INVERSE PROBLEMS USING NODAL POSITION DATA - UNIQUENESS RESULTS, ALGORITHMS AND BOUNDS

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ABSTRACT:

In this paper we present results for the inverse problem where the data is nodal positions. In the specific results to be stated here the solutions are spacially varying parameters, i.e., coefficients in differential operators of second order. We will also discuss future research goals for this type of inverse problem.

[1] This work is supported by the Centre for Mathematical Analysis, Australian National University and by NSF Grant No. DMS-8701585.

[2] This work was supported by Centre for Mathematical Analysis, Australian National University and by ONR Grant No. N00014-84-k-516, and by AFOSR Grant No. AFOSR-86-0180.

Introduction

In the following Sections we present uniqueness results, algorithms, and numerical calculations for inverse problems using nodal position and antinodal position data. We consider three second order eigenvalue problems

$$(1) \qquad (py_x)_x + \lambda\rho\, y = 0\,, \qquad 0 < x < L$$

$$(2) \qquad y_x(0) = y_x(L) = 0$$

$$(3) \qquad u_{ss} + (\mu - q)u = 0\,, \qquad 0 < s < 1$$

$$(4) \qquad u_s(0) - h\,u(0) = 0\,, \qquad u_s(1) + H\,u(1) = 0$$

$$(5) \qquad (P\,u_r)_r + (\ell+2)(\ell-1)\,P(Rv^2-Q)u = 0\,, \qquad R_c < r < R_e$$

$$(6) \qquad u_r(R_c) = u_r(R_e) = 0$$

We will seek unknown coefficients $p > 0$, $\rho > 0$, $P > 0$ and q. The positive coefficients R and Q will be assumed known.

The inverse spectral problem which is often considered for these problems see, e.g., [1] - [9] has as data the eigenvalues plus possibly norning constants for the mode shapes. The norming constants are ratios of values of the mode shapes at the end points or are L^2 norms of the mode shapes. In this paper we make different measurements of the mode shapes. The measurements are the positions of zeros or position of extreme points (maxima or minima) of the mode shapes. We show that even for a small number of measurements we obtain quite accurate values of the unknown parameters.

In the first two problems the data will be nodal position data. Problem (1) - (2) is the eigenvalue problem obtained for longitudinal vibrations of a beam, see [10]. It is also the eigenvalue problem for torsional vibrations of a beam of circular cross section, see [11], or it is the eigenvalue problem for a string. Problem (1) - (2) has eigenvalues $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with

eigenfunctions $y_n(x)$, $n = 0,1,2 \ldots$. The nth eigenfunction, $y_n(x)$, has n zeros, $n = 0,1,2 \ldots$. The zeros of the eigenfunctions are the nodal positions; a subset of these nodal positions are the data for the inverse problem considered here.

In Sections 1 and 2 we present uniqueness theorems which show that any dense set of nodal positions on the interval [0,L] will determine the unknown coefficients uniquely. These results are an extension of the uniqueness result for a second order Dirichlet problem presented by McLaughlin [12]. In Sections 3 and 4 we present algorithms, bounds and numerical calculations. In these sections we consider nodal position data which comes from a particular experiment. In this experiment the vibrating system is excited at the n^{th} natural frequency. The n nodal positions x_j^n, $j = 1,\ldots,n$, are then measured for the corresponding mode shape. This data is the data used for the reconstruction algorithms. We show that even for $n = 10$ the results are (surprisingly) good.

The specific problems of the form (1) - (2) for which calculations are made are $p \equiv 1$ with variable ρ and $\rho \equiv 1$ with variable p. Problem (3) - (4) is primarily of interest because it is obtained from Problem (1) - (2) by Liouville transformation. All of the essential asymptotics that are required need only be obtained for Problem (3) -(4). We note that an exact solution method is presented for problem (3) - (4) when q is a piecewise constant function.

In Section 5 we consider problem (5) - (6). Here the eigenvalue problem is that for each integer $\ell \geq 2$ we seek values of ν^2 for which we have a nontrivial solution. An example of where this problem occurs is for the toroidal vibrations for a spherically symmetric earth, see [6], [8], [13]. In this paper it is assumed that and Q are known, and that the ratio /Q is

strictly increasing. We seek the coefficient P as a solution of the inverse problem. This problem is different from problems (1) - (2) and (3) - (4) in that for low order eigenvalues the function $R\,\nu^2 - Q$ has a simple zero, i.e., the differential equation has a simple turning point. It is precisely the mode shapes for which the turning point exists that are of interest to us. The data will be antinodal positions. In the case that is considered in Section 5 the antinodal position of interest is the position of the zero of the derivative, $(ru)_r$, which is closest to the turning point. We show these antinodal positions are dense in $R_c < r < R_e$ and this property yields the uniqueness result.

This is the beginning of a large study of inverse problems where the data is nodal position or antinodal position data. In the future we will determine algorithms for problem (1) - (2) for finding (simultaneously) both coefficients p and ρ. We will seek results for second order systems of ordinary differential equations and for second order partial differential equations.

