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Integration Structures

IGOR KLUVÁNEK

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PREFACE

The term "structure" in the title is used in the Bourbakist sense. Chapter 0 is devoted to the exposition of a certain notorious failure, or inadequacy, of currently used integration structures, including those presented in Book 5 of the Bourbaki treatise and in the well-known text of P.R. Halmos. In Section G of that chapter, the nature of the integration structures presented here is briefly described. This description is amplified somewhat in the pre-ambles to Chapters 2, 3 and 4.

To indicate the issue involved, I would first repeat what was already said by people learned in these matters and was even recorded, for example by my distinguished friends in their book which is listed among the references as item [9]. Namely, the problem of integration with respect to a vector valued measure which has finite (and σ-additive) variation is trivial. For, we do not in fact integrate with respect to such a vector measure; we integrate with respect to its variation which is a true (positive) measure. The integral with respect to the given vector measure is then a uniquely determined (vector valued) continuous linear functional in the space of functions integrable with respect to the variation.

In contrast, the problem of integration with respect to a vector valued measure having infinite variation seems to be nontrivial even when the range-space is one-dimensional. For, such a vector measure does not generate a continuous linear functional in the space of integrable functions with respect to any positive measure. I should note here, perhaps, that, since the appearance of the work of R.G. Bartle, N. Dunford and J.T. Schwartz, listed among the references as item [2], integration with respect to certain vector measures of possibly infinite variation can be reduced, using duality, to integration with respect to families of scalar valued measures of finite variation. However, this device is surely not available for all measures of infinite variation; for example, it is not available for measures with values in a finite-dimensional space.

From a sufficiently abstract point of view, the integration structures presented here can be seen as instances of a single general structure. That structure is intended to make trivial also the problem of integration 'with respect to measures of infinite variation'. It represents a construction of a complete normed function space - which of course cannot be an L^1-space in general - such that a given vector measure generates a continuous linear functional in it. Indeed, if we do not succeed in making this problem trivial, then, in my view, we do not have a chance to tackle successfully those problems whose solutions for measures of finite variation are so brilliantly exposed by my distinguished friends in the mentioned book.

These remarks indicate, I hope, that I opted for an approach to this problem which is different from the approaches found in the literature. That explains, to some extent, the list of references or, rather, the obvious omissions from it. So, for example, the works of R.H. Cameron and his collaborators are not mentioned although a considerable proportion of my motivation derives from the problems arising in connection with the Feynman integral. Or, the names of R. Henstock and J. Kurzweil do not appear here even though my theme concerns non-absolutely convergent integrals. Similarly, in Chapter 5, I introduce bilinear integrals, but the works of R.G. Bartle and of I. Dobrakov are not referred to. This presents for me a certain difficulty, even embarrassment. It is true that I have not discovered nontrivial relationship, at the technical level, between the results presented here and those results reported in the literature that concern similar themes but were obtained from different perspectives. On the other hand, I am also aware of the fact that I reached the point of view presented here only because I was influenced - possibly and admittedly only indirectly - by the works of the mentioned and of many other authors.

I have still greater difficulties with giving due credit and expressing my gratitude to friends and colleagues who assisted me by their thoughts not available publicly. It is simply impossible for me to trace all such influences, not to speak of their explicit articulation. What is more, in some cases in which I would be able to do so, I do not know the names of the persons who assisted me in this manner. They are, for example, the referees of my journal articles, even, or especially, those which (happily) were not accepted for publication.

However, I am too conscious of the generous help rendered to me by Brian Jefferies, Susumu Okada and Werner Ricker not to mention them by name. I wish I were able to express better my gratitude for their criticism of my numerous attempts at the realization of this project and for helping me to maintain the confidence in its viability.

I am delighted that I am able to put on record my gratitude to Neil Trudinger who created the possibility for me to work on this project at the Centre for Mathematical Analysis and to make the results of my effort available to the public in this form.

In my endeavour to facilitate the reading of the text I was greatly assisted by Dorothy Nash. I would like to thank her for the expert advice about lay-out, for the understanding, even anticipation, of my intentions, for the initiative with which she explored the possibilities of the available equipment for their realization and, generally, for her pleasant cooperation.

Canberra, November 1988. I.K.

LIST OF SYMBOLS

$A(W)$	6A, 162
$A(P)$	6A, 162, 146
$\mathrm{BL}(E)$	6A, 161
$\mathrm{BV}^{\Phi}(\mathcal{Q},E)$	4J, 139
$\mathrm{BV}^{\Phi}(\Delta,E)$	4J, 139
$\mathrm{BV}^{\infty}(\mathcal{Q},E)$	4G, 134
$B^{*}(E,F)$	1C, 34
$\mathcal{B}(\Lambda)$	7A, 181
e_s	7B, 186
$\mathcal{J}_n$	6G, 176
$\mathcal{K}_n(\Gamma)$	6G, 176
$L^p(\iota)$	3C, 85
$L(P)$	6C,167
$L(\rho,\mathcal{K})$	2A, 52
$L(\rho,\mathcal{K},E)$	5A, 146
$L^{\Phi}(\iota)$	3C, 91
$\mathcal{L}^{\infty}(P)$	6B, 166
$\mathcal{L}(P)$	6C, 167
$\mathcal{L}(\iota)$	3B, 81
$\mathcal{L}(\rho,\mathcal{K})$	2A, 51
$\mathcal{L}(\rho,\mathcal{K},E)$	5A, 145
$\mathcal{L}^{\Phi}(\iota)$	3C, 91
$\mathcal{L}^{\infty}(P)$	6B, 166
M_t	7A, 183
$M_t\varphi$	7A, 184
$M_\iota(f,\mathcal{P})$	4D, 121
$M_\iota(f,X)$	4D, 121
M_{Φ}	1G, 49; 3C, 89
$\mathcal{M}^p(\Gamma)$	6F, 174
$P(W)$	7A, 181
P^p_{Γ}	6F, 175
q_ρ	2A, 52; 5A, 146
$q_{\rho,\mathcal{K}}$	2A, 52
$\mathcal{Q}_{\Gamma}$	1D, 40
$\mathcal{R}(\iota)$	3B, 82
$\mathcal{R}^p(\Gamma)$	6F, 175
$\mathrm{sim}(Q)$	1B, 31
$\mathrm{sim}(\mathcal{K})$	1D, 37
$\mathrm{sim}(\mathcal{K},E)$	1D, 37
$\mathrm{var}(f)$	6G, 176
$v_p(\mu,\Delta)$	4A, 108
$v(\mu)$	3F, 99
$v(\mu,f)$	3F, 99
$v(\mu,X)$	3F, 99
$v(\mu,\Delta)$	4A, 108
$v_{\Phi}(\mu)$	4A,108
$v_{\Phi}(\mu)$	4A, 108
$v_{\Phi}(\mu,\mathcal{P};X)$	4A, 107
$v_{\Phi}(\mu,\Delta)$	4A, 108
$v_{\Phi}(\mu,\Delta;X)$	4A, 107
$v_{\infty}(\mu;X)$	4E, 124
$V_{\Phi}(\xi,\Delta)$	4J, 140
$V^0_{\Phi}(\xi,\Delta)$	4J, 141
$V_{\infty}(\xi)$	4G, 134
$Y(t_1,\ldots,t_k;B_1,\ldots,B_k)$	7A, 183
$\mathcal{Z}_\rho$	2B, 54

(Δ_2)	1G, 48	$\|f\|^0_{\Phi,\iota}$	3C, 91
$\mu_{\rho,b}$	5B, 149	$\|f\|_\infty$	6B, 165
$\mu_\rho(f)$	3A, 78	$\|\mathcal{P}\|_\iota$	4B, 112
$\Pi(\mathcal{Q})$	1D, 40	$\|x\|_\Phi$	1G, 49
ρ_P	6B, 164	$\|x\|^0_\Phi$	1G, 49
$\Sigma(\mathcal{Q})$	1D, 39	$E\otimes F$	1C, 35
F'	1B, 32	$g\otimes h$	5C, 150
$[f],[f]_P$	6B, 166	$x\otimes y$	1C, 35
$\|\mu\|$	3F, 99	$\sigma\otimes\tau$	5C, 150
$\|f\|_{p,\iota}$	3C, 85	$E\hat{\otimes}F$	1C, 36
$[f]_\rho$	2A, 52; 5A, 146	$\mathcal{Q}\cap\Upsilon$	1D, 39
$\|f\|_{\Phi,\iota}$	3C, 91	$\mathcal{P}_1\prec\mathcal{P}_2$	1D, 40

$\int_\Omega f\mathrm{d}_\rho\mu,\ \int_\Omega f(\omega)\mu(\mathrm{d}_\rho\omega)$ 3A, 78; 5C, 150

$\int_\Omega b(f,\mathrm{d}_\rho\mu)\ ,\ \int_\Omega b(f(\omega),\mu(\mathrm{d}_\rho\omega))$ 5C, 150

O. BY WAY OF INTRODUCTION

Amounts (the extensive quantities), modelled by set functions, are in general the primary quantities and states (densities or the intensive quantities), modelled by point functions, the derived ones. Therefore, mathematical models constructed in terms of integrals have conceptual and often also technical priority with respect to ones constructed in terms of derivatives.

A relatively detailed description - with references but without proofs - of a mathematical model of homogeneous isotropic diffusion and of its superposition with a process of creation and/or destruction, illustrates this point. It also gives us an opportunity to introduce problems for which the classical (Lebesgue) integration structure is inadequate and to make a suggestion about the nature of this inadequacy.

So, this chapter represents what is commonly, but inaccurately, called the motivation for the material presented in the subsequent chapters. Also, in Section G, the nature of that material is briefly described and so, the way of approaching the problems introduced in this chapter is indicated. This chapter does not form a part of the systematic exposition though; no reference to it is made in the subsequent chapters.

A. In this section, a mathematical model of homogeneous isotropic diffusion is described.

Let E be the Banach space of all real or complex Borel measures in $\mathbb{R}^3$, that is, real or complex valued σ-additive set functions whose domain is the σ-algebra, $\mathcal{B}$, of all Borel sets in $\mathbb{R}^3$. The norm, $\|\varphi\|$, of an element φ of E is the total variation of φ. By $\mathrm{BL}(E)$ is denoted the algebra of all bounded linear operators on E. By I is denoted the identity operator on E.

Now, assume that the space, represented as $\mathbb{R}^3$, is filled with a solvent into which some soluble substance was added. The distribution of that substance is represented by a (real) positive element of E. Its value on the whole space, which is equal to its norm, is the total amount of the substance added.

For every $t \geq 0$, $x \in \mathbb{R}^3$ and $B \in \mathcal{B}$, let the number $k_t(x,B)$ have the following interpretation: If, at the time 0, a unit amount of the diffusing substance is placed at the point x, then, at the time t, the amount of that substance found in the set B is precisely $k_t(x,B)$. Consistently with this interpretation we assume that

(i) $k_0(x,B) = \delta_x(B)$, for every $x \in \mathbb{R}^3$ and $B \in \mathcal{B}$, that is, $k_0(x,B) = 1$, if $x \in B$, and $k_0(x,B) = 0$, if $x \notin B$;

(ii) for every $t \geq 0$ and every $x \in \mathbb{R}^3$, the set function $B \mapsto k_t(x,B)$, $B \in \mathcal{B}$, represents a probability measure on $\mathcal{B}$, that is, a non-negative element of E such that $k_t(x,\mathbb{R}^3) = 1$;

(iii) for every $t \geq 0$ and every $B \in \mathcal{B}$, the function $x \mapsto k_t(x,B)$, $x \in \mathbb{R}^3$, is $\mathcal{B}$-measurable.

The set function $B \mapsto k_t(x,B)$, $B \in \mathcal{B}$, of the requirement (ii) is the distribution of the diffusing substance at the time t provided that a unit amount of the substance is situated at the point x at the time 0. So, the requirement (ii) respects the principle of the conservation of mass. By (i), the requirement (iii) is automatically satisfied for $t = 0$. Without imposing some condition, such as (iii), on the studied kernel not even the most basic analytic techniques would be applicable to it and it would be difficult to interpret it as describing any physical process. On the other hand, the condition (iii) suffices for drawing useful conclusions from the principles of the conservation of mass and of the superposition.

So, assume that at time 0 the distribution of the diffusing substance is represented by the measure $\varphi \in E$, $\varphi \geq 0$. For a fixed $B \in \mathcal{B}$ and $t \geq 0$, let $\mu(X) = \mu_{\varphi,t,B}(X)$ be the amount of the substance which, at the time 0, was in a set $X \in \mathcal{B}$ and at the time t, is found in the set B. Then the principles of superposition and conservation of mass applied to the given situation imply that μ is an additive set function such that

$$\varphi(X)\inf\{k_t(x,B) : x \in X\} \;\leq\; \mu(X) \;\leq\; \varphi(X)\sup\{k_t(x,B) : x \in X\},$$

for every $X \in \mathcal{B}$. It then follows, from (ii) and (iii), that

$$\mu(X) = \int_X k_t(x,B)\varphi(\mathrm{d}x) ,$$

for every $X \in \mathcal{B}$. In particular, $\mu(\mathbb{R}^3)$ is the total amount of the diffusing substance found in the set B at the time t. Hence,

(iv) if the distribution of the diffusing substance at the time 0 is represented by a measure $\varphi \in E$, then the distribution of this substance at a time $t \geq 0$ is represented by the measure $\psi \in E$ given by

$$\psi(B) = \int_{\mathbb{R}^3} k_t(x,B)\varphi(\mathrm{d}x) , \tag{A.1}$$

for every $B \in \mathcal{B}$.

For every $t \geq 0$, let $S(t) : E \to E$ be the map such that, for every $\varphi \in E$, the element $\psi = S(t)\varphi$ of E is given by (A.1). Then, by (i), $S(0) = I$. Furthermore, by (ii), $S(t)$ is a continuous linear map of E into E of norm equal to 1 .

Now we restrict our attention to a time-homogeneous, space-homogeneous and isotropic diffusion. The time-homogeneity is expressed by the condition that $S(t+s) = S(t)S(s)$, for every $s \geq 0$ and $t \geq 0$, that is, the map $t \mapsto S(t)$, $t \geq 0$, from $[0,\infty)$ into $\mathrm{BL}(E)$, is a semigroup of operators. It means that the conditions of diffusion, that is, the properties of the environment and the diffusing substance influencing the diffusion, do not change in time. By (A.1), it can be stated explicitly by requiring that

$$\int_{\mathbb{R}^3} k_{t+s}(x,B)\varphi(\mathrm{d}x) = \int_{\mathbb{R}^3} k_t(x,B) \int_{\mathbb{R}^3} k_s(y,\mathrm{d}x)\varphi(\mathrm{d}y) ,$$

for every $\varphi \in E$. This requirement is of course equivalent to the statement that

(v) the equality

$$k_{t+s}(x,B) = \int_{\mathbb{R}^3} k_t(y,B)k_s(x,\mathrm{d}y) \tag{A.2}$$

holds for every $s \geq 0$, $t \geq 0$ and $B \in \mathcal{B}$.

