z)}{P_m(z)P_n(z)} \delta_\kappa\,. \tag{9}$$

Now we extend X to $l^1(N)$ via (9). If $\mu = \sum a_\kappa \delta_\kappa$ and $\nu = \sum b_n \delta_n$, then

$$\mu \times \nu = \sum a_\kappa b_n \delta_\kappa \times \delta_n\,. \tag{10}$$

N endowed with the hypergroup structure defined by (9) and (10) will be denoted by $N_{u,v}$.

Random walks on $N_{1,1}$ were studied in great detail in [1] and in [2], where, among other things, the central limit theorem and the law of large numbers were proved. A random walk Z_n on $N_{u,v}$ is defined by assigning the initial measure $\mu_0 = \sum C_\kappa \delta_\kappa$, $\sum C_\kappa = 1$, $C_\kappa \geq 0$, and the transition probabilities

$$Pr(Z_{n+1} = \kappa \mid Z_n = h) = \delta_h \times \mu_0(\kappa)\,.$$

The connection with the processes $|Y_n|$ is given by the following theorem.

THEOREM 3. *Let $u, v > 0$. Let μ_0 as above and define a pseudoisotropic probability measure μ on S_1 by*

$$\mu = \sum_{\kappa=0}^{\infty} \frac{C_\kappa}{(uv)^{n/2} P_n(z)} \xi_n\,.$$

Then the processes $|Y_n|$ and Z_n are equivalent Markov processes on the nonnegative integers.

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