Section 1

We begin with the following two eigenvalue problems:

(1) $$(py_x)_x + \lambda\rho\, y = 0\,, \quad p > 0\,, \quad \rho > 0\,, \quad 0 < x < L\,,$$

(2) $$y_x(0) = y_x(L) = 0\,,$$

and

(3) $$u_{ss} + (\lambda - q)\, u = 0\,, \quad 0 < s < 1\,,$$

(4) $$u_s(0) - h\, u(0) = 0\,, \quad u_s(1) + H\, u(1) = 0\,.$$

The real coefficients p and ρ depend on x and q depends on s; h and H are constants. For both problems there exists a sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \ldots$ with $\lim_{n\to\infty} \lambda_n = \infty$. The n^{th} eigenfunction has n zeros, n = 0, 1,2,... The position of a zero of an eigenfunction will be called a node or nodal position. In the inverse problems discussed here the data will be a subset of these nodal positions. What are sought are the spacially varying parameters p, ρ, and q. We present uniqueness theorems, algorithms, and where possible sharp bounds for the algorithms. The proofs are based on asymptotic forms for the eigenvalues and the nodal positions.

To obtain these asymptotic forms we will assume that p and ρ have absolutely continuous first derivatives, i.e., p', $\rho' \in L^1(0,L)$ while $q \in L^1(0,1)$. Under these assumptions problem (1) - (2) can be transformed, using the Liouville transformation

$$s = \frac{1}{K}\int_0^x \sqrt{\frac{\rho}{p}}\,dx \qquad \text{where} \qquad K = \int_0^L \sqrt{\frac{\rho}{p}}\,dx$$

and u = fy, where $f = (\rho p)^{\frac{1}{4}}$, to a problem of the form (3) - (4). In this transformed problem $q = f_{ss}/f$, $h = f_s(0)/f(0)$, and $H = -f_s/f$ evaluated at s = 1. Explicitly we have

(3') $$u_{ss} + (\mu - f_{ss}/f)u = 0\,, \quad 0 < s < 1$$

(4') $$[u_s - \{fs/f\}u]\Big|_{s=0} = 0\,, \quad [u_s - \{fs/f\}u]\Big|_{s=1} = 0\,,$$

Letting the eigenvalues of (1) - (2) be $\{\lambda_n\}_{n=0}^{\infty}$ and the eigenvalues of (3') - (4') be $\{\mu_n\}_{n=0}^{\infty}$, we have the relationship $\lambda_n = \mu_n/K^2$, $n = 0,1,2,\ldots$. The nodal positions of (1) - (2) and (3') - (4') are also simply related. Using the notation that x_j^n , s_j^n are the jth nodal positions of the nth mode, $j = 1,2,\ldots,n$, for (1) - (2) and (3') - (4') respectively, we have the formula

$$s_j^n = \frac{1}{K}\int_0^{x_j^n} \sqrt{\frac{\rho(x)}{p(x)}}\,dx \tag{7}$$

The fundamental asymptotic forms can then be obtained for the problem (3) - (4) and the asymptotics for (1) - (2) follow as a corollary.

We state the fundamental lemma: (the proof will be presented elsewhere)

Lemma Let $q \in L^1(0,1)$ and set $\omega(\delta) = \sup_s \int_s^{s+\delta} |q(t)|dt$. Assume $A \geq 1$ and that $|h|, |H|, \int_0^1 |q|dt \leq A/3$. Set

$$a^n = h + H + (1/2)\int_0^1 [1 + \cos(2n\pi t)]q(t)dt \tag{8}$$

$$b_j^n = h + (1/2)\int_0^{(j-\frac{1}{2})/n} [1 + \cos(2n\pi t)]q(t)dt - a^n[j - (\tfrac{1}{2})]/n \tag{9}$$

for $j = 1,2,\ldots,n$. If $n \geq 5A$ then there exists a constant θ, which depends on n, and a constant γ, which depends on n and j, such that $|\gamma| < 0.7$ and $|\theta| < 1$ and

$$\sqrt{\mu_n} = n\pi + a^n/(n\pi) + \gamma[A/n\pi]^2 \tag{10}$$

$$s_j^n = [j - (\tfrac{1}{2})]/n + b_j^n/(n\pi)^2 + \theta[(1/n^2\pi^2)\omega\left[\frac{1.1A}{(n\pi)^2}\right] + \frac{1.4A^2}{(n\pi)^3} \tag{11}$$

Section 2

Before we state the uniqueness theorems, there are several relevant remarks to make. First of all formula (10) has a simpler form when we assume that $q \in L^2(0,1)$. In this case we can show that when $\|q\|_2 \leq \frac{A}{3}$.

$$s_j^n = \left[j - \left(\frac{1}{2}\right)\right]/n + b_j^n/(n\pi)^2 + \theta[1.8\ A^2/(n\pi)^3]$$

where again $|\theta| < 1$ and θ depends on n and j. More important is that we have shown that when q is in $L^p(0,1)$ for any $p \geq 1$, then for large n the set of nodes for a single eigenfunction are roughly evenly spaced on [0,1]. This says that the set of all nodes for all of the eigenfunctions forms a dense set on [0,1]. Dense subsets of the full set of nodes can be determined by choosing, e.g., the first node in [0,1/2], the second in [1/2,1]. The third, fourth, fifth and sixth nodes are chosen in the intervals [0,1/4], [1/4,1/2], [1/2,3/4] and [3/4,1] and so on. A systematic approach in which one determines a dense subset by selecting one prescribed node from each eigenfunction has been given by McLaughlin [12]. The selection of a set of nodes which is dense on [0,1] is an important hypothesis in the uniqueness theorems.

Having established the fundamental lemma we can now state the uniqueness theorems:

Theorem 1: Let $p \equiv 1$ and $\rho'' \in L^1(0,L)$ in (1) - (2). Then ρ is uniquely determined (up to a multiplicative constant) by any dense subset of the nodes, $\{x_j^n\}$, $n \geq 1$, $j = 1,2,\ldots,n$.