The requirement of the space-homogeneity means that the properties of diffusion are the same around every point of the space $\mathbb{R}^3$. Expressed formally,

(vi) for every $t \geq 0$, there is a measure $\kappa_t \in E$ such that $k_t(x,B) = \kappa_t(B-x)$, for every $x \in \mathbb{R}^3$ and every $B \in \mathcal{B}$.

Recall that $B - x = \{y - x : y \in B\}$ for any $B \subset \mathbb{R}^3$ and $x \in \mathbb{R}^3$. By (ii), κ_t is a probability measure in $\mathbb{R}^3$, that is a non-negative element of E such that $\kappa_t(\mathbb{R}^3) = 1$, for every $t \geq 0$. If the requirement (vi) is in force, then the equality (A.2) takes the form

$$\text{(A.3)} \qquad \kappa_{t+s}(B) = \int_{\mathbb{R}^3} \kappa_t(B-x)\kappa_s(\mathrm{d}x) ,$$

for every $s \geq 0$, $t \geq 0$ and $B \in \mathcal{B}$.

The isotropy means that the diffusion is the same in every direction. In formal terms, it reduces to the requirement that

(vii) for every $t \geq 0$, the measure κ_t is invariant with respect to the rotations of the space $\mathbb{R}^3$ about the origin.

If we add to all these requirements also a certain requirement of continuity, then the maps $S(t) : E \to E$, $t \geq 0$, describing the process of diffusion, are determined up to a positive parameter - the diffusion constant - which characterizes the speed of this process. In fact, the following theorem, due to G.A. Hunt, holds.

THEOREM 0.1. *Let* κ_t, $t \geq 0$, *be rotationally invariant probability measures on* $\mathbb{R}^3$ *such that the equality* (A.1) *holds for every* $s \geq 0$, $t \geq 0$ *and* $B \in \mathcal{B}$. *Assume that, for every* $\epsilon > 0$,

$$\lim_{t \to 0+} \frac{1}{t} \kappa_t(\{x : |x| \geq \epsilon\}) = 0 .$$

Then, either $\kappa_t = \delta_0$ *for every* $t \geq 0$, *or there exists a constant* $D > 0$ *such that* $\kappa_0 = \delta_0$ *and*

$$\kappa_t(B) = \frac{1}{(4\pi Dt)^{3/2}} \int_B \exp\left[-\frac{|x|^2}{4Dt}\right] \mathrm{d}x \tag{A.4}$$

for every $t > 0$ *and every* $B \in \mathcal{B}$.

This theorem appeared in greater generality in [25]. It is also presented in Section 2 of Chapter IV of H. Heyer's book [22]. A convenient proof, of such degree of generality that corresponds to the formulation given here, can be found in the notes, [53], on Brownian motion by E. Nelson.

Given a $D > 0$, let

$$p(t,x) = \frac{1}{(4\pi Dt)^{3/2}} \exp\left[-\frac{|x|^2}{4Dt}\right] \tag{A.5}$$

for every $t \geq 0$ and $x \in \mathbb{R}^3$. The formula (A.4) says that the function $x \mapsto p(t,x)$, $x \in \mathbb{R}^3$, is the density of the measure κ_t , for every $t \geq 0$. The function p itself is the solution of the Cauchy problem

$$\dot{u}(t,x) = D\Delta u(t,x) \ , \ t \geq 0 \ , \ x \in \mathbb{R}^3 \ ; \ \lim_{t \to 0+} \int_B u(t,x)\mathrm{d}x = \delta_0(B) \ , \ B \in \mathcal{B} \ .$$

It is useful to note, for the indicated physical interpretation, that the dimension (unit of measurement) of the constant D is the reciprocal of the unit of time. The values of the measures κ_t , $t \geq 0$, are dimensionless numbers. In fact, if a measure $\varphi \in E$ represents the distribution of the diffusing substance at time 0 , then its values are given in a unit of mass. Further, at any time $t \geq 0$, the distribution of the substance is represented by the measure $\psi = S(t)\varphi$, where

$$\psi(B) = \int_B \kappa_t(B-x)\varphi(\mathrm{d}x)$$

for every $B \in \mathcal{B}$, and the values of ψ are of course too given in that unit of mass. Consequently, the values of p are given in the reciprocal of a unit of volume.

Using the notation (A.5), the semigroup $S : [0,\infty) \to \mathrm{BL}(E)$, describing the considered physical process of diffusion, can be expressed in the following concise form: $S(0) = I$ and

$$(S(t)\varphi)(B) = \int_B \left[\int_{\mathbb{R}^3} p(t,x-y)\varphi(\mathrm{d}y) \right] \mathrm{d}x \,, \tag{A.6}$$

for every $t > 0$, $\varphi \in E$ and $B \in \mathcal{B}$.

B. Now we describe a mathematical model of a chemical reaction.

We have in mind the following (idealized) situation. The space, $\mathbb{R}^3$, is filled with a medium (solvent) in which another substance is distributed. The distribution of this substance is represented by a non-negative or sometimes an arbitrary real valued element of E. The substance reacts with the environment or is in an unstable state so that it changes and thereby increases or decreases in amount. At the same time, we assume that the concentration is so small that the reaction does not alter the environment. On the other hand, we assume that the reaction-rate is proportional to the concentration of the reacting substance and admit that the coefficient of the proportion varies with place and possibly also with time.

To arrive at a formal description of such a process, we assume that, for every $t \geq 0$, an operator $T(t) \in \mathrm{BL}(E)$ is given which has the following meaning. If a measure $\varphi \in E$ represents the distribution of the reacting substance at the time 0, then $T(t)\varphi$ represents the distribution of the reacting substance at the time $t \geq 0$.

Consistently with this interpretation, we assume that $T(0) = I$, the identity operator.

The assumption that the reaction-rate is proportional to the concentration of the reacting substance is then expressed by assuming that a function V on $[0,\infty[\times \mathbb{R}^3$ is given such that

$$(\dot{T}(t)\varphi)(B) = \int_B V(t,x)(T(t)\varphi)(\mathrm{d}x) \tag{B.1}$$

for every $t \geq 0$, every $\varphi \in E$ and every $B \in \mathcal{B}$.

Besides (B.1) we assume that

(B.2) $$\lim_{t \to 0+} T(t)\varphi = \varphi$$

for every $\varphi \in E$.

The conditions (B.1) and (B.2) strongly suggest the presence of the exponential function about. It actually enters formally in the following way.

For every $B \in \mathcal{B}$, let $P(B) \in \mathrm{BL}(E)$ be the operator defined by

$$(P(B)\varphi)(X) = \varphi(B \cap X) = \int_X \chi_B(x)\varphi(\mathrm{d}x)$$

for every $\varphi \in E$ and every $X \in \mathcal{B}$.

Then, clearly,

(i) $P(\mathbb{R}^3) = I$;

(ii) $P(B \cap C) = P(B)P(C)$ for every $B \in \mathcal{B}$ and $C \in \mathcal{B}$;

(iii) if $\varphi \in E$ and $B_j \in \mathcal{B}$, $j = 1,2,\ldots$, are pair-wise disjoint sets whose union is the set B, then

$$P(B)\varphi = \sum_{j=1}^{\infty} P(B_j)\varphi .$$

Given a $\mathcal{B}$-measurable function W on $\mathbb{R}^3$, we denote by

$$P(W) = \int_{\mathbb{R}^3} W(x)P(\mathrm{d}x)$$

the operator, whose domain is the set of all measures $\varphi \in E$ such that W is φ-integrable, such that

$$(P(W)\varphi)(X) = \int_X W(x)\varphi(\mathrm{d}x) = \int_{\mathbb{R}^3} W(x)(P(\mathrm{d}x)\varphi)(X)$$

for every $X \in \mathcal{B}$.

In plain Slovak, $P(W)\varphi$ is the indefinite integral of the function W with respect to φ interpreted of course as an element of the space E. For this reason, some authors, in their depravity, denote $P(W)$ simply as W, i.e. $W = P(W)$. So, $W\varphi$ then stands for the indefinite integral of W with respect to φ. If φ is absolutely continuous, then so is $P(W)\varphi$ and the density of $P(W)\varphi$ is equal to the (point-wise) product of W and the density of φ.

Note that the domain of the operator $P(W)$ is a vector subspace of E. If W is bounded then the domain of $P(W)$ is the whole of E and $\|P(W)\| = \sup\{\|W(x)\| : x \in \mathbb{R}^3\}$.

It is immediate that

(i) $P(cW) = cP(W)$ for any number c and a measurable function W;

(ii) $P(W_1 + W_2) \supset P(W_1) + P(W_2)$ for any measurable functions W_1 and W_2; and

(iii) $P(W_1 W_2) \supset P(W_1)P(W_2)$ for any measurable functions W_1 and W_2.

Using this machinery, we deduce from (B.1) and (B.2) that

$$T(t) = P(\exp\left[\int_0^t V(s,\cdot)\mathrm{d}s\right], \tag{B.3}$$

for every $t \geq 0$, which means just that

$$(T(t)\varphi)(X) = \int_X \exp\left[\int_0^t V(s,x)\mathrm{d}s\right]\varphi(\mathrm{d}x)$$

for every φ belonging to the domain of the operator (0.9) and every $X \in \mathcal{B}$.

More generally, let

$$T(t,s) = P\left[\exp\left[\int_s^t V(r,\cdot)\mathrm{d}\rho\right]\right]$$

for any $0 \leq \sigma \leq t$. The interpretation of the operators $T(t,s)$ is clear.

C. In this section we describe a mathematical model of evolution of the distribution of a substance which simultaneously undergoes the processes of diffusion and a chemical reaction.

What we are set up to do is to produce a family of operators $U(t) \in \mathrm{BL}(E)$, $t \geq 0$, which have the following meaning: If $\varphi \in E$ is the distribution, at the time 0, of a substance which diffuses in $\mathbb{R}^3$ and also is subject to a reaction which causes its creation or destruction, then $U(t)\varphi$ is the distribution of this substance at any time $t \geq 0$.

For the sake of simplicity, we will assume the diffusion to be time-homogeneous, space-homogeneous and isotropic, as in Section A, so that there is a constant $D > 0$ such that the semigroup of operators $S : [0,\infty) \to \mathrm{BL}(E)$ describing it is given by (A.6), for every $t \geq 0$, $\varphi \in E$ and $B \in \mathcal{B}$. Further, we will assume that the reaction-rate does not change in time so that the process of reaction is described by the semigroup $T : [0,\infty) \to \mathrm{BL}(E)$, where $T(t) = \exp(tP(V))$, for every $t \geq 0$, and V is a function on $\mathbb{R}^3$. This is a special case of the situation discussed in Section B, in particular the formula (B.9), when the function V does not depend on time.

Then, of course, $U(0) = I$. For $t \geq 0$, we can expect that $U(t)$ will be well approximated by the operators of the form

$$\text{(C.1)} \qquad U_\alpha(t) = S(t-t_n)\,T(t_n-t_{n-1})S(t_n-t_{n-1}) \ldots \\ \ldots T(t_3-t_2)S(t_3-t_2)\,T(t_2-t_1)S(t_2-t_1)\,T(t_1)S(t_1)\,,$$

where α is a sufficiently fine partition of the interval $[0,t]$ given by the points $0 = t_0 < t_1 < t_2 < \ldots < t_{n-1} < t_n \leq t$.

Let us introduce a mathematical structure in which this suggestion can be conveniently explored.

For a given $t \geq 0$, let Υ_t be the set of all continuous maps $v : [0,t] \to \mathbb{R}^3$. The elements of Υ_t are usually referred to as paths in $\mathbb{R}^3$ based on the interval $[0,t]$.

Let $\mathcal{R}_t$ be the family of all sets

(C.2) $$Y = \{v \in \Upsilon_t : v(t_j) \in B_j\,,\ j = 1,2,\ldots,n\}$$

such that n is a natural number, $0 \le t_1 < t_2 < \ldots < t_{n-1} < t_n \le t$ and $B_j \in \mathcal{B}$ for every $j = 1,2,\ldots,n$.

Then $\mathcal{R}_t$ is a semiring of sets in Υ_t. Let

$$M_t(Y) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1)$$

for every set $Y \in \mathcal{R}_t$ given in the form (C.2). Then $M_t : \mathcal{R}_t \to L(\mathrm{E})$ is an additive set function.

Before returning to our problem, let us note a point about integration with respect to M_t. Namely, if $0 \le t_1 \le t$ and W_1 is a function on $\mathbb{R}^3$ and if $h_1(v) = W_1(v(t_1))$, for every $v \in \Upsilon_t$, then

$$\int_{\Upsilon_t} h_1 \mathrm{d}M_t = \int_{\Upsilon_t} W_1(v(t_1))M_t(\mathrm{d}v) = S(t-t_1)P(W_1)S(t_1)\,,$$

provided the function h_1 is M_t-integrable. Similarly, if $0 \le t_1 < t_2 \le t$ and W_1 and W_2 are functions on $\mathbb{R}^3$ and if $h_2(v) = W_1(v(t_1))\,W_2(v(t_2))$, for every $v \in \Upsilon_t$, then

$$\int_{\Upsilon_t} h_2 \mathrm{d}M_t = \int_{\Upsilon_t} W_1(v(t_1))\,W_2(v(t_2))M_t(\mathrm{d}v) = S(t-t_2)P(W_2)S(t_2-t_1)P(W_1)S(t_1)\,,$$

provided the function f is M_t-integrable. And so on.

You may note that we have not yet specified what we mean by integrability with respect to M_t. The presented statements and their obvious inductive extensions simply mean that if the integral with respect to M_t is introduced with the slightest regard to reasonableness, then these formulas must be true. Moreover, the function h_2, say, should be M_t-integrable on Υ_t if the function $(x_1,x_2) \mapsto W_1(x_1)\,W_2(x_2)$, $(x_1,x_2) \in \mathbb{R}^3 \times \mathbb{R}^3$, is integrable on $\mathbb{R}^3 \times \mathbb{R}^3$ with respect to the additive set function

$$B_1 \times B_2 \mapsto S(t-t_2)P(B_2)S(t_2-t_1)P(B_1)S(t_1)\,,\quad B_1 \in \mathcal{B}\,,\ B_2 \in \mathcal{B}\,.$$

So, the operator (C.1) can be written as

$$U_\alpha(t) = \int_{\Upsilon_t} \left[\prod_{j=1}^{n} \exp(V(v(t_j))(t_j - t_{j-1}))\right] M_t(\mathrm{d}v) =$$

$$= \int_{\Upsilon_t} \left[\exp\left[\sum_{j=1}^{n} V(v(t_j))(t_j - t_{j-1})\right]\right] M_t(\mathrm{d}v) .$$

Accordingly, we define

$$e_t(v) = \exp\left[\int_0^t V(v(s))\mathrm{d}s\right] , \tag{C.3}$$

for every $v \in \Upsilon_t$. Then we would expect that

$$U(t) = \int_{\Upsilon_t} e_t(v) M_t(\mathrm{d}v) \tag{C.4}$$

for every $t \geq 0$. Let us show that this expectation is warranted.

First, the formula (C.4) means to say that if $u(t) = U(t)\varphi$, for any given $\varphi \in E$, then

$$u(t) = \int_{\Upsilon_t} \left[\exp\left[\int_0^t V(v(s))\mathrm{d}s\right]\right] M_t(\mathrm{d}v)\varphi . \tag{C.5}$$

That is, $u(t)$ is equal to the integral of the function e_t with respect to the E-valued additive set function $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$. Comments about integration with respect to this set function are postponed into the next section.

Now, assuming $t \geq 0$ given, let

$$f(s,v) = V(v(s))\exp\left[\int_0^s V(v(r))\mathrm{d}r\right] , \tag{C.6}$$

for every $s \in [0,t]$, and $v \in \Upsilon_t$. Then

$$\int_{\Upsilon_t} f(s,v) M_t(\mathrm{d}v)\varphi = S(t-s)P(V)u(s) ,$$

for every $s \in [0,t]$, and

$$\int_0^t f(s,v)\mathrm{d}s = \exp\left[\int_0^t V(v(s))\mathrm{d}s\right] - 1 ,$$

for every $v \in \Upsilon_t$. Therefore, by the Fubini theorem,

$$u(t) - S(t)\varphi = \int_{\Upsilon_t} \left[\exp\left[\int_0^t V(v(s))\mathrm{d}s\right] - 1\right] M_t(\mathrm{d}v)\varphi =$$

$$= \int_{\Upsilon_t} \left[\int_0^t f(s,v)\mathrm{d}s\right] M_t(\mathrm{d}v)\varphi = \int_0^t \left[\int_{\Upsilon_t} \varphi(s,v) M_t(\mathrm{d}v) f\right] \mathrm{d}\sigma =$$

$$= \int_0^t S(t-s)P(V)u(s)\mathrm{d}s ,$$

or

(C.7) $$u(t) = S(t)\varphi + \int_0^t S(t-s)P(V)u(s)\mathrm{d}s .$$

If the function V is not 'too large', then, for any $t > 0$, the measure $u(t) \in E$, given by (C.5), is absolutely continuous (with respect to the Lebesgue measure in $\mathbb{R}^3$). (This is of course obvious for $V = 0$.) If we then abuse the notation and denote by $x \mapsto u(t,x)$, $x \in \mathbb{R}^3$, the density of $u(t)$, we can re-write the integral equation (C.7) as

$$u(t,x) = \int_{\mathbb{R}^3} p(t,x-y)\varphi(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^3} p(t-s,x-y)\, V(y) u(s,y)\mathrm{d}y\mathrm{d}s , \quad x \in \mathbb{R}^3 ,$$

which represents the initial-value problem

(C.8)
$$\dot{u}(t,x) = D\Delta u(t,x) + V(x)u(t,x) , \; t \geq 0 , \; x \in \mathbb{R}^3 ;$$
$$\lim_{t\to 0+} \int_B u(t,x)\mathrm{d}x = \varphi(B) , \; B \in \mathcal{B} .$$

Our original problem of the superposition of diffusion and a chemical reaction is most commonly formulated as this initial-value problem.

It is clear that formula (C.5) has certain advantages against the integral equation (C.7) and the problem (C.8). For it represents $u(t)$ in a form which allows various calculations and estimations which are not possible directly from (C.7) or (C.8). Secondly, (C.5) may have a good meaning also when (C.7) or (C.8) do not have a solution (in some sense) or cannot even be written down.

D. Generalizations, considererd in [31] and [34], of the situation discussed in the previous section give us the opportunity to introduce problems for which the classical integration theory is inadequate.

Let E be an arbitrary Banach space, $\mathrm{BL}(E)$ the space of bounded linear operators on E and $S : [0,\infty) \to \mathrm{BL}(E)$ a continuous semigroup of operators. So,

(i) $S(t+s) = S(t)S(s)$, for every $s \geq 0$ and $t \geq 0$;

(ii) $S(0) = I$, the identity operator; and

(iii) $\lim_{t\to 0+} S(t)\varphi = \varphi$, for every $\varphi \in E$.

Let Λ be a locally compact Hausdorff space, $\mathcal{B}(\Lambda)$ the σ-algebra of Baire sets in Λ and $P : \mathcal{B}(\Lambda) \to \mathrm{BL}(E)$ a σ-additive spectral measure. That is to say,

(i) $P(B \cap C) = P(B)P(C)$, for every $B \in \mathcal{B}(\Lambda)$ and $C \in \mathcal{B}(\Lambda)$;

(ii) $P(\Omega) = I$; and

(iii) the set function $B \mapsto P(B)\varphi$, $B \in \mathcal{B}(\Lambda)$, is σ-additive, for every $\varphi \in E$.

For a Baire function W on Λ, we denote by

$$P(W) = \int_\Lambda W \mathrm{d}P$$

the operator such that

$$P(W)\varphi = \int_\Lambda W(x)P(\mathrm{d}x)\varphi$$

for every $\varphi \in E$ for which the right-hand side exists as integral with respect to the E-valued measure $B \mapsto P(B)\varphi$, $B \in \mathcal{B}(\Lambda)$. (A standard reference for integration with respect to spectral measures and also with respect to Banach valued measures is the monograph [14] of N. Dunford and J.T. Schwartz.) The operator $P(W)$ is bounded, that is, belongs to $\mathrm{BL}(E)$, if and only if the function W is essentially bounded. In general, $P(W)$ is a densely defined closed operator on E.

Given a Baire function V on Λ, assume that the function $\exp V$ is essentially bounded on Λ. Then, for every $t \geq 0$, the function $\exp(tV)$ too is essentially bounded. In that case, let

$$T(t) = P(\exp(tV)),$$

for every $t \geq 0$. The resulting map $T : [0,\infty) \to \mathrm{BL}(E)$ is a continuous semigroup of operators such that $P(V)$ is its infinitesimal generator. That is

$$P(V)\varphi = \lim_{t \to 0+} \frac{1}{t}(T(t)\varphi - \varphi)$$

for every φ in the domain of $P(V)$. Then we also write

$$T(t) = \exp(tP(V))$$

for every $t \geq 0$, as customary in the theory of continuous semigroups.

The semigroups S and T are interpreted as describing two evolution processes in which an element φ of the space E is transformed, during a time-interval of duration $t \geq 0$, into the element $S(t)\varphi$ and $T(t)\varphi$, respectively. Our problem is to determine the element of the space E into which a given element φ evolves in a time $t \geq 0$ if both these processes go on simultaneously. In other words, we wish to construct a semigroup U which describes the superposition of the processes described by the semigroups S and T.

This problem is traditionally formulated in terms of differential equations. Let

$$A\varphi = \lim_{t \to 0+} \frac{1}{t}(S(t)\varphi - \varphi)$$

for every $\varphi \in E$ for which this limit exists. The operator A, the infinitesimal generator of the semigroup S, is not bounded in general.

So, we are seeking a semigroup whose infinitesimal generator is $A + P(V)$, that is, a solution of the initial-value problem

(D.1) $$\dot{U}(t) = AU(t) + P(V)U(t), \quad t \geq 0; \quad U(0+) = I.$$

In other words, we look for the fundamental solution of the differential equation

$$\dot{u}(t) = Au(t) + P(V)u(t)\ ,\quad t \geq 0$$

with the unknown E-valued function, u, right-continuous at 0.

This problem is non-trivial because, strictly speaking, it is not even unambiguously formulated. The point is that the operator $A + P(V)$ is not necessarily the infinitesimal generator of a continuous semigroup of operators. On the other hand, this operator may have an extension which is an infinitesimal generator, but such an extension may not be unique. It is conceivable that the obvious generalizations of the objects introduced in the previous section would be helpful in clarifying the issues involved in this problem and in solving it.

For a $t \geq 0$, let Υ_t be a 'sufficiently rich' set of maps $v : [0,t] \to \Lambda$, to be called paths in Λ. Let $\mathcal{I}_t$ be the family of all sets (C.2) for arbitrary $n = 1,2,\ldots$, $0 \leq t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n \leq t$ and $B_j \in \mathcal{B}(\Lambda)$, $j = 1,2,\ldots,n$. Let

$$M_t(Y) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1)$$

for any such set Y.

Then a heuristic argument, similar to that presented in the previous section, suggests that the operators $U(t)$ can be expressed by the means of the Feynman-Kac type formula:

(D.2) $$U(t) = \int_{\Upsilon_t} \left[\exp\left[\int_0^t V(v(r))\mathrm{d}r\right]\right] M_t(\mathrm{d}v)\ ,$$

for every $t \geq 0$. In fact, an integral equation for U can be derived in an manner precisely analogous to that of deriving (C.7). Namely, assume that $t \geq 0$, that the function e_t is given by (C.3) for every $v \in \Upsilon_t$ and that the function f is given by (C.6) for every $s \in [0,t]$ and $v \in \Upsilon_t$. Then

$$\int_{\Upsilon_t} f(s,v)M_t(\mathrm{d}v) = S(t-s)P(V)U(s)$$

for every $s \in [0,t]$ and

$$\int_0^t f(s,v)\mathrm{d}s = \exp\left[\int_0^t V(v(r))\mathrm{d}r\right] - 1 \,,$$

for every $v \in \Upsilon_t$. Therefore, by the Fubini theorem (!),

$$U(t) - S(t) = \int_{\Upsilon_t} \left[\exp\left[\int_0^t V(v)r))\mathrm{d}r\right] - 1\right] M_t(\mathrm{d}v) =$$

$$= \int_{\Upsilon_t} \left[\int_0^t f(s,v)\mathrm{d}s\right] M_t(\mathrm{d}v) = \int_0^t \left[\int_{\Upsilon_t} f(s,v) M_t(\mathrm{d}v)\right] \mathrm{d}s =$$

$$= \int_0^t S(t-s)P(V)U(s)\mathrm{d}s \,.$$

The obtained integral equation,

$$U(t) = S(t) + \int_0^t S(t-s)P(V)U(s)\mathrm{d}s \,, \tag{D.3}$$

replaces the initial-value problem (D.1).

It goes without saying that once we have a solution of the problem (D.1), then we have solutions of the initial-value problems

$$\dot{u}(t) = Au(t) + P(V)u(t) \,,\ t \geq 0 \,;\ \ u(0+) = \varphi \,, \tag{D.4}$$

for all $\varphi \in E$. Indeed, it suffices to put $u(t) = U(t)\varphi$, for every $t \geq 0$, where U is a solution of (D.1). On the other hand, the point of the formula (D.2), or the formula (C.5) for a given $\varphi \in E$, is that $U(t)$ or $u(t)$ could possibly be defined by these formulas even when the initial-value problems (D.1) or (D.4) do not have a solution or perhaps could not even be meaningfully formulated.

The question then arises whether the formulas (C.5) and (D.2) can be put on a solid footing. Or, rather, whether a formal framework can be erected in which these formulas have a good meaning and the conditions for a legitimate use of the operations lading to them can be formulated.

Now, integration with respect to the $\mathrm{BL}(E)$-valued set function M_t is reduced to integration with respect to the E-value set functions $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, for every $\varphi \in E$.

Accordingly, the equality (D.2) is defined to mean that

$$U(t)\varphi = \int_{\Upsilon_t} \left[\exp\left[\int_0^t V(v(r))\mathrm{d}r\right]\right] M_t(\mathrm{d}v)\varphi ,$$

for every $\varphi \in E$, where the integral is understood with respect to the E-valued set function. This reduction is analogous to integration with respect to spectral measures. It has the advantage that one may attempt the construction of a solution of the problem (D.4), for some $\varphi \in E$, by the means of the formula (C.5) and thus avoid the fundamental solution. In fact, it is conceivable that the integral (C.5) may exist for some $\varphi \in E$ while the integral (D.2) does not.

So, there remains the problem how to integrate with respect to the E-valued set functions $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, for $\varphi \in E$.

EXAMPLE 0.2. In the case when S is the diffusion semigroup (see Section A) and T the creation/destruction process semigroup (see Section B), the means for an easy solution of this problem are provided by the Wiener measure. In fact, given a set $Y \in \mathcal{R}_t$ of the form (C.2), the number

$$w(Y) = \int_{\mathbb{R}^3} \int_{B_{n-1}} \int_{B_{n-1}} \cdots \int_{B_2} \int_{B_1} p(t-t_n, y-x_n) p(t_n-t_{n-1}, x_n-x_{n-1}) \cdots$$

$$\cdots p(t_2-t_1, x_2-x_1) p(t_1, x_1) \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_{n-1} \mathrm{d}x_{n-1} \mathrm{d}x_n \mathrm{d}y = (M_t(Y)\delta_0)(\mathbb{R}^3) ,$$

where the kernel p is given by (A.5), is equal to the Wiener measure (or variance 2D per unit of time) of the set Y. To be sure, w is a probability measure on the whole of the σ-algebra $\mathcal{S}_t$ generated by the family of sets $\mathcal{R}_t$.