Analogue 1: Let $\rho \equiv 1$, $p'' \in L^1(0,L)$ in (1) - (2). Then p is uniquely determined (up to a multiplicative constant) by a dense subset of the nodes, $\{x_j^n\}$, $n \geq 1$, $j = 1,2,\ldots,n$.

Analogue 2: Let $q \in L^1(0,1)$, and h and H are real constants in (3) - (4). Then

h, H and $q - \int_0^1 q\,dt$ are uniquely determined by a dense subset of the nodes, $\{s_j^n\}$, $n \geq 1$, $j = 1,\ldots,n$.

Theorem 2: Let p'', $\rho'' \in L'(0,1)$ in (1) - (2). Then the pair ρ, p are uniquely determined up to two constants by a dense subset of nodal positions, $\{x_j^n\}$, $n \geq 1$, $j = 1,2,\ldots,n$.

Remark: The constant in Theorem 1 and Analogue 1 is the intergral

$$K = \int_0^L \sqrt{\frac{\rho(x)}{p(x)}}\,dx$$

The two constants in Theorem 2 are the constant K above and the integral

$$c = \int_0^L \left[\frac{\left[\left(\sqrt[4]{\rho p} \right)_x \sqrt{\frac{p}{\rho}} \right]}{\sqrt[4]{\rho p}} x \right] dx$$

It is important to observe that the constant K can be given directly or can be determined by knowing the value of p and ρ at a single point or by knowing, e.g., $\int_0^L \rho(x)\,dx$ and $\int_0^L \frac{1}{p(x)}\,dx$. For a vibrating string or vibrating beam problem where ρ is density per unit length the first integral corresponds to knowing the total weight. The constant, K, can never be determined by the nodal positions. To see this we observe that we can always scale ρ by multiplying it by any positive constant ρ_0 or scale p by multiplying it by a positive constant p_0, giving the eigenvalue problem

$$\left(p_0 p y_x\right)_x + \nu\, \rho_0 \rho y = 0\,, \qquad \nu = \frac{\lambda p_0}{\rho_0}\,,$$

$$y_x(0) = y_x(L) = 0\,.$$

This scaling changes the eigenvalues but not the eigenfunctions, and hence

the nodal positions are unchanged by this scaling.

This argument suggests that an alternative method of calculating K, namely by evaluating the limit $\lim_{n\to\infty} \frac{n^2\pi^2}{\lambda_n} = K^2$ if the eigenvalues are also known.

The statement of Analogue 2 also contains an arbitrary constant, the value of the integral $\int_0^1 q\ dt$. This too can be determined from the eigenvalues if they are known by observing that the limit $\lim_{n\to\infty} \{\mu_n - n^2\pi^2\} = h + H + \frac{1}{2}\int_0^1 q\ dt$. Since the constants h and H are determined by the nodal positions the above limit yields the required integral. Finally in Theorem 2 the constants are the value of K and the constant c which is the analog of the integral $\int_0^1 q\ dt$ written in the dependent variables p and ρ and the independent variable x. The two unknown constants in Theorem 2 can then be determined, when the eigenvalues are known, from the limits $\lim_{n\to\infty} \frac{n^2\pi^2}{\lambda_n} = K^2$ and $\lim_{n\to\infty} [\lambda_n K^2 - n^2\pi^2] = c$.

These uniqueness theorems show that a dense subset of nodal positions contain enough information to determine (up to two arbitrary constants) two spacially varying parameters in a second order problem. It can be shown that there is a dense set where only one nodal position is chosen from each mode shape, see [12]. This is in contrast to the fact that the set of all the eigenvalues is not enough to determine even one coefficient, unless an additional constraint is added, e.g., that the coefficient is assumed to be symmetric about the midpoint.

We present only the proof of Theorem 1. Analogue 1 is proved similarly. The proof of Analogue 2 is similar to that in [12]. The proof of Theorem 2 combines the ideas of both proofs and will be presented elsewhere. Note that the proof presented below does not require the full detail of the bounds in the Lemma. We need only that $s_j^n = [j-(1/2)]/n + o(1)$ as $n\to\infty$.

Proof of Theorem 1: Suppose that $\rho \equiv 1$ and that $x_j^n(\rho_i)$ are the nodal positions for (1) - (2) when ρ is replaced by ρ_i, $i = 1,2$. Suppose that for a dense subset of nodal positions $x_j^n(\rho_1) = x_j^n(\rho_2)$. Select a fixed but arbitrary $x \in [0,L]$ and let $\{x^m\}$ be a subsequence of the dense set of nodal positions such that $x^m \to x$. Then from the asymptotic forms for the corresponding s^m we have

$$\frac{1}{K_1}\int_0^{x^m}\sqrt{\rho_1}\,dx - \frac{1}{K_2}\int_0^{x^m}\sqrt{\rho_2}\,dx = o(1)$$

as $m\to\infty$. Taking the limit as $m\to\infty$ we have

$$0 = \lim_{m\to\infty}\left[\frac{1}{K_1}\int_0^{x^m}\sqrt{\rho_1}\,dx - \frac{1}{K_2}\int_0^{x^m}\sqrt{\rho_2}\,dx\right] = \int_0^x\left[\frac{\sqrt{\rho_1}}{K_1} - \frac{\sqrt{\rho_2}}{K_2}\right]dx$$

Since x was arbitrary and ρ_1 and ρ_2 are continuous, we have

$$\frac{\sqrt{\rho_1}}{K_1} - \frac{\sqrt{\rho_2}}{K_2} \equiv 0$$

and the ratio $\dfrac{\rho_1}{\rho_2}$ = constant.