Now, for a set $Y \subset \Upsilon_t$ and $x \in \mathbb{R}^3$, let $Y - x$ be the set of all paths $s \mapsto v(s) - x$, $s \in [0,t]$, such that $v \in Y$. Let $w_x(Y) = w(Y-x)$ for every $Y \in \mathcal{R}_t$ and $x \in \mathbb{R}^3$. Then w_x is a probability measure on $\mathcal{S}_t$ such that $w_x(Y) = (M_t(Y)\delta_x)(\mathbb{R}^3)$ for every $Y \in \mathcal{R}_t$. Furthermore, if $\varphi \in E$, let

$$w_\varphi(Y) = \int_{\mathbb{R}^3} w_x(Y)\varphi(\mathrm{d}x) = \int_{\mathbb{R}^3} w(Y-x)\varphi(\mathrm{d}x)$$

for every $Y \in \mathcal{S}_t$. Then w_φ is a real or complex valued σ-additive measure on $\mathcal{S}_t$ such that $w_\varphi(Y) = (M_t(Y)\varphi)(\mathbb{R}^3)$ for every $Y \in \mathcal{R}_t$. Hence, if μ is the variation of the measure φ, then w_μ is a finite positive measure on $\mathcal{S}_t$ such that $|(M_t(Y)\varphi)(B)| \le w_\mu(Y)$ for every $Y \in \mathcal{R}_t$ and every $B \in \mathcal{B}(\mathbb{R}^3)$. So, if $\varphi \in E$ is real-valued then the norm of the element $M_t(Y)\varphi$ of E, that is, the total variation of this measure, is not greater than $w_\mu(Y)$, for any $Y \in \mathcal{R}_t$. If $\varphi \in E$ is complex then the norm of $M_t(Y)\varphi$ is not greater than $2w_\mu(Y)$, say, for every $Y \in \mathcal{R}_t$. Consequently, there exists a unique continuous linear map $i_\varphi : \mathcal{L}^1(w_\mu) \to E$ such that $i_\varphi(f) = M_t(Y)\varphi$, whenever f is the characteristic function of a set $Y \in \mathcal{R}_t$. Therefore, we may declare a function f on Υ_t to be integrable with respect to the E-valued set function $Y \mapsto M_t(Y)\varphi$, $Y \in \mathcal{R}_t$, if it is w_μ-integrable and define

$$\int_{\Upsilon_t} f(v) M_t(\mathrm{d}v)\varphi = i_\varphi(f)$$

for every $f \in \mathcal{L}^1(w_\mu)$.

EXAMPLE 0.3. Let $E = L^2(\mathbb{R}^3)$. The Fourier-Plancherel transform of an element $\varphi \in E$ is denoted by $\hat{\varphi}$. Let m be a (strictly) positive number. For every real t, let $S(t) : E \to E$ be the map uniquely determined by the requirement that

$$(S(t)\varphi)^\wedge(\xi) = \exp\left[-\frac{t\,\mathrm{i}}{2m}|\xi|^2\right]\hat{\varphi}(\xi)$$

for every $\varphi \in E$ and (almost) every $\xi \in \mathbb{R}^3$. The Plancherel theorem implies that $S(t) : E \to E$ is a unitary operator and the resulting map $S : \mathbb{R} \to \mathrm{BL}(E)$ is a continuous group of operators.

For every $t \ne 0$ and $x \in \mathbb{R}^3$, let

(D.5) $$p(t,x) = \frac{1}{(2\pi \mathrm{i} t/m)^{3/2}} \exp\left[\mathrm{i}m\frac{|x|^2}{2t}\right].$$

Then,

$$(S(t)\varphi)(x) = \int_{\mathbb{R}^3} p(t,x-y)\varphi(y)\mathrm{d}y$$

for every $\varphi \in L^1 \cap L^2(\mathbb{R}^3)$. The kernel (D.5) is obtained from (A.5) by substituting $D = 2/m\mathrm{i}$.

The group $t \mapsto S(t)$, $t \in (-\infty,\infty)$, is called the Schrödinger group. It is interpreted as the description of the motions (evolutions) of a free non-relativistic quantum mechanical particle of mass m with three degrees of freedom. The states of such a particle are determined by elements of the space E with norm equal to 1. The word 'free' indicates that no external forces are acting on the particle. Then, if the particle is at a state φ at time $t = 0$, then, at any other time $t \in (-\infty,\infty)$, the particle was or will be at the state $S(t)\varphi$.

Let $\Lambda = \mathbb{R}^3$. For every $B \in \mathcal{B}(\mathbb{R}^3) = \mathcal{B}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B. Then, clearly, $P : \mathcal{B} \to \mathrm{BL}(E)$ is a spectral measure. If W is a measurable function on $\mathbb{R}^3$, then $P(W)$ is the operator of multiplication by W. Therefore, one usually writes simply W instead of $P(W)$.

Now, let V be a real-valued function on $\mathbb{R}^3$ interpreted as the potential of the forces acting on the particle. Let

$$T(t) = \exp(-\mathrm{i}tP(V))$$

for every $t \in (-\infty,\infty)$. The group T describes the fictitious motions of the particle under the influence of the forces with potential V assuming that 'inertial motions' are suspended.

The superposition, U, of these groups S and T describes the real motions of the particle in the force-field of potential V. The group U can be considered the fundamental solution of the equation

$$\dot{u}(t) = \frac{\mathrm{i}}{2m}\Delta u(t) - \mathrm{i}\,Vu(t)\ ,\quad t \in (-\infty,\infty)\ .$$

That is to say, if $\varphi \in E$ and $u(t) = U(t)\varphi$, for every $t \in (-\infty,\infty)$, then

(D.6) $\dot{u}(t,x) = \frac{i}{2m}\Delta u(t,x) - i V(x)u(t,x)$, $t \in (-\infty,\infty)$, $x \in \mathbb{R}^3$; $u(0,x) = \varphi(x)$, $x \in \mathbb{R}^3$,

assuming that φ is represented by a sufficiently smooth function and the potential V is not 'too bad'.

However, there are considerable difficulties associated with the construction of the semigroup U by the means of the formula (D.2) because it is not at all clear how to integrate with respect to the $\mathrm{BL}(E)$-valued set function M_t. Indeed, for most vectors $\varphi \in E$ and $\psi \in E$, the scalar valued set functions $Y \mapsto \langle \psi, M_t(Y)\varphi \rangle$, $Y \in \mathcal{R}_t$, (the scalar product in E) has infinite variation on every 'non-trivial' set in $\mathcal{R}_t$.

E. Because of its significance, Example 0.3 deserves further comments.

Although the problems posed by Example 0.3 are much more difficult to handle, historically it precedes Example 0.2. In his Thesis, [15], R.P.Feynman suggested the replacement of the initial-value problem (D.6) by the formula

$$\text{(E.1)} \qquad u(t,x) = \int_{\Upsilon_t} \exp\left[\frac{i}{\hbar}\left[\frac{m}{2}\int_0^t |\dot{v}(s)|^2 ds - \int_0^t V(v(s))ds\right]\right] \varphi(v(0))\mathcal{D}(v)$$

(with some insignificant changes of notation) which is to be understood as

$$\text{(E.2)} \qquad u(t,x) = \lim_{n\to\infty} \left[\frac{2\pi i \hbar t}{nm}\right]^{-3n/2} \overbrace{\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3}}^{n \text{ times}}$$

$$\exp\left[\frac{i}{\hbar}\left[\frac{mt}{2n}\sum_{k=1}^{n} |x_k - x_{k-1}|^2 - \frac{t}{n}\sum_{k=1}^{n} V(x_k)\right]\right] \varphi(x_0)dx_0 dx_1 \dots dx_{n-1} ,$$

where $x_n = x$. The possibilities of an approach to quantum mechanics based on this suggestion are systematically explored in the book [16] by R.P. Feynman and A.R. Hibbs.

The formula (E.1) has a great heuristic value. Its attractiveness to physicists is to a considerable degree based on the fact that, apart from the factor $i/\hbar$, the argument of the exponential function is equal to the classical action along the trajectory v. This heuristic value seems to be responsible for the resilience of this formula, its popularity and that of its variants and generalizations, in spite of serious conceptual difficulties associated with it.

The main difficulty presented by (E.1) is that the integration 'with respect to the variable v' over the space, Υ_t, of paths in $\mathbb{R}^3$ refers to integration with respect to the infinite product of copies of the Lebesgue measure in $\mathbb{R}^3$ indexed by all the time-instants from the interval $[0,t]$. However, such an infinite-dimensional analogue of the Lebesgue measure does not exist. This is caused by the fact that the measure of the whole space $\mathbb{R}^3$ is infinite so that the measure of any (presumably measurable) set in Υ_t would be either 0 or ∞. This state of affairs cannot be remedied by admitting into Υ_t more (or even all) maps $v : [0,t] \to \mathbb{R}^3$ besides the continuous ones. This difficulty is intrinsic and directly insurmountable. Therefore, (E.1) cannot be taken as anything more than a suggestive way of writing (E.2)

By interpreting (E.1) as the limit (E.2), the mentioned difficulty is to a certain degree circumvented together with that which is related to the existence of the derivatives $\dot{v}(s)$ for $v \in \Upsilon_t$ and $s \in [0,t]$. However, it should be born in mind that the integrals with respect to $x_1, x_2, \ldots, x_{n-1}$ are not absolutely convergent because the integrand has constant absolute value. So, one cannot arbitrarily change the order of integrations.

There is considerable literature devoted to definitions of the Feynman integral, interpreted as the limit (E.2), exploiting, roughly speaking, a suitable summability method for the calculation of the finite-dimensional integrals in (E.2) and/or its approximation which facilitates the subsequent passage to the limit.

In a somewhat different manner, a rigorous meaning can be given to (E.1) by constructing the superposition U of the semigroups S and T, defined in Example 0.3, through approximation of the operators $U(t)$ by operators of the form (C.1). In fact, H.F. Trotter, [65], and T. Kato, [27], have found conditions under which the limit

$$U(t)\varphi = \lim_{n\to 0}(S(tn^{-1})\,T(tn^{-1}))^n\varphi$$

exists for every $\varphi \in E = L^2(\mathbb{R}^3)$. A variant of this approach is used by G.N. Gestrin, in his paper [17].

However, with this approach integration over the function space Υ_t is to some extent suppressed and with it the heuristic value of (E.1) diminishes. In a way, the same can be said about many definitions of (E.1) using sequential limits. A certain useful compromise in this direction is achieved by E. Nelson in his influential paper [52]. He uses the Trotter-Kato formula to guarantee the construction, by the means of integrals over the space Υ_t of continuous paths, of the operators $U_\zeta(t)$ analogous to $U(t)$ but with the mass m replaced by complex numbers ζ with positive imaginary parts. For any $\varphi \in E$, the so obtained E-valued function $\zeta \mapsto U_\zeta(t)\varphi$ is then analytic in the upper complex half-plane and the vector $U(t)\varphi = U_m(\varphi)$ is obtained as the boundary value of this function at m, that is, as the non-tangential limit for $\zeta \mapsto m$, $\mathrm{Im}\zeta > 0$. Unfortunately, the boundary value exists only for almost every m (in the sense of the Lebesgue measure).

Nelson's approach led to considerable insight into the situation, especially in the cases of some badly behaved functions V, but still, it did not solve completely the problem of maintaining the heuristic value of the formula (E.1) and, at the same time, turning it into a sufficiently flexible and reliable analytic tool. It seems that a solution of this problem cannot be tied too closely to the specific properties of the Schrödinger group. A structure or a method is called for which is applicable in a wider class of cases. A hint that such a structure might exist can be derived from the work of Mark Kac. He noted that, if the factor $i/\hbar$ is dropped from the exponent in (E.1), then the integral can be given a perfect meaning in terms of the Wiener measure. (Cf. the exposition in [26] Chapter IV.) Of course, by dropping the factor $i/\hbar$ we switch to a different problem. One of the possible physical meanings of the new problem is described in Section C ; to another one is devoted the book [61] of B. Simon (see also its review [54] by E. Nelson).

The 'derivation' of the equation (D.3), or (C.7), shows that an integration scheme which allows 'integration with respect to sufficiently wild set functions of infinite variation' and for which a Fubini-type theorem holds, would do for the required structure. Such an integration scheme is presented in Chapters 2 and 3.

F. To emphasize that the difficulties observed in Example 0.3 are caused neither by the fact that the underlying space, Υ_t, is infinite-dimensional nor the fact that the values of the integrator belong to an infinite-dimensional space, in this section we mention a classical case in which both, the underlying space and the space of values, are one-dimensional, none-the-less the same difficulties as in Example 0.3 occur. In fact, if the function g has infinite variation in every non-degenerate sub-interval of the interval $[a,b]$, then the difficulties associated with the (definition, existence, properties,... of the) Stieltjes integral

$$\int_a^b f(x)\mathrm{d}g(x) \tag{F.1}$$

are in principle the same as with

$$\int_{\Upsilon_t} f(v)\langle \psi, M_t(\mathrm{d}v)\varphi\rangle ,$$

if E, M_t, φ, ψ etc. have the same meaning as in Example 0.3.

Stieltjes integrals (F.1) are the subject of attention for many reasons. Perhaps the most prominent among them is the exploration of the possibilities of integration with respect to (individual) sample paths of stochastic processes, such as the Wiener process, and of the analysis of the solutions of stochastic differential equations. In spite of marked successes, such as that of H. Sussmann, [63], the progress in this direction seems still not satisfactory.

An interesting approach to integrals (F.1), exploiting the moduli of continuity of the functions f and g, was initiated by L.C. Young in [69]. The best result is due to A. Beurling, [3], who used a most ingenious method for introducing integrals of this type. Unfortunately, Beurling's method is difficult to extend to cases in which the interval $[a,b]$ of the real-line is replaced by a more general space. Secondly, it does not provide a complete metric in the space of functions integrable with respect to a fixed integrator.

G. The classical theory of absolutely convergent integrals proved to be inadequate in the situations described in previous sections. So, it is desirable to produce a more general theory of integration which would be applicable not only in the classical situations but also in the situations similar to those mentioned above.

Because a generalized theory necessarily lacks certain features of a more specific one, the question arises: which aspects of the classical theory of integration should be considered so essential that also the more general theory must retain them? This question is a result of two related concerns, namely that about the actual erection of the new theory and that of its usefulness. That is, we wish to choose those aspects of the classical theory on the basis of which the new theory could be conveniently developed and, at the same time, would guarantee that the new, more general, theory would be sufficiently powerful in the situations for which it is intended. Such a choice is of course a matter of an interpretation of the integration theory.

A short reflection would reveal that an interpretation which is formulated in terms of a particular method, or procedure, used for introducing integrable functions and/or integral, is not really helpful. Then the most fruitful of the 'objective' interpretations of the integration theory, that is, those which are independent of any such procedure, seems to be one that characterizes the L^1-space as the completion of simple functions (continuous functions,...) in the L^1-norm. The point of a particular construction of integral is in showing that the completion is represented by functions on the original underlying space or equivalence classes of such functions.

This interpretation can be further refined by noting that there exist families of functions which generate the L^1-space and are not necessarily vector spaces. That is, the L^1-space is the completion of the linear hull of such a family and its norm is the largest norm with a given restriction to the generating family of functions. Characteristic functions of sets belonging to a sufficiently rich family of measurable sets can serve as a typical example. To make this remark more perspicuous, we recall the following fact concerning the classical integration theory.

Let Ω be an abstract space and $\mathcal{Q}$ a semiring of its subsets. For simplicity, a subset of Ω and its characteristic function are denoted by the same symbol. Let ι be a real-valued (finite) non-negative σ-additive set function on $\mathcal{Q}$.

A function, f, on Ω is integrable with respect to (the measure generated by) ι if and only if there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

$$\text{(G.1)} \qquad \sum_{j=1}^{\infty} |c_j| \iota(X_j) < \infty$$

and

$$\text{(G.2)} \qquad f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\text{(G.3)} \qquad \sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Moreover, the L^1-norm,

$$\|f\| = \int_{\Omega} |f| \, d\iota ,$$

of such a function f is equal to the infimum of the sums (G.1) taken for all such choices of the numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$. A proof of this fact is given in Section 2E below. The vector space of all (individual) functions integrable with respect to ι is denoted by $\mathcal{L}(\iota)$.

In the case when Ω is an interval of the real-line, $\mathcal{Q}$ consists of intervals and ι is the Lebesgue measure, it can be easily visualized. This case is commented on in more detail in the Preface to the book [50] of J. Mikusiński and in [33]. More comments can be found in Section 3B. Now we mention just a straightforward but important consequence of this fact.

If μ is an additive set function on $\mathcal{Q}$ (which may, possibly, be vector valued) such that $|\mu(X)| \leq \iota(X)$, for every set $X \in \mathcal{Q}$, then there exists a unique linear functional, $\ell = \ell_\mu$, on $\mathcal{L}(\iota)$ such that $\ell(X) = \mu(X)$, for every $X \in \mathcal{Q}$, and

$|\ell(f)| \leq \|f\|$, for every function $f \in \mathcal{L}(\iota)$. In particular, the integral with respect to ι is the linear functional, ℓ , on $\mathcal{L}(\iota)$ such that $\ell(X) = \iota(X)$, for every $X \in \mathcal{Q}$, and $|\ell(f)| \leq \|f\|$, for every $f \in \mathcal{L}(\iota)$.

Now, assume that μ is an additive set function on $\mathcal{Q}$ and that there does not exist a finite σ-additive set function, ι , such that $|\mu(X)| \leq \iota(X)$, for every $X \in \mathcal{Q}$. In the previous sections, we have shown that such set functions occur abundantly and are of considerable interest. In such case, μ does not generate a continuous linear functional on any L^1-space containing the characteristic functions of all sets from $\mathcal{Q}$. Nevertheless, there may still exist a complete normed space (strictly speaking, a seminormed space), $\mathcal{L}$, consisting of functions on Ω and containing the characteristic functions of sets belonging to $\mathcal{Q}$, such that μ can be extended to a continuous linear functional on $\mathcal{L}$.

So, we may look for a non-negative set function, ρ , on $\mathcal{Q}$, which is a restriction to $\mathcal{Q}$ (interpreted of course as a family of functions) of the norm on some such space $\mathcal{L}$, such that $|\mu(X)| \leq \rho(X)$, for every $X \in \mathcal{Q}$. If the space $\mathcal{L}$ is generated by $\mathcal{Q}$, that is, it is the completion of the linear hull of $\mathcal{Q}$, and if the norm on $\mathcal{L}$ is the smallest norm coinciding with ρ on $\mathcal{Q}$, then μ can be uniquely extended to a continuous linear functional on the whole of $\mathcal{L}$.

Let us turn the tables and call a non-negative set function, ρ , on $\mathcal{Q}$ an integrating gauge, if

$$\rho(X) \leq \sum_{j=1}^{\infty} |c_j| \rho(X_j)$$

for any set $X \in \mathcal{Q}$, numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

$$X(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for ever $\omega \in \Omega$ satisfying the inequality (G.3).

Given an integrating gauge, ρ , on $\mathcal{Q}$, let $\mathcal{L}(\rho,\mathcal{Q})$ be the family of all functions, f , on Ω for which there exist numbers, c_j , and sets, $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

G.4)
$$\sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty$$

and the equality (G.2) holds for every $\omega \in \Omega$ for which the inequality (G.3) does. For any such function, $f \in \mathcal{L}(\rho,\mathcal{Q})$, let

$$q(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(X_j) ,$$

where the infimum is taken over all choices of numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, satisfying condition (G.4), such that the equality (G.2) holds for every $\omega \in \Omega$ for which the inequality (G.3) does.

It turns out that $\mathcal{L}(\rho,\mathcal{Q})$ is a vector space and q is a norm (strictly speaking, a seminorm) under which the space $\mathcal{L}(\rho,\mathcal{Q})$ is complete and the linear hull of $\mathcal{Q}$ is dense in it. Moreover, if μ is an additive set function on $\mathcal{Q}$ such that $|\mu(X)| \le \rho(X)$, for every $X \in \mathcal{Q}$, then there exists a unique linear functional, ℓ, on $\mathcal{L}(\rho,\mathcal{Q})$ such that $\ell(X) = \mu(X)$, for $X \in \mathcal{Q}$, and $|\ell(f)| \le q(f)$, for every $f \in \mathcal{L}(\iota,\mathcal{Q})$.

Now the problem naturally arises of producing a sufficient supply of integrating gauges. Of the various ways of solving this problem, let us mention the following one. If ι is a finite non-negative σ-additive set function on $\mathcal{Q}$ and φ a continuous, increasing and concave function on $[0,\infty)$ such that $\varphi(0) = 0$, then the set function ρ, defined by $\rho(X) = \varphi(\iota(X))$ for every $X \in \mathcal{Q}$, is an integrating gauge on $\mathcal{Q}$.

To show the usefulness of this construction, let us indicate how it solves the problem of Stieltjes integration with respect to functions of infinite variation. So, let, for example, $\Omega = (0,1]$, let $\mathcal{Q}$ be the family of all intervals $(s,t]$ such that $0 \le s \le t \le 1$, let g be a function on $[0,1]$ such that $|g(t)-g(s)| \le |t-s|^{\frac{1}{2}}$, for any $s \in [0,1]$ and $t \in [0,1]$, and let $\mu(X) = g(t) - g(s)$, for any $X = (s,t] \in \mathcal{Q}$. If we define $\rho(X) = (\iota(X))^{\frac{1}{2}}$, for every $X \in \mathcal{Q}$, where ι is the Lebesgue measure, then we obtain an integrating gauge, ρ, on $\mathcal{Q}$. Now we can define

$$\int_0^1 f \mathrm{d}\mu = \int_0^1 f(x) \mathrm{d}g(x) ,$$

for any function $f \in \mathcal{L}(\rho,\mathcal{Q})$, to be the value, $\ell(f)$, of the continuous linear functional, ℓ, on $\mathcal{L}(\rho,\mathcal{Q})$ such that $\ell(X) = \mu(X)$, for every $X \in \mathcal{Q}$.

Other applications are presented in Chapter 7 in which we return to the problems described in this chapter. Hopefully, they suffice as an indication that the attention payed to the introduced notions is warranted. Nevertheless, it is natural and convenient to introduce a still more general structure which generalizes simultaneously integrals with respect to σ-additive set functions and Daniell integrals. To do so, it suffices to replace the family of characteristic functions of sets belonging to $\mathcal{Q}$ by any sufficiently rich family, $\mathcal{K}$, of functions on Ω. It is assumed that a functional, ρ, to be called an integrating gauge, is given on $\mathcal{K}$ such that there exists a complete (semi)normed space $\mathcal{L}(\rho,\mathcal{K})$, consisting of functions on Ω such that the linear hull of $\mathcal{K}$ is dense in $\mathcal{L}(\rho,\mathcal{K})$ and the norm of $\mathcal{L}(\rho,\mathcal{K})$ is the smallest norm whose values on $\mathcal{K}$ coincide with those of ρ. The construction of $\mathcal{L}(\rho,\mathcal{K})$ is of course analogous to that of $\mathcal{L}(\rho,\mathcal{Q})$; it is briefly described in the pre-amble to Chapter 2. The definition of integral is sketched in the pre-amble to Chapter 3. Some of the possibilities inherent in this more general structure are exploited in Chapter 6 which deals with the spectral theory of operators. Not without interest may also be the fact, adverted to in Chapter 3, that many classical function spaces may be defined as instances of the space $\mathcal{L}(\rho,\mathcal{K})$, for suitable choices of $\mathcal{K}$ and ρ.

1. PRELIMINARIES, NOTATION, CONVENTIONS

Even though the notation and conventions adopted here are fairly standard, slight variations that occur in the literature can cause inconvenience to the reader. So, the problem of making the whole text sufficiently self-contained is solved by placing this chapter at the beginning. None-the-less the chapter can be used as an appendix, that is, the reader may refer to it only as the need arises. To facilitate such usage, frequent references to this one are made in the subsequent chapters.

A. The need to treat real and complex vector spaces separately will only seldom arise. Therefore, the real or complex numbers will be referred to simply as numbers or scalars.

To maintain the perspicuity of the notation pertaining to vector valued functions and integrals, the multiplication by scalars of elements of a vector space will be written commutatively. That is to say, if E is a vector space, we shall write interchangeably $cx = xc$, for any scalar c and a vector $x \in E$.

By a seminorm on a vector space E is meant a function $q : E \to [0,\infty)$ such that $q(x+y) \leq q(x) + q(y)$, for every $x \in E$ and $y \in E$, and $q(cx) = |c|\, q(x)$, for every $x \in E$ and a number c. So, a seminorm has all the properties of a norm with the only exception that its value may be equal to zero on a non-zero element of E.

The study of spaces of individual integrable functions, rather than those of the equivalence classes of such functions, makes it convenient to consider general seminormed and not just normed spaces. To be sure, a seminormed space is a vector space together with a specified seminorm on it. A majority of concepts referring to normed spaces are with obvious modifications applicable to seminormed spaces. The occasional difficulties are caused mainly by the non-uniqueness of limits and similar objects.

So, let E be a seminormed space with the seminorm q.

A set $S \subset E$ is called bounded if $\{q(x) : x \in S\}$ is a bounded set of numbers.

A set $S \subset E$ is dense in E if, for every $x \in E$ and $\epsilon > 0$, there exists an $y \in S$ such that $q(x-y) < \epsilon$.

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of E is said to be convergent if there exists an element x of E such that

$$\lim_{n\to\infty} q(x-x_n) = 0 .$$

In that case, x is said to be a limit of the sequence $\{x_n\}_{n=1}^{\infty}$. We write

$$x = \lim_{n\to\infty} x_n .$$

If y too is a limit of this sequence, then $q(x-y) = 0$.

A sequence, $\{x_n\}_{n=1}^{\infty}$, of elements of E is said to be Cauchy if, for every $\epsilon > 0$, there is a δ such that $q(x_n - x_n) < \epsilon$, for every $n > \delta$ and $m > \delta$.

If we want to be specific, we speak of q-bounded sets, q-convergent sequences, and so on.

The space E is said to be complete if every Cauchy sequence of its elements is convergent.

We shall reserve the term "Banach space" to denote a complete normed space. So, E is a Banach space if q is actually a norm, that is, the equality $q(x) = 0$ implies that $x = 0$, and if E is complete.

The norm of an unspecified Banach space will be mostly denoted as modulus.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of the seminormed space E is said to be conditionally (or simply) summable if the sequence $\{s_n\}_{n=1}^{\infty}$, where

$$s_n = \sum_{j=1}^{n} x_j$$

for every $n = 1,2,\ldots$, is convergent. If s is a limit of the sequence $\{s_n\}_{n=1}^{\infty}$, then we write

$$s = \sum_{j=1}^{\infty} x_j$$

and call the element s a sum of the sequence $\{x_j\}_{j=1}^{\infty}$.

The sequence $\{x_j\}_{j=1}^{\infty}$ is said to be unconditionally summable if, for all choices of $\epsilon_j = 0$ or 1, $j = 1,2,\ldots,$ the sequence $\{\epsilon_j x_j\}_{j=1}^{\infty}$ is conditionally summable.

The sequence $\{x_j\}_{j=1}^{\infty}$ is said to be absolutely summable if

$$\sum_{j=1}^{\infty} q(x_j) < \infty \tag{A.1}$$

and if it is summable.

The following two statements are designated as propositions with their own numbers only to give them prominence. Their proofs are of course omitted.

PROPOSITION 1.1. *The seminormed space E is complete if and only if every sequence, $\{x_j\}_{j=1}^{\infty}$, of its elements which satisfies condition* (A.1) *is summable.*

PROPOSITION 1.2. *Let E be a Banach space with the norm q. Let H be a dense vector subspace of E. Then every element, x, of the space E can be expressed as the sum of some elements, x_j, of H, $j = 1,2,\ldots,$ satisfying condition* (A.1). *Furthermore,*

$$q(x) = \inf \sum_{j=1}^{\infty} q(x_j)\ ,$$

where the infimum is taken over all expressions of x as the sum of elements x_j of H, $j = 1,2,\ldots,$ satisfying (A.1).

B. Let F be a vector space. Let $Q \subset F$; the set Q is not assumed to be a vector space.

The linear hull of Q will be denoted by $\mathrm{sim}(Q)$. That is, $x \in \mathrm{sim}(Q)$ if and only if there exist a (strictly) positive integer n, numbers c_j and elements, x_j, of Q, $j = 1,2,\ldots,n$, such that

$$x = \sum_{j=1}^{n} c_j x_j\ . \tag{B.1}$$

Elements of the space F that belong to $\mathrm{sim}(Q)$ are called Q-simple. This notation and terminology originated in elementary integration theory and will be mainly used in that context. (See Section D below.)

Let E be another vector space. A map $\mu : Q \to E$ will be called linear if

$$\sum_{j=1}^{n} c_j \mu(x_j) = 0$$

for any $n = 1,2,\ldots,$ numbers c_j and elements x_j of Q, $j = 1,2,\ldots,n$, such that

$$\sum_{j=1}^{n} c_j x_j = 0 .$$

A map $\mu : Q \to E$ is linear if and only if there exists a linear map $\tilde{\mu} : \mathrm{sim}(Q) \to E$ such that $\tilde{\mu}(x) = \mu(x)$ for every $x \in Q$. If it exists, such a linear map $\tilde{\mu}$ is unique. Therefore, following the custom, we shall not distinguish, in terminology and notation, between a linear map $\mu : Q \to E$ and the linear map on $\mathrm{sim}(Q)$ into E that extends μ.