Section 3

In this section the algorithms will be given for three inverse nodal problems. It will be assumed that only a finite number of measurements can be made. The previous uniqueness theorems suggest that data which is uniformly distributed on the interval [0,L] for problem (1) - (2) or on the interval [0,1] for problem (3) - (4) will give the best approximation for the true unknown parameter. The particular choice that is made here is motivated by a particular physical experiment. In this experiment the system is excited at a natural frequency. Measurements are then made of all of the nodal positions of the mode shape corresponding to that frequency; usually measurements are made of the eigenfrequency as well. The data is then the length of the interval between nodal positions plus the eigenfrequency itself. The algorithms will give the unknown coefficient at a point between consecutive nodes as a function of the distance between the nodal position and the eigenfrequency.

Heuristic arguments can be given for the algorithm even in the absence of smoothness of the coefficients. This is done for the algorithm for problem (3) - (4). When the coefficient, q, in (3) - (4) is in, say $H^2(0,1)$, bounds can be determined for the error. This requires more detailed asymptotics than are given in the Lemma and will be presented elsewhere. The asymptotics given in the Lemma are sufficient to both motivate our algorithm and to establish good bounds for problem (1) - (2). These bounds will be given here although proofs will not be given.

We begin with two special cases of problem (1) - (2). These are the two special cases given in Theorem 1 and Analogue 1, i.e., i) find $\rho(x)$ when $p \equiv 1$ and, (ii) find $p(x)$ when $\rho \equiv 1$. We motivate the algorithms as follows. First, since n is fixed we will simplify the notation and let $x_j^n = x_j$ and $s_j^n = s_j$,

$j = 1,2,\ldots n$. In problem (1) - (2) formulas (7) and (11) can be combined so that when $p'' \in L^1(0,L)$ and $\rho'' \in L^1(0,L)$ we have

$$\frac{1}{K}\int_{x_j}^{x_{j+1}} \sqrt{\frac{\rho(x)}{p(x)}}\, dx = \frac{1}{n} + o\left(\frac{1}{n^2}\right), \qquad j = 1,\ldots,n-1$$

and formula (10) yields

$$\sqrt{\lambda_n} = n\pi/K + O(1/n),$$

or

$$n/K = \sqrt{\lambda_n}/\pi + O(1/n).$$

We rewrite the above formula by dividing by the length of the interval between the nodes to obtain

$$\frac{1}{(x_{j+1} - x_j)}\int_{x_j}^{x_{j+1}} \sqrt{\frac{\rho(x)}{p(x)}}\, dx = \frac{\pi}{\sqrt{\lambda_n}(x_{j+1} - x_j)} + \text{error} \qquad j = 1,2,\ldots,n-1.$$

Approximating the term on the left by the value of the integrand at the midpoint $\bar{x}_j = \frac{x_{j+1} + x_j}{2}$, we have

$$\sqrt{\frac{\rho(\bar{x}_j)}{p}} = \frac{\pi}{\sqrt{\lambda_n}(x_{j+1} - x_j)} + \text{error}.$$

Hence if $\rho \equiv 1$ then the algorithm for p will be

$$\sqrt{p(\bar{x}_j)} = \frac{\sqrt{\lambda_n}(x_{j+1} - x_j)}{\pi}, \qquad j = 1,\ldots,n-1 \tag{12}$$

and if $p = 1$ the algorithm for ρ is

$$\sqrt{\rho(\bar{x}_j)} = \frac{\pi}{\sqrt{\lambda_n}(x_{j+1} - x_j)}, \qquad j = 1,\ldots,n-1 \tag{13}$$

These algorithms can be shown to give very accurate results at the

midpoint $\bar{x}_j$, $j = 1,2,\ldots,n-1$. We give, below, bounds which display this accuracy, and in the next section we display specific numerical output. The bounds are given in terms of the modulus of continuity for p" and for ρ". We present the following theorems.

Theorem 3: Suppose that $p \equiv 1$. Let ρ be twice differentiable and assume that $L^{-2}\, m \leq \rho \leq L^{-2}M$ for $0 \leq x \leq L$ and that $|\rho'|, \int_0^L |\rho''| \leq L^{-3}M$. Set

$$E = \frac{\sqrt{M}}{L^3}\left[\frac{M}{m}\right]^{11/2}\left[0.3\,\frac{L^4}{M}\,\frac{L}{n}\,\tilde{\omega}\left[\frac{L}{n}\right] + (0.9)\left[\frac{L}{n}\right]^2\right]$$

where $\tilde{\omega}(\delta) = \sup_x \int_x^{x+\delta} |\rho''|$ is the modulus of continuity for $\int_0^x |\rho''|$. Let $x_1,\ldots,x_n$ be the nodes for the n^{th} eigenfunction with eigenvalue λ_n. If $n \geq 9\left(\frac{M}{m}\right)^{5/2}$ then

$$\left|\sqrt{\rho\left(\tfrac{1}{2}(x_{j+1} + x_j)\right)} - \frac{\pi}{\sqrt{\lambda_n}(x_{j+1} - x_j)}\right| \leq E$$

for $j = 1,2,\ldots,n$.

Analogue 3: Suppose that $\rho \equiv 1$. Let p be twice differentiable. Assume that $L^2 m \leq p \leq L^2 M$, for $0 \leq x \leq L$ and $|p'|, \int_0^L |p''| \leq L\,M$. Set

$$E = \frac{\sqrt{M}}{L}\left[\frac{M}{m}\right]^4\left[0.3\,\frac{1}{M}\,\frac{L}{n}\,\tilde{\omega}\left[\frac{L}{n}\right] + 0.4\left[\frac{L}{n}\right]^2\right]$$

where $\tilde{\omega}(\delta) = \sup_x \int_x^{x+\delta} |p''|dx$ is the modulus of continuity for $\int_0^x |p''|dx$. Let $x_1,\ldots,x_n$ be the nodes for the n^{th} eigenfunction with eigenvalue λ_n. If $n \geq 5\left(\frac{M}{m}\right)^2$

$$\left| \sqrt{p\left(\tfrac{1}{2}(x_{j+1}+x_j)\right)} - \frac{\sqrt{\lambda_n}(x_{j+1}-x_j)}{\pi} \right| \leq E$$

for $j = 1,\ldots,n-1$.

In the next section we exhibit the numerical results. We note that we have given algorithms for determining the parameters p and ρ at the midpoints between two nodes. We have not presented an algorithm for finding p or ρ in the nodal domains $[0,x_1]$ and $[x_n,L]$. A first guess for an algorithm might be, e.g., that $\sqrt{\rho}\ (\frac{1}{2}\ x_1) = 2\pi/(\sqrt{\lambda_n}\ x_1)$ with a similar algorithm for $\sqrt{p}\ (\frac{1}{2}(L+x_n))$. We have shown that this is not the best choice. We will show in a future paper an algorithm which gives an approximation to $\sqrt{\rho}$ (or $\sqrt{p}$) at a point other than the midpoint of $[0,x_1]$ and $[x_n,L]$. The bounds will be consistent with those given in this paper.

We now present an algorithm for finding q in problem (3) - (4) at the points $s = \frac{j}{n}$, $j = 1,2,\ldots,n-1$. The algorithm is the one suggested by the Lemma. We give only a heuristic argument since the asymptotics of the Lemma are not detailed enough to present bounds. As in the previous algorithms the data will be the nodal positions s_j^n, $j = 1,\ldots,n$ for only one eigenfunction, that is, the n^{th} eigenfunction. For simplification, since n is fixed we will write s_j for s_j^n.

We observe that

$$s_{j+1} - s_j - \frac{1}{n} = \frac{1}{2(n\pi)^2} \int_{\frac{(j-\frac{1}{2})}{n}}^{\frac{(j+\frac{1}{2})}{n}} q(t)dt + \text{error}.$$

Multiplying by $2n^3\pi^2$ we obtain

$$2n^3\pi^2\left[s_{j+1} - s_j - \frac{1}{n}\right] = n\int_{\frac{(j-\frac{1}{2})}{n}}^{\frac{(j+\frac{1}{2})}{n}} q(t)dt + \text{error}.$$

If we replace the integral on the right hand side by the value of the integrand at the midpoint we obtain the algorithm

$$2n^3\pi^2\left[s_{j+1} - s_j - \frac{1}{n}\right] = q\left(\frac{j}{n}\right), \quad j = 1,\ldots,n. \tag{14}$$

It is not at all clear from the asymptotics given in the Lemma that this algorithm will give a good approximation to q. We have shown by numerical calculations that this does give very good results when q" is integrable. Calculations will not be given here.

We also present an exact solution in the case where q is not smooth but can be represented by a piecewise constant function. To be specific, in the case where h = H = 0, an exact function q which yields nodal positions $0 < s_1 < s_2 < \ldots < s_n < 1$ can be given by the piecewise constant function

$$q(s) = \begin{cases} \lambda_n - \dfrac{\pi^2 4}{(s_1)^2} & 0 < s < s_1, \\ \lambda_n - \dfrac{\pi^2}{(s_{j+1} - s_j)^2} & s_j < s < s_{j+1},\ j = 1, \ldots, n-1 \\ \lambda_n - \dfrac{4\pi^2}{[1 - (s_n)]^2} & s_n < s < 1 \end{cases}$$

Section 4:

In this section we display examples of the numerical output for the reconstruction of $p(x)$ in

$$(15) \qquad (pu_x)_x + \lambda u = 0 , \quad 0 < x < L ,$$
$$u_x(0) = u_x(L) = 0 ,$$

and the reconstruction of $\rho(x)$ in

$$(16) \qquad u_{xx} + \lambda \rho u = 0 , \quad 0 < x < L ,$$
$$u_x(0) = u_x(L) = 0 .$$

We will in each of our examples, choose $L = 1$. The data is a single frequency and all the nodal positions of the mode shape for that frequency. Our aim is to display how well $p(x)$ and $\rho(x)$ are calculated by the formulas (12) and (13) given in the previous section. The formulas are for $p(x)$ and $\rho(x)$ at the mid-point between two nodes. Our choice of problems to consider here is fortunate. We will show that the nodal positions change a great deal when p or ρ changes so that the information we seek is, in some sense, accessible. In a later paper we will examine similar output for the potential q in (3)-(4) and see that the changes in the nodal positions are much smaller as q changes. While in this latter case we get very good recovery of q with accurate synthetic numerical data, it would be more difficult to recover q accurately from, say, experimental data.