If E is the one-dimensional vector space, that it, the space of scalars, then a linear map $\mu : Q \to E$ is called a linear function, or a linear functional. The vector space of all linear functions on the whole of F is called the algebraic dual space to F and denoted by F^*.

Assume now that E and F are seminormed spaces with the seminorms p and q, respectively. Then we can speak about the continuity of a map $\mu : F \to E$ at a point $x \in F$. To be sure, such a map is continuous at a point $x \in F$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $p(\mu(y) - \mu(x)) < \epsilon$ for every $y \in F$ for which $q(y-x) < \delta$.

As in the case of normed spaces, for a linear map $\mu : F \to E$, the following statements (i), (ii) and (iii), are equivalent:

(i) There is a point in F at which μ is continuous.

(ii) The map μ is continuous at every point of the space F.

(iii) There is a constant $k \geq 0$ such that $p(\mu(x)) \leq kq(x)$, for every $x \in F$.

So, it is quite unambiguous to say simply about a linear map on a (whole) vector space that it is continuous.

The vector space of all continuous linear functionals on a seminormed space F is called the continuous dual space to F, or just the dual of F, and denoted by F'.

If we define $q'(x') = \sup\{|x'(x)| : q(x) \leq 1\}$, for every $x' \in F'$, Then q' is a norm on F' which makes of F' a Banach space.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of a seminormed space F is said to be conditionally weakly summable if there exists an element s of F such that

$$\sum_{j=1}^{\infty} x'(x_j) = x'(s) ,$$

for every $x' \in F'$.

A sequence, $\{x_j\}_{j=1}^{\infty}$, of elements of a seminormed space is said to be unconditionally weakly summable if, for any choice of $\epsilon_j = 0$ or 1, $j = 1,2,\ldots$, the sequence $\{\epsilon_j x_j\}_{j=1}^{\infty}$, is conditionally weakly summable.

PROPOSITION 1.3. *Any unconditionally weakly summable sequence of elements of a seminormed space is unconditionally summable.*

This proposition is known in the literature as the Orlicz-Pettis lemma. A special case of it appeared in the early work of W. Orlicz on trigonometric series. However, the first published proof for an arbitrary Banach space is due to B.J. Pettis, [57]. Several other proofs were invented since; see, for example, [9], Corollary 4.4 and the remarks on p.34, and [23], Lemma 3.2.1 and Theorem 3.2.3. It is a matter of a mere routine to weaken the assumptions so as to allow an arbitrary seminormed space.

We are now going to modify a classical lemma of H. Hahn, see e.g. [23], Theorem 2.7.7, about the construction of a continuous linear functional from its values on a subset of a Banach space. The modification consists in relaxing the assumptions on the functional if the norm of the given Banach space satisfies a certain, rather stringent, condition. The condition says that it is the largest norm on the space with a given restriction on the given subset. So, the resulting proposition turns out to be rather trivial. However, it applies to the usual constructions of L^1-spaces, some of their generalizations, and to the projective tensor products of pairs of Banach spaces.

PROPOSITION 1.4. *Let F be a Banach space with the norm q and let $Q \subset F$. Assume that* $\mathrm{sim}(Q)$ *is dense in F and that, for every $x \in \mathrm{sim}(Q)$,* (B.2) $$q(x) = \inf \sum_{j=1}^{n} |c_j| q(x_j) ,$$

where the infimum is taken over all expressions of x *in the form* (B.1) *with arbitrary* $n = 1,2,\ldots,$ *numbers* c_j *and elements* $x_j \in Q$, $j = 1,2,\ldots,n$.

Let E *be a Banach space with the norm denoted as the modulus. Let* $\mu : Q \to E$ *be a linear map such that* $|\mu(x)| \leq q(x)$, *for every* $x \in Q$.

Then there exists a unique linear map $\tilde{\mu} : F \to E$ *such that* $\tilde{\mu}(x) = \mu(x)$, *for every* $x \in Q$, *and* $|\tilde{\mu}(x)| \leq q(x)$, *for every* $x \in F$.

Proof. Let $\mu_1 : \mathrm{sim}(Q) \to E$ be the unique linear extension of μ . Then

$$|\mu_1(x)| = \left| \sum_{j=1}^{n} c_j \mu(x_j) \right| \leq \sum_{j=1}^{n} |c_j| \, |\mu(x_j) \leq \sum_{j=1}^{n} |c_j| q(x_j)$$

for every $x \in \mathrm{sim}(Q)$ and every expression of x in the form (B.1). Consequently, by the assumption, $|\mu_1(x)| \leq q(x)$, for every $x \in \mathrm{sim}(Q)$. So, there exists a unique linear map $\tilde{\mu} : F \to E$ such that $\tilde{\mu}(x) = \mu_1(x)$, for every $x \in \mathrm{sim}(Q)$, and $|\tilde{\mu}(x)| \leq q(x)$, for every $x \in F$.

C. Let Ξ and Υ be any non-empty sets. Let $\Omega = \Xi \times \Upsilon$ be their Cartesian product. If f is a function on Ω with values in a given Banach space and $\xi \in \Xi$, then by $f(\xi,\cdot)$ is denoted the function on Υ whose value at any point $v \in \Upsilon$ is equal to $f(\xi,v)$. Similarly, for any given $v \in \Upsilon$, by $f(\cdot,v)$ is denoted the function on Ξ whose value at any $\xi \in \Xi$ is $f(\xi,v)$.

Now, let E , F and G be vector spaces. A map $b : E \times F \to G$ is said to be bilinear if, for every $x \in E$, the map $b(x,\cdot) : F \to G$ is linear and also, for every $y \in F$, the map $b(\cdot,y) : E \to G$ is linear. If G happens to be the space of scalars, we speak of a bilinear function.

Let $B(E,F)$ be the vector space of all bilinear functions on $E \times F$. Let $B^*(E,F)$ be its algebraic dual; that is, $B^*(E,F)$ is the vector space of all linear functions on $B(E,F)$.

For each $x \in E$ and $y \in F$, let $x \otimes y$ denote the linear function on $B(E,F)$ whose value at any element, b, of $B(E,F)$ is equal to $b(x,y)$. The map $(x,y) \mapsto x \otimes y$, $x \in E$, $y \in F$, is an injection of $E \times F$ into $B^*(E,F)$; it identifies $E \times F$ with a subset of $B^*(E,F)$ which we denote by Q. The vector space, $\mathrm{sim}(Q)$, spanned by Q is denoted by $E \otimes F$ and is called the tensor product of the spaces E and F. The map $(x,y) \mapsto x \otimes y$, $x \in E$, $y \in F$, is called the canonical bilinear map of $E \times F$ into $E \otimes F$.

It is immediate that (i) $c(x \otimes y) = (cx) \otimes y = x \otimes (cy)$, for any number c and vectors $x \in E$ and $y \in F$. Also (ii) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$, for any $x_1 \in E$, $x_2 \in E$ and $y \in F$; and, similarly (iii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, for any $x \in E$ and $y_1 \in F$ and $y_2 \in F$. So, an element, z, of $B^*(E,F)$ belongs to $E \otimes F$ if and only if there is an integer $n = 1,2,\ldots$ and vectors $x_j \in E$ and $y_j \in F$, $j = 1,2,\ldots,n$, such that

$$z = \sum_{j=1}^{n} x_j \otimes y_j . \tag{C.1}$$

Alternatively, the tensor product, $E \otimes F$, of the vector spaces E and F can be defined as the set of all formal linear combinations of the products $x \otimes y$, with $x \in E$ and $y \in F$, reduced so that the identities (i), (ii) and (iii) hold. More precisely, we define V to be the vector space whose basis is $E \times F$ and V_0 to be the subspace of V spanned by the elements of the form $(0,y)$, $(x,0)$, $(x_1+x_2,y) - (x_1,y) - (x_2,y)$, $(x,y_1+y_2) - (x,y_1) - (x,y_2)$, $(cx,y) - c(x,y)$ and $(x,cy) - c(x,y)$, with an arbitrary number c, vectors x, x_1 and x_2 in E and vectors y, y_1 and y_2 in F. Then the space $E \otimes F$ is isomorphic (as a vector space) with the quotient space V/V_0 under the linear map that associates any element $x \otimes y$ of $E \otimes F$, $x \in E$, $y \in F$, with the element $(x,y) + V_0$ of the space V/V_0.

Assume now that E and F are normed spaces with norms p and q, respectively. Let the norm r on $E \otimes F$ be defined by

$$r(z) = \inf \sum_{j=1}^{n} p(x_j) q(y_j) ,$$

for every $z \in E \otimes F$, where the infimum is taken over all expressions of z in the form

(C.1) with arbitrary $n = 1,2,\ldots,$ $x_j \in E$ and $y_j \in F$, $j = 1,2,\ldots,n$. Clearly, $r(x\otimes y) = p(x)q(y)$, for every $x \in E$ and $y \in F$. In fact, r is the largest norm on $E \otimes F$ having this property.

By $E \hat{\otimes} F$ is denoted the completion of the space $E \otimes F$ in the norm r. The Banach space $E \hat{\otimes} F$ is called the (complete) projective tensor product of the normed spaces E and F.

PROPOSITION 1.5. *For every element, z, of the complete projective tensor product, $E \hat{\otimes} F$, of the spaces E and F, there exist elements, x_j, of the space E and elements, y_j, of the space F, $j = 1,2,\ldots,$ such that*

$$\sum_{j=1}^{\infty} p(x_j)q(y_j) < \infty \tag{C.2}$$

and

$$z = \sum_{j=1}^{\infty} x_j \otimes y_j . \tag{C.3}$$

Moreover, the norm of z in the space $E \hat{\otimes} F$ is equal to the infimum of the numbers (C.2) *subject to the expression of z in the form* (C.3).

Proof. It follows directly from Proposition 1.2.

Let now G be a Banach space with the norm denoted as modulus. A bilinear map $b : E\times F \to G$ is continuous if and only if there is a constant $k \geq 0$ such that

$$|b(x,y)| \leq kp(x)q(y) , \tag{C.4}$$

for every $x \in E$ and $y \in F$.

PROPOSITION 1.6. *If $b : E\times F \to G$ is a continuous bilinear map, then there exists a unique continuous linear map $\mu : E \hat{\otimes} F \to G$ such that $\mu(x\otimes y) = b(x,y)$, for every $x \in E$ and $y \in F$. Furthermore, if* (C.4) *holds for every $z \in E$ and $y \in F$, then $|\mu(z)| \leq kr(z)$, for every $z \in E \hat{\otimes} F$.*

Proof. It follows from Proposition 1.4.

This is all that will be needed in the sequel about tensor products. For further general facts and facts concerning the relation of tensor products with vector integration, the interested reader is referred to [9], Chapter VIII.

D. We say that $\mathcal{K}$ is a nontrivial family of functions on a space Ω if Ω is a nonempty set and $\mathcal{K}$ is a set of scalar valued functions whose domain is Ω such that the zero function belongs to $\mathcal{K}$.

Any such nontrivial family, $\mathcal{K}$, is considered to be a subset of the vector space of all scalar valued functions on Ω. So, the symbol $\mathrm{sim}(\mathcal{K})$ has an unambiguous meaning introduced in Section B; viz., it denotes the linear hull of $\mathcal{K}$. Functions belonging to $\mathrm{sim}(\mathcal{K})$ are called $\mathcal{K}$-simple.

Clearly, $\mathcal{K}$ is a vector space if and only if $\mathrm{sim}(\mathcal{K}) = \mathcal{K}$. If $\mathcal{K}$ is a vector space whose elements are real-valued and if, with every function $f \in \mathcal{K}$, also the function $|f|$, that is, the function $\omega \mapsto |f(\omega)|$, $\omega \in \Omega$, belongs to $\mathcal{K}$, then $\mathcal{K}$ is called a vector lattice.

The notion of a $\mathcal{K}$-simple function is extended so as to permit consideration of vector valued functions. Namely, let $\mathcal{K}$ be a nontrivial family of functions on a space Ω and let E be a Banach space. By $\mathrm{sim}(\mathcal{K},E)$ is denoted the vector space spanned by all the E-valued functions cf, where $c \in E$ and $f \in K$. That is to say, $\mathrm{sim}(\mathcal{K},E)$ consists of all functions $f : \Omega \to E$ for which there exist a positive integer n, elements c_j of E and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,n$, such that

$$f = \sum_{j=1}^{n} c_j f_j .$$

Functions belonging to $\mathrm{sim}(\mathcal{K},E)$ are called $(\mathcal{K},E)$-simple.

To save subscripts and circumlocution, subsets of Ω will be identified with their characteristic functions. Accordingly, a family, $\mathcal{Q}$, of subsets of Ω is called a paving in Ω if it is a nontrivial family of functions on Ω, that is, characteristic functions of sets from $\mathcal{Q}$ a nontrivial family of functions on Ω. So, a family of subsets of Ω is a paving in Ω if it contains the empty set.

The paving $\mathcal{Q}$ is said to be multiplicative if it contains the intersection of any two of its members.

The paving $\mathcal{Q}$ in Ω is called a quasiring of sets in the space Ω if, for any sets X and Y belonging to $\mathcal{Q}$, the intersection $X \cap Y$ is equal to the union of a finite collection of pair-wise disjoint sets belonging to $\mathcal{Q}$ and also the difference $Y \setminus X$ is equal to the union of a finite collection of pair-wise disjoint sets from $\mathcal{Q}$.

The paving $\mathcal{Q}$ in Ω is called a semiring of sets in the space Ω if, for every $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$, there exist a positive integer n and pair-wise disjoint sets $Z_j \in \mathcal{Q}$, $j = 0,1,\dots,n$, such that

$$X \cap Y = Z_0 \,, \; Y \setminus X = \bigcup_{j=1}^{n} Z_j$$

and the union

$$\bigcup_{j=0}^{k} Z_j$$

belongs to $\mathcal{Q}$, for every $k = 0,1,\dots,n$. The notion of a semiring is due to J. von Neumann who uses the term half-ring; see [55], Definition 10.1.5. The importance of semirings will become apparent in the next section; cf., in particular, Proposition 1.9.

Every semiring is a quasiring, but it is not difficult to exhibit quasirings which are not semirings.

A quasiring of sets in Ω which contains the union of any finite collection of its members is called a ring of sets in the space Ω. A ring of sets which contains the union of any sequence of its members is called a σ-ring. A ring of sets which contains the intersection of any sequence of its members is called a δ-ring. A ring (quasiring, semiring, σ-ring) of sets in Ω which contains Ω as one of its members is called an algebra (quasialgebra, semialgebra, σ-algebra, respectively) of sets in the space Ω.

By a σ-ideal in the space Ω we understand a family of subsets of Ω that is closed under taking countable unions and subsets, that is it contains all the subsets of the union of any sequence of its elements. A family of sets with this property is in fact a σ-ideal of the Boolean algebra of all subsets of Ω ; so, this terminology represents a slight abuse of the language.

If $\mathcal{S}$ σ-algebra of sets in the space Ω , a function, f , on Ω is said to be $\mathcal{S}$-measurable, if every set of the form $\{\omega : f(\omega) \in U\}$, where U is an open set of scalars, belongs to $\mathcal{S}$.

The least σ-algebra of sets in a topological space Ω that contains all open sets is called the Borel σ-algebra in Ω ; its elements are called Borel sets. The least σ-algebra of sets in a topological space Ω that contains all sets of the form $\{\omega \in \Omega : f(\omega) \in U\}$, where f a real valued continuous function on Ω with compact support and U an open subset of $\mathbb{R}$, is called the Baire σ-algebra of sets in Ω ; its elements are called Baire sets.

If $\mathcal{Q}$ is a paving in the space Ω and $\Upsilon \subset \Omega$, $\Upsilon \neq \emptyset$, then the family $\mathcal{Q} \cap \Upsilon = \{X \cap \Upsilon : X \in \mathcal{Q}\}$ is a paving in the space Υ . If $\mathcal{Q}$ is a quasiring then so is $\mathcal{Q} \cap \Upsilon$. Similarly for a semiring, ring, algebra, σ-ring, δ-ring and σ-algebra.

If $\mathcal{Q}$ is a quasiring of sets in the space Ω then every $\mathcal{Q}$-simple function has an expression

$$f = \sum_{j=1}^{n} c_j X_j , \tag{D.1}$$

where the n is a positive integer, the c_j are numbers and the X_j are pair-wise disjoint sets belonging to $\mathcal{Q}$, $j = 1,2,\ldots,n$. The family, $\mathcal{R}$, of all sets belonging to $\mathrm{sim}(\mathcal{Q})$, that is, sets whose characteristic functions re $\mathcal{Q}$-simple, is the ring of sets generated by $\mathcal{Q}$. So, every element of $\mathcal{R}$ is equal to the union of a finite collection of pair-wise disjoint sets from $\mathcal{Q}$.

Let $\mathcal{Q}$ be an arbitrary paving in the space Ω . By $\Sigma(\mathcal{Q})$ will be denoted the set of all families of pair-wise disjoint non-empty sets belonging to $\mathcal{Q}$.

A family of sets, $\mathcal{P}$, belonging to $\Sigma(\mathcal{Q})$ is called a $\mathcal{Q}$-partition (of Ω), if the union of all sets that belong to $\mathcal{P}$ is equal to Ω and, for every $X \in \mathcal{Q}$, the sub-family,

$$\{Y \in \mathcal{P} : Y \cap X \neq \emptyset\} ,$$

of $\mathcal{P}$ consisting of sets having non-empty intersection with X, is finite. The set of all $\mathcal{Q}$-partitions is denoted by $\Pi(\mathcal{Q})$.

Let $\mathcal{P}_1 \in \Pi(\mathcal{Q})$ and $\mathcal{P}_2 \in \Pi(\mathcal{Q})$. If, for every set $Y \in \mathcal{P}_2$, there exists a (necessarily unique) set $X \in \mathcal{P}_1$ such that $Y \subset X$, we say that the partition $\mathcal{P}_2$ is a refinement of the partition $\mathcal{P}_1$ and write $\mathcal{P}_1 \prec \mathcal{P}_2$.

We say that a set $\Gamma \subset \Pi(\mathcal{Q})$ is directed (by the relation of refinement) if, for every $\mathcal{P}_1 \in \Gamma$ and $\mathcal{P}_2 \in \Gamma$, there exists a partition $\mathcal{P}_3 \in \Gamma$ such that $\mathcal{P}_1 \prec \mathcal{P}_3$ and $\mathcal{P}_2 \prec \mathcal{P}_3$.

If $\mathcal{Q}$ is a multiplicative quasiring, then the set, $\Pi(\mathcal{Q})$, of all partitions is directed.

If $\mathcal{Q}$ is an arbitrary paving and Γ is a directed subset of $\Pi(\mathcal{Q})$, then the paving

$$\mathcal{Q}_\Gamma = \{\emptyset\} \cup \bigcup_{\mathcal{P}\in\Gamma} \mathcal{P},$$

to which belong the empty set and all the sets forming the partitions belonging to Γ, is a multiplicative quasiring of sets.

E. Let E be a vector space.

If $\mathcal{K}$ is a nontrivial family of functions on a given space and $\mu : \mathcal{K} \to E$ a map, the question whether the map μ is linear or not has a meaning. Indeed, the notion of a linear map was introduced in Section B. If $\mathcal{K}$ satisfies some additional hypotheses, then it may be possible to simplify the condition of linearity. It is obviously so when $\mathcal{K}$ happens to be a vector space. Less obvious simplifications are possible for some kinds of pavings.

An E-valued map whose domain is a paving is usually called an E-valued set function. The real or complex valued set functions are referred to simply as set functions, and so are E-valued set functions whenever the space E is specified otherwise or irrelevant.

Let $\mathcal{Q}$ be a paving in a space Ω and $\mu : \mathcal{Q} \to E$ a set function. Let n be a positive integer. The set function μ is said to be n-additive if

$$\mu(X) = \sum_{j=1}^{n} \mu(X_j)$$

for any set $X \in \mathcal{Q}$ and pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\dots,n$, such that

$$X = \bigcup_{j=1}^{n} X_j .$$

If μ is n-additive, for every $n = 1,2,\dots,$ we say that it is additive.

PROPOSITION 1.7. *If $\mathcal{Q}$ is a quasiring of sets, then a set function $\mu : \mathcal{Q} \to E$ is linear if and only if it is additive.*

Proof. Any linear set function is additive. So, let $\mathcal{Q}$ be a quasiring of sets and $\mu : \mathcal{Q} \to E$ an additive set function. If a function $f \in \text{sim}(\mathcal{Q})$ is expressed in the form (D.1), let

$$\tilde{\mu}(f) = \sum_{j=1}^{n} c_j \mu(X_j) .$$

The additivity of μ implies that this definition is unambiguous. It is then straightforward that the resulting map $\tilde{\mu} : \text{sim}(\mathcal{Q}) \to E$ is linear and that $\tilde{\mu}(X) = \mu(X)$ for every $X \in \mathcal{Q}$.

This proposition implies that, if $\mathcal{Q}$ is a quasiring of sets and $\mathcal{R}$ is the ring of sets generated by $\mathcal{Q}$, then any additive set function $\mu : \mathcal{Q} \to E$ has a unique additive extension on the whole of $\mathcal{R}$; that is, there exists a unique additive set function $\tilde{\mu} : \mathcal{R} \to E$ such that $\tilde{\mu}(X) = \mu(X)$, for $X \in \mathcal{Q}$.

If $\mathcal{Q}$ happens to be a semiring, then the condition of linearity can be simplified still further.

PROPOSITION 1.8. *If $\mathcal{Q}$ is a semiring of sets, then a set function $\mu : \mathcal{Q} \to E$ is additive if and only if it is 2-additive.*

Proof. Let $\mathcal{Q}$ be a semiring of sets and $\mu : \mathcal{Q} \to E$ a 2-additive set function. As μ is trivially 1-additive, for an inductive proof, assume that $k \geq 1$ is an integer and that μ is k-additive.

Let $X \in \mathcal{Q}$ and let $X_i \in \mathcal{Q}$, $i = 0,1,2,\dots,k$, be pair-wise disjoint sets whose union is equal to X. By the definition of a semiring there exist a natural number m and pair-wise disjoint sets $Z_j \in \mathcal{Q}$, $j = 0,1,2,\dots,m$, such that $Z_0 = X \cap X_0 = X_0$,

$$\bigcup_{i=1}^{k} X_i = X \setminus X_0 = \bigcup_{j=1}^{m} Z_j$$

and, for every $l = 0,1,2,\dots,m$, the set

$$W_l = \bigcup_{j=0}^{l} Z_j$$

belongs to $\mathcal{Q}$. Then, clearly, $W_0 = X_0$, $W_l = W_{l-1} \cup Z_l$ and $W_{l-1} \cap Z_l = \emptyset$, for every $l = 1,2,\dots,m$, and $W_m = X$.

Now, by the 2-additivity of μ, we have $\mu(W_l) = \mu(W_{l-1}) + \mu(Z_l)$, for every $l = 1,2,\dots,m$. Therefore, $\mu(W_1) = \mu(W_0) + \mu(Z_1) = \mu(Z_0) + \mu(Z_1)$; $\mu(W_2) = \mu(W_1) + \mu(Z_2) = \mu(Z_0) + \mu(Z_1) + \mu(Z_2)$; and so on. Hence, by finite induction ending at $l = m$,

$$\text{(E.1)} \qquad \mu(X) = \mu(W_m) = \sum_{j=0}^{m} \mu(Z_j) .$$

Furthermore, for any $i = 1,2,\dots,k$, we have $X_i \cap W_0 = X_i \cap X_0 = \emptyset$, $X_i \cap W_l = (X_i \cap W_{l-1}) \cup (X_i \cap Z_l)$ and $(X_i \cap W_{l-1}) \cap (X_i \cap Z_l) = \emptyset$, for every $l = 1,2,\dots,m-1$, and $X_i \cap W_m = X_i \cap X = X_i$. Therefore, $\mu(X_i \cap W_l) = \mu(X_i \cap W_{l-1}) + \mu(X_i \cap Z_l)$, for every $l = 1,2,\dots,m$, and, hence, by finite induction,

$$\text{(E.2)} \qquad \mu(X_i) = \mu(X_i \cap W_m) = \sum_{j=1}^{m} \mu(X_i \cap Z_j)$$

for every $i = 1,2,\dots,k$.

On the other hand,

$$Z_j = \bigcup_{i=1}^{k} (X_i \cap Z_j)$$

for every $j = 1,2,\dots,k$, and the sets $X_i \cap Z_j$, $i = 1,2,\dots,k$, are pair-wise disjoint. Hence,

$$\mu(Z_j) = \sum_{i=1}^{k} \mu(X_i \cap Z_j) , \tag{E.3}$$

for every $j = 1,2,\ldots,m$, because, by the assumption, the set function μ is k-additive.

So, by (E.1), (E.2) and (E.3),

$$\mu(X) = \sum_{j=0}^{m} \mu(Z_j) = \mu(Z_0) + \sum_{j=1}^{m} \mu(Z_j) =$$

$$= \mu(X_0) + \sum_{j=1}^{m} \sum_{i=1}^{k} \mu(X_i \cap Z_j) = \mu(X_0) + \sum_{i=1}^{k} \sum_{j=1}^{m} \mu(X_i \cap Z_j) =$$

$$= \mu(X_0) + \sum_{i=1}^{k} \mu(X_i) = \sum_{i=0}^{k} \mu(X_i) .$$

That is, μ is $(k+1)$-additive.

It may be interesting to note that this proposition does not hold for quasirings instead of semirings.

EXAMPLE 1.9. Let $\Omega = \{1,2,3\}$ and let $\mathcal{Q} = \{\emptyset,\{1\},\{2\},\{3\},\Omega\}$. Then $\mathcal{Q}$ is a quasiring of sets in the space Ω . Let $\mu(\emptyset) = 0$, $\mu(\{1\}) = \mu(\{2\}) = \mu(\{3\}) = \mu(\Omega) = 1$. Then obviously, $\mu(X) = \mu(Y) + \mu(Z)$, for any sets X , Y and Z belonging to $\mathcal{Q}$, such that $Y \cap Z = \emptyset$ and $X = Y \cup Z$. However, μ is not additive.

The surprisingly nontrivial Proposition 1.8 expresses a property of semirings that makes them preferable to quasirings. It is due to J. von Neumann, [55], Theorem 10.1.12; see also [19], Exercise 5 in §7. However, some naturally occurring pavings in torus-like spaces are only quasirings.

F. Let $\mathcal{Q}$ be a paving in a space Ω . Let E be a normed vector space. A set function $\mu : \mathcal{Q} \to E$ is said to be σ-additive if

$$\mu(X) = \sum_{j=1}^{\infty} \mu(X_j)$$

for any set $X \in \mathcal{Q}$ and pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that

$$X = \bigcup_{j=1}^{\infty} X_j .$$

PROPOSITION 1.10. *Let $\mathcal{Q}$ be a quasiring of sets and $\mathcal{R}$ the ring generated by $\mathcal{Q}$. Let $\mu : \mathcal{Q} \to E$ be an additive set function and $\tilde{\mu} : \mathcal{R} \to E$ its additive extension.*

The set function $\tilde{\mu}$ is σ-additive if and only if μ is σ-additive.

Proof. It follows directly from the fact that every set in $\mathcal{R}$ can be written as the union of a finite collection of a pair-wise disjoint sets from $\mathcal{Q}$.

Demonstration of the σ-additivity of a given set function may not be a simple matter, not even if the set function is scalar valued. In fact, the problem of σ-additivity of vector valued set functions is often reduced, *via* the Orlicz-Pettis lemma, say, to the problem of σ-additivity of some scalar valued set functions and even positive real valued ones. The basic source of positive σ-additive set functions is the theorem of A.D. Alexandrov; see [14], Theorem III.5.13 and the remarks in Section III.15 (p.233), and also [55], Theorem 10.1.20. Because of its importance, we present here an elementary proof of an extended and, at the same time, simplified version of this theorem.

A paving $\mathcal{C}$ is called compact if

(F.1) $$\bigcap_{n=1}^{\infty} C_n \neq \emptyset$$

for any sets $C_n \in \mathcal{C}$, $n = 1,2,\ldots$, such that

(F.2) $$\bigcap_{n=1}^{k} C_n \neq \emptyset ,$$

for every $k = 1,2,\ldots$.

More appropriately, instead of "compact", we should have used - as some authors actually do - the term "semicompact" or "sequentially compact". The proof of the following lemma is taken from [56], Lemma I.6.1.

LEMMA 1.11. *Let $\mathcal{C}$ be a compact paving. Let $\mathcal{D}$ be a paving whose elements are the unions of finite collections of sets from $\mathcal{C}$. Then the paving $\mathcal{D}$ too is compact.*

Proof. Let $D_n \in \mathcal{D}$, $n = 1,2,\ldots$, be sets such that

$$\bigcap_{n=1}^{k} D_n \neq \emptyset \tag{F.3}$$

for every $k = 1,2,\ldots$. The proof will be accomplished if we show that the intersection of all the sets D_n, $n = 1,2,..$, is not empty.