Our calculations of both the eigenvalues and the nodal positions for both problems (15) and (16) were done using the modified Prufer transformation, see Paine [14], p. 16. In this calculation we use the fact that p' and ρ' are continuous and make the following change of variables. Let $(\lambda p \rho)^{\frac{1}{2}} u = r \sin\theta$ and $pu' = r \cos\theta$. Then

$$\frac{d\theta'}{dt} = \sqrt{\frac{\lambda\rho}{p}} + \frac{1}{4} \frac{p'}{p} + \frac{\rho'}{\rho} \sin 2\theta$$

with initial condition $\theta(0) = \pi/2$. We determine the eigenvalues by satisfying the boundary condition at the right end point. We must have $\theta(1) = \pi/2 + n\pi$ for the nth eigenfunction. The j^{th} node occurs when $\theta(x_j) = j\pi$. We will illustrate how the nodal positions are changed by a change in the coefficient. We recall that if $p \equiv 1$, $\rho \equiv 1$ then the nodal positions for the nth mode shape are at $(j - \frac{1}{2})\frac{1}{n}$, $j = 1,\ldots,n$. We show first the change in nodal position when either p or ρ is replaced by the same function. That is we choose p or ρ (but not both) to be the function $1 + (1.5x)^2(1-x)$ and present the numerical output for the nodes for $n = 10$ and for $\omega_{10} = \sqrt{\lambda_{10}}$.

	$p \equiv 1$, $\rho = 1 + (1.5x)^2(1-x)$	$\rho \equiv 1$, $p = 1 + (1.5x)^2(1-x)$
ω_{10}	57.731119	34.121966
x_1	.054382	.046091
x_2	.161905	.139045
x_3	.266536	.234046
x_4	.367797	.331887
x_5	.465991	.432845
x_6	.561848	.536632
x_7	.656348	.642345
x_8	.750717	.748434
x_9	.846591	.852707
x_{10}	.946502	.952392

Table 1: Two problems with 10 nodes.

In the next example the coefficients p and ρ are chosen so that both

problems (15) and (16) have the same eigenvalues. Our choices are $p \equiv 1$, $\rho = (e-1)/[1+(e-1)x]$ and $\rho \equiv 1$, $p = (e-1)e^{-x}$. The nodal positions are displayed for $n = 10$. In each case we have $\sqrt{\lambda_{10}} = \omega_{10} = 31.739122$. We point out that even though all the eigenvalues are the same for both problems, the nodal positions are significantly different.

	$p(x) = (e-1)e^{-x}$	$\rho(x) = (e-1)/[1 + (e-1)x]$
ω_{10}	31.739122	31.739122
x_1	.063441	.038117
x_2	.185363	.118523
x_3	.300276	.203827
x_4	.408942	.294029
x_5	.512007	.389131
x_6	.610018	.489131
x_7	.703449	.594029
x_8	.792709	.703827
x_9	.878154	.818523
x_{10}	.960097	.938117

Table 2: Two (isospectral) problems with 10 nodes.

To exhibit the results of the algorithm we begin with the function

$$f(x) = \begin{cases} 1 & .00 < x < .45 \\ 1 + \frac{1}{2}\sin\left(10\pi\left(x - \frac{1}{2}\right)\right) & .45 < x < .55 \\ 2 & .55 < x < 1.00 \end{cases}$$

which has a continuous first derivative but discontinuous second derivative.

We do our calculations with the function

$$\rho(x) = f(x) + (1-x^2)$$

The bounds given in Theorems 3 and Analogue 3 show that the error is of order $\frac{1}{n^2}$ when p" or $\rho'' \in L^\infty$. To estimate our error we will use the discrete L^1 and the discrete L^2 norm as

$$\|e\|_q = \left[\sum_{j=1}^{n-1} (x_{j+1} - x_j)|e(\bar{x}_j)|^q\right]^{\frac{1}{q}}$$

for q = 1,2. For $p \equiv 1$, $e(\bar{x}_j) = \rho(\bar{x}_j) - \frac{\pi^2}{\lambda_n(x_{j+1} - x_j)^2}$, and for $\rho \equiv 1$ we have $e(\bar{x}_j) = p(\bar{x}_j) - \frac{\lambda_n(x_{j+1} - x_j)^2}{\pi^2}$, j = 1,2,... For the discrete L^∞ norm use

$$\|e\|_\infty = \max_{1 \leq j \leq n-1} |e(\bar{x}_j)|$$

Tables 3 and 4 give the $\omega_n = \sqrt{\lambda_n}$, the midpoints between the nodal positions, and the calculated and exact solutions for n = 5 and n = 10 for the function $\rho(x) = f(x) + (1-x^2)$. Note again that for $p \equiv 1$ and $\rho \equiv 1$ the midpoints have the value $\bar{x}_j = \frac{j}{n}$, j = 1,...,n-1.

$\rho(x) = f(x) + (1-x^2)$				
$\omega_5 = 12.301853$			exact	calculated
$\bar{x}_1$	.210184	$\rho(\bar{x}_1)$	1.4558	1.4544
$\bar{x}_2$	.425244	$\rho(\bar{x}_2)$	1.3191	1.3677
$\bar{x}_3$	.622289	$\rho(\bar{x}_3)$	2.1127	2.1120
$\bar{x}_4$	.948877	$\rho(\bar{x}_4)$	1.8536	1.8530

Table 3: Calculation for n = 5.

$\rho(x) = f(x) + (1-x^2)$				
$\omega_{10} = 24.403179$			exact	calculated
$\bar{x}_1$	.105321	$\rho(\bar{x}_1)$	1.4889	1.4885
$\bar{x}_2$	.211444	$\rho(\bar{x}_2)$	1.4553	1.4549
$\bar{x}_3$	.319254	$\rho(\bar{x}_3)$	1.3981	1.3977
$\bar{x}_4$	.429707	$\rho(\bar{x}_4)$	1.3154	1.3209
$\bar{x}_5$	.530336	$\rho(\bar{x}_5)$	2.1263	2.0809
$\bar{x}_6$	.619206	$\rho(\bar{x}_6)$	2.1166	2.1164
$\bar{x}_7$	.709000	$\rho(\bar{x}_7)$	1.9973	1.9971
$\bar{x}_8$	.801783	$\rho(\bar{x}_8)$	1.9973	1.8570
$\bar{x}_9$	.898495	$\rho(\bar{x}_9)$	1.8571	1.6926

Table 4: Calculation for n = 10.

We expect the largest errors to occur near the midpoint of the interval since the function f(x) increases rapidly there. We observe, however, that even for n = 10 the calculations are (surprisingly) good even near $x = \frac{1}{2}$.

We now display the L^1, L^2 and L^∞ errors. The quadratic convergence is observable only for a large number of nodes. Table 5 presents the results where again we choose $\rho(x) = f(x) + (1-x^2)$.

	n = 5	n = 10	n = 20	n = 40
L^1 error	.0112	.0049	.0013	.0004
L^2 error	.0227	.0136	.0040	.0013
L^∞ error	.0485	.0456	0135	.0061

Table 5: Error in reconstruction of $\rho(x) = f(x) + (1-x^2)$.

For completeness, we now display the values of the midpoints and the calculated and exact solutions for the first two examples of this section for n = 10. For each example we also give $\omega_n = \sqrt{\lambda_n}$.

In the Table 6 we let $\rho = 1 + (1.5x)^2(1-x)$ in problem (16) and display $\omega_{10} = \sqrt{\lambda_{10}}$, the midpoints, $\bar{x}_j$, and the exact and calculated (using formula (13)) values for $\rho(\bar{x}_j)$, $j = 1,\ldots, 9$.

$\rho(x) = 1 + (1.5x)^2(1-x)$				
$\omega_{10} = 57.731119$			exact	calculated
$\bar{x}_1$	.1081	$\rho(\bar{x}_1)$	1.0235	1.0241
$\bar{x}_2$	.2142	$\rho(\bar{x}_2)$	1.0811	1.0815
$\bar{x}_3$	.3172	$\rho(\bar{x}_3)$	1.1546	1.1546
$\bar{x}_4$	.4169	$\rho(\bar{x}_4)$	1.2280	1.2279
$\bar{x}_5$	.5139	$\rho(\bar{x}_5)$	1.2889	1.2885
$\bar{x}_6$	.6091	$\rho(\bar{x}_6)$	1.3263	1.3258
$\bar{x}_7$	.7035	$\rho(\bar{x}_7)$	1.3302	1.3294
$\bar{x}_8$	.7987	$\rho(\bar{x}_8)$	1.2890	1.2881
$\bar{x}_9$	.8965	$\rho(\bar{x}_9)$	1.1871	1.1860

Table 6: Calculation of ρ when n = 10.

In the Table 7 we let $p(x) = 1 + (1.5x)^2(1-x)$ in problem (15) and display $\omega_{10} = \sqrt{\lambda_{10}}$, the midpoints, $\bar{x}_j$, and the exact and calculated (using formula (12).) values for $\rho(\bar{x}_j)$, $j = 1,\ldots, 9$.

$p(x) = 1 + (1.5x)^2(1-x)$				
$\omega_{10} = 34.121966$			exact	calculated
$\bar{x}_1$	.0926	$p(\bar{x}_1)$	1.0175	1.0193
$\bar{x}_2$	.1865	$p(\bar{x}_2)$	1.0637	1.0647
$\bar{x}_3$	.2830	$p(\bar{x}_3)$	1.1292	1.1293
$\bar{x}_4$	.3824	$p(\bar{x}_4)$	1.2032	1.2024
$\bar{x}_5$	.4847	$p(\bar{x}_5)$	1.2724	1.2707
$\bar{x}_6$	.5895	$p(\bar{x}_6)$	1.3210	1.3183
$\bar{x}_7$	.6954	$p(\bar{x}_7)$	1.3314	1.3277
$\bar{x}_8$	.8006	$p(\bar{x}_8)$	1.2876	1.2827
$\bar{x}_9$	.9026	$p(\bar{x}_9)$	1.1786	1.1723

Table 7: Calculation of p when n = 10.

It should be noted again that we have not given a calculation or a formula for $\rho(x)$ or $p(x)$ at the midpoints in the first interval $[0,x_1]$ or in the very last interval $[x_n, 1]$. The above analysis suggest formulas of the form, say, $\rho\left(\frac{x_1}{2}\right) = \frac{4\pi^2}{\lambda_n (x_1)^2}$. We have shown this is not the best choice. In a future paper we will present a formula, plus bounds, for ρ or p at a point other than the midpoint.