For every $n = 1,2,\ldots$, let m_n be a natural number and C_n^j, $j = 1,2,\ldots,m_n$, sets from $\mathcal{C}$ such that

$$D_n = \bigcup_{j=1}^{m_n} C_n^j .$$

Let $M_n = \{1,2,\ldots,m_n\}$, for every $n = 1,2,\ldots$. Let J be the set of all sequences $\iota = \{\iota_n\}_{n=1}^{\infty}$ such that $\iota_n \in M_n$ for every $n = 1,2,\ldots$. Finally, for every $k = 1,2,\ldots$, let J_k be the set of all sequences $\iota \in J$ such that

$$\bigcap_{n=1}^{k} C_n^{\iota_n} \neq \emptyset . \tag{F.4}$$

It then follows immediately that,

(i) if $\iota \in J_k$, $\kappa \in J$ and $\kappa_n = \iota_n$, for every $n = 1,2,\ldots,k$, then $\kappa \in J_k$;

(ii) if p and q are natural numbers such that $p \leq q$, then $J_q \subset J_p$.

Moreover, by the distributive law,

$$\bigcap_{n=1}^{k} D_n = \bigcup_{\iota \in J} \left[\bigcap_{n=1}^{k} C_n^{\iota_n} \right] .$$

Therefore, by (F.3), (F.4) holds for at least one $\iota \in J$. So,

(iii) $J_k \neq \emptyset$ for every $k = 1,2,\ldots$.

Our next aim is to prove that there exists a sequence $\iota \in J$ which belongs to J_k for every $k = 1,2,\ldots$. Such a sequence is constructed inductively.

First, using (iii), for every $k = 1,2,\ldots,$ we fix an element ι^k of the set J_k.

(I) Because the first terms, ι_1^k, of the sequences ι^k, $k = 1,2,\ldots,$ all belong to the finite set M_1, there exist an element ι_1 of M_1 such that the set, S_1, of all natural numbers k for which $\iota_1^k = \iota_1$, is infinite.

(II) Assume that p is a natural number and that for every $n = 1,2,\ldots,p$, ι_n is an element of M_n such that the set, S_p, of all natural numbers k such that $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, is infinite. Because the $(p+1)$-st terms, ι_{p+1}^k, of the sequences ι^k, $k = 1,2,\ldots,$ belong to the finite set M_{p+1}, there exists an element ι_{p+1} of M_{p+1} such that the set S_{p+1}, of those elements k of the set S_p for which $\iota_{p+1} = \iota_{p+1}^k$, is infinite. Then $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, $p+1$, whenever $k \in S_{p+1}$.

So, a sequence $\iota = \{\iota_n\}_{n=1}^{\infty}$ is constructed such that, for every $p = 1,2,\ldots,$ the set S_p of natural numbers k such that $\iota_n = \iota_n^k$, for every $n = 1,2,\ldots,p$, is infinite. Consequently, for every natural number p, there exists a natural number $q \geq p$ such that $\iota_n = \iota_n^q$, for every $n = 1,2,\ldots,p$. But then, by (ii), $\iota^q \in J_p$. Hence, by (i), $\iota \in J_p$. Because the constructed sequence, ι, belongs to J_k, (F.4) holds for every $k = 1,2,\ldots$. Consequently,

$$\bigcap_{n=1}^{\infty} C_n^{\iota_n} \neq \emptyset ,$$

because the paving $\mathcal{C}$ is compact, and the intersection of the sets D_n, $n = 1,2,\ldots,$ cannot be empty either.

Let μ be a non-negative real valued additive set function on $\mathcal{Q}$ and $\mathcal{C}$ a paving in Ω. The set function μ is said to be $\mathcal{C}$-regular if, for every $X \in \mathcal{Q}$ and every $\epsilon > 0$, there exist a set $C \in \mathcal{C}$ and a set $Y \in \mathcal{Q}$ such that

$$Y \subset C \subset X \text{ and } \mu(X) - \mu(Y) < \epsilon .$$

PROPOSITION 1.12. *Let $\mathcal{Q}$ be a quasiring of sets and $\mathcal{C}$ a compact paving in the space Ω. Any $\mathcal{C}$-regular non-negative real valued additive set function on $\mathcal{Q}$ is σ-additive.*

Proof. Let μ be such a set function. Without a loss of generality we will assume that $\mathcal{Q}$ is a ring of sets. For, if it is not the case, let $\tilde{\mu}$ be the additive extension of μ on the ring, $\mathcal{R}$, generated by $\mathcal{Q}$, and $\mathcal{D}$ the paving consisting of the unions of all finite collections of sets from $\mathcal{C}$. Then $\tilde{\mu}$ is, obviously, $\mathcal{D}$-regular, because every set from $\mathcal{R}$ is the union of a finite collection of sets from $\mathcal{Q}$, and, by Lemma 1.11, the paving $\mathcal{D}$ is compact.

So, let $X_n \in \mathcal{Q}$ be sets such that $X_n \supset X_{n+1}$ and $\mu(X_n) \geq \alpha$, for some $\alpha > 0$, $n = 1,2,\dots$. Let $C_n \in \mathcal{C}$ and $Y_n \in \mathcal{Q}$ be sets such that

$$Y_n \subset C_n \subset X_n \text{ and } \mu(X_n) - \mu(Y_n) < 2^{-n}\alpha,$$

$n = 1,2,\dots$. Let

$$Z_k = \bigcap_{n=1}^{k} Y_n,$$

for every $k = 1,2,\dots$. Then, by the assumption that $\mathcal{Q}$ is a ring, $Z_k \in \mathcal{Q}$, and

$$\mu(X_k) - \mu(Z_k) \leq \sum_{n=1}^{k} (\mu(X_n) - \mu(Y_n)) < \alpha,$$

so that $Z_k \neq \emptyset$ and (F.2) holds for every $k = 1,2,\dots$. By the compactness of $\mathcal{C}$, (F.1) holds, and, consequently,

$$\bigcap_{n=1}^{\infty} X_n \neq \emptyset,$$

which implies the σ-additivity of μ, because $\mathcal{Q}$ is a ring of sets.

G. By a Young function we shall understand a real valued function, Φ, on the interval $[0,\infty)$ that is continuous, strictly increasing and convex and satisfies the conditions

$$\lim_{t\to 0+} \frac{\Phi(t)}{t} = 0 \quad\text{and}\quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty.$$

It follows that $\Phi(0) = 0$ and $\Phi(t) > 0$ for $t > 0$.

Proofs of the following two propositions can be found in [38], I.1.5 and I.2.2, respectively.

PROPOSITION 1.13. *A function,* Φ *, on* $[0,\infty)$ *is a Young function if and only if there exists a non-decreasing function,* φ *, on* $[0,\infty)$ *such that* $\varphi(0) = 0$ *,* $\varphi(s) > 0$ *for* $s > 0$ *,* $\varphi(s) \to \infty$ *as* $s \to \infty$ *, and*

$$\Phi(t) = \int_0^t \varphi(s)\mathrm{d}s \,, \tag{G.1}$$

for every $t \geq 0$ *. Moreover, if* φ *is right-continuous at every point of the interval* $[0,\infty)$ *, then it is unique.*

The Young function, Φ , is said to satisfy condition (Δ_2) for large values of the argument if there exist numbers $k > 0$ and $a \geq 0$ such that

$$\Phi(2t) \leq k\Phi(t) \,, \tag{G.2}$$

for every $t \in [a,\infty)$.

The Young function, Φ , is said to satisfy condition (Δ_2) for small values of the argument if there exist numbers $k > 0$ and $a > 0$ such that (G.2) holds for every $t \in [0,a]$.

If a Young function satisfies condition (Δ_2) for small and also for large values of the argument, we say that it satisfies condition (Δ_2) .

Let Φ be a Young function. The function Ψ defined by

$$\Psi(t) = \sup\{st - \Phi(s) : s \geq 0\} \,,$$

for every $t \geq 0$, that is, the Legendre transform of Φ , is called the function complementary to Φ .

PROPOSITION 1.14. *Let* Φ *be a Young function and let* φ *be the right-continuous function in* $[0,\infty)$ *such that* (G.1) *holds for every* $t \geq 0$ *. Let*

$$\psi(t) = \sup\{s : \varphi(s) \leq t\}$$

for every $t \in [0,\infty)$ *. Then the function* Ψ *, complementary to* Φ *, is given by*

$$\Psi(t) = \int_0^t \psi(s)\mathrm{d}s ,$$

for every $t \geq 0$.

The function Ψ , complementary to a Young function, Φ , is again a Young function and the function complementary to Ψ is Φ . If Φ and Ψ is a pair of mutually complementary Young functions, then the inequality, called the Young inequality,

$$st \leq \Phi(s) + \Psi(t)$$

holds for every $s \geq 0$ and $t \geq 0$.

Given a Young function, Φ , and an integer $n \geq 1$, let

$$M_\Phi(x) = \sum_{j=1}^{n} \Phi(|x_j|) ,$$

for every vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{C}^n$.

The following proposition is known; its proof can be found, for example, in [51], 3.32. It is of course a special case of an inequality valid in general Orlicz spaces. (See Section 3C below.)

PROPOSITION 1.15. *For every vector* $x \in \mathbb{C}^n$, *let*

$$\|x\|_\Phi = \inf\{k : k>0 , M_\Phi(k^{-1}x) \leq 1\}$$

and, if $x = (x_1, x_2, \ldots, x_n)$,

$$\|x\|_\Phi^0 = \sup\{|\sum_{j=1}^{n} x_j y_j| : y = (y_1, y_2, \ldots, y_n) \in \mathbb{C}^n , M_\Psi(y) \leq 1\} ,$$

where Ψ *is the function complementary to* Φ .

Then the functions $x \mapsto \|x\|_\Phi$ *and* $x \mapsto \|x\|_\Phi^0$, $x \in \mathbb{C}^n$, *are norms on* $\mathbb{C}^n$, *each making of* $\mathbb{C}^n$ *a Banach space, such that*

$$\|x\|_\Phi \leq \|x\|_\Phi^0 \leq 2\|x\|_\Phi ,$$

for every $x \in \mathbb{C}^n$.

2. INTEGRATING GAUGES

An integration theory involves two constructions, namely that of the space of integrable function and that of the integral. These two constructions are often carried out simultaneously. However, having in mind the generalizations pursued here, it is desirable to keep them at least conceptually separated. In this chapter, spaces of integrable functions are introduced; integrals will be dealt with in the next one.

We start with a family of functions, $\mathcal{K}$, defined on a space Ω, which contains the zero-function but is not necessarily a vector space, and a non-negative real valued functional, ρ, on $\mathcal{K}$, called a gauge, such that $\rho(0) = 0$. Then we introduce the vector space $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$ of functions, f, on Ω which can be expressed in the form

$$(*) \qquad f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for all $\omega \in \Omega$ subject to certain exceptions, where c_j are numbers and f_j functions belonging to $\mathcal{K}$, $j = 1,2,\dots$, such that

$$(\ddagger) \qquad \sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty .$$

The equality $(*)$ is not required to hold for those points $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| = \infty ,$$

even if the sum on the right in $(*)$ exists as the limit of the sequence of partial sums; the values of f at such points are arbitrary. For the seminorm, $q(f)$, of such a function f we take the infimum of the numbers $(\ddagger)$. The space $\mathcal{L}$ is complete in this seminorm and the linear hull of $\mathcal{K}$ is dense in it. Of course, to avoid the obvious pathology that the seminorm of some functions $f \in \mathcal{K}$ with $\rho(f) > 0$ collapses to 0, some conditions have to be imposed on the gauge ρ. Accordingly, the gauge ρ is called integrating if $q(f) = \rho(f)$, for every function $f \in \mathcal{K}$.

If $\mathcal{K}$ is the family of characteristic functions of sets from a σ-algebra, say, and ρ is a measure on it, then this construction gives us precisely the family of functions integrable with respect to ρ and the corresponding seminorm of convergence in mean. Similarly, if $\mathcal{K}$ is a vector lattice and $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$, where ι is a Daniell integral on $\mathcal{K}$, then $\mathcal{L}$ is the family of all ι-integrable functions. Other choices of $\mathcal{K}$ and ρ lead to other classical and less classical spaces some of which will be described in the next chapter.

A. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω. (See Section 1D.) A non-negative real valued functional ρ on $\mathcal{K}$ such that $\rho(0) = 0$ will be called a gauge on $\mathcal{K}$. Good examples of gauges to keep in mind, in what follows, are seminorms on vector spaces of functions and (finite) non-negative additive, or just sub-additive, set functions on quasirings of sets. (Recall that we identify sets with their characteristic functions.)

The following definition can be viewed as the abstract core of the construction of the space of integrable functions and its L^1-seminorm from a given elementary measure or content.

Let ρ be a gauge on the family of functions $\mathcal{K}$. A function f on Ω will be called integrable with respect to ρ, or, briefly, ρ-integrable, if there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

(A.1) $$\sum_{j=1}^{\infty} |c_j|\rho(f_j) < \infty$$

and

(A.2) $$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega)$$

for every $\omega \in \Omega$ for which

(A.3) $$\sum_{j=1}^{\infty} |c_j f_j(\omega)| < \infty .$$

The family of all (individual) functions integrable with respect to ρ is denoted by $\mathcal{L}(\rho,\mathcal{K})$.

For any function $f \in \mathcal{L}(\rho,\mathcal{K})$, let

$$q_\rho(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(f_j) ,$$

where the infimum is taken over all choices of the numbers c_j and the functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.1), such that the equality (A.2) holds for every $\omega \in \Omega$ for which the inequality (A.3) does.

Clearly, $\mathcal{L}(\rho,\mathcal{K})$ is a vector space such that $\mathrm{sim}(\mathcal{K}) \subset \mathcal{L}(\rho,\mathcal{K})$. (See Section 1D.) Also, it is not difficult to see that q_ρ is a seminorm on $\mathcal{L}(\rho,\mathcal{K})$; it is called the seminorm generated by the gauge ρ. Consequently, we can speak of q_ρ-Cauchy and q_ρ-convergent sequences of functions from $\mathcal{L}(\rho,\mathcal{K})$.

The ρ-equivalence class of a function $f \in \mathcal{L}(\rho,\mathcal{K})$, consisting of all functions $g \in \mathcal{L}(\rho,\mathcal{K})$ such that $q_\rho(