Section 5

In this section we present a uniqueness result where the data is antinodal positions. The mathematical model considered is the eigenvalue problem.

$$(5) \qquad (P\,u_r)_r + (\ell+2)(\ell-1)P\,(\mathcal{R}\nu^2 - Q)u = 0, \qquad R_c \leq r \leq R_e$$

$$(6) \qquad u_r(R_c) = 0 = u_r(R_e)$$

where $P = P(r)$, $\mathcal{R} = \mathcal{R}(r)$, and $Q = Q(r)$ are all positive smooth functions, and the ratio $\mathcal{R}/Q$ is strictly increasing. For each integer $\ell \geq 2$, the eigenvalue problem is to seek values of $(\nu_{\ell,n})^2$, $n = 1,2,\ldots$ for which (5) - (6) has a nontrivial solution. The new feature for this set of eigenvalue problems is that for each ℓ the equation (5) has a simple turning point, $\tilde{r}_{\ell,n}$, for eigenvalues of 'low enough' order. The inverse problem is to determine the function P when $\mathcal{R}$ and Q are known.

The data considered here for the inverse problem is antinodal data for the eigenfunction in the presence of a turning point. To explain this more carefully we recall that when equation (5) has a turning point, $\tilde{r}_{\ell,n}$, the eigenfunction $u_{n,\ell}$ is strictly increasing for $R_c < r \leq \tilde{r}_{\ell,n}$, and oscillatory for $\tilde{r}_{\ell,n}, < r < R_e$. There is a first point, $\bar{r}_{\ell,n}$, to the right of $\tilde{r}_{\ell,n}$ where $\frac{d}{dr}(ru_{n,\ell}) = 0$. For the inverse problem considered here it is a subset of the antinodal positions, $\{\bar{r}_{\ell,n}\}_{\substack{0\leq n<\infty \\ 2\leq \ell<\infty}}$, which are to be the data.

A specific geophysical example where the eigenvalue problem (5) - (6) occurs is for the toroidal modes for a spherically symmetric earth, see [6], [8], [13]. In this case we begin with the eigenvalue problem

$$(\mu r^2 y_r)_r + \mu[\omega^2 r^2 \tfrac{\rho}{\mu} - \ell(\ell+1) - \tfrac{\mu_r r}{\mu}]\, y = 0$$

$$(y_r - \tfrac{y}{r})(R_c) = (y_r - \tfrac{y}{r})\,(R_e) = 0$$

where $\mu = \mu(r)$ is the rigidity of the earth, $\rho = \rho(r)$ is the density and $\sqrt{\frac{\mu}{\rho}} = \beta(r)$ is the shear wave velocity. The radius R_c is the radius of the core and R_e is the radius of the earth. When we make the change of dependent variables $u = \frac{y}{r}$ we obtain the boundary value problem

$$(17) \qquad (r^4 \mu u_r)_r + r^4 \mu \left[\omega^2 \frac{\rho}{\mu} - \frac{(\ell-1)(\ell+2)}{r^2}\right] u = 0 ,$$

$$(18) \qquad u_r(R_c) = u_r(R_e) = 0$$

The boundary value problem (17),(18) is now of the form (5),(6) when $\ell > 1$ and $P = r^4\mu$, $R = \frac{\rho}{\mu} = \frac{1}{\beta^2}$, $Q = \frac{1}{r^2}$, and $\nu^2 = \frac{\omega^2}{(\ell-1)(\ell+2)}$. We observe that $(ru)_r = 0$ when $y_r = 0$. It should also be noted that in general at the antinodal position the eigenfunction $y_{n,\ell}(\bar{r}_{\ell,n})$ has its maximum displacement.

The uniqueness theorem as well as a preliminary lemma will be presented without proof. It should be noted in advance that detailed asymptotics for the eigenvalues $(\nu_{\ell,n})^2$ is not required for the uniqueness proof. We state a preliminary lemma which gives the needed denseness result.

Lemma: Consider the eigenvalue problem (5), (6). Suppose that P, R, Q are all positive funtions on $R_c \leq r \leq R_e$ with integrable second derivatives. Suppose that the ratio R/Q is strictly increasing on $R_c \leq r \leq R_e$. Then the set of turning points $\{\tilde{r}_{\ell,n}\}$ is dense on $R_c \leq r \leq R_e$. The set of antinodal positions $\{\bar{r}_{\ell,n}\}$ is also dense on $R_c \leq r \leq R_e$.

The uniqueness theorem now follows.

Theorem 4: Consider the eigenvalue problem (5), (6). Suppose that P and R and Q are all positive functions on $R_c \leq r \leq R_e$ with integrable second derivatives. Suppose that the ratio R/Q is strictly increasing on $R_c \leq r \leq R_e$ and that R and

Q are known functions. Suppose that a dense subset of the antinodes $\{\bar{r}_{\ell,n}\}$ are given as well as the corresponding eigenvalues $\{\nu_{\ell,n}{}^2\}$. Then P is uniquely determined up to one arbitrary constant.

Remark: If P_1 and P_2 are two coefficients for two eigenvalue problems of the form (5), (6), then the hypothesis of the theorem is that for a dense subset of the antinodes $r_{\ell,n}(P_1) = r_{\ell,n}(P_2)$ and that the corresponding eigenvalues satisfy $[\nu_{\ell,n}(P_1)]^2 = [\nu_{\ell,n}(P_2)]^2$. Then the proof shows that P_1 and P_2 satisfy the same second order ordinary differential equation. The arbitrary constant corresponds to assigning the ratios $P_{1,r}/P_1$ and $P_{2,r}/P_2$ at $r = R_e$.

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ACKNOWLEDGEMENT

The authors greatly benefited by conversations with R.S. Anderssen and Brian Kennett during the 1986 Special Program on Inverse Problems at the Centre for Mathematical Analysis, Australian National University.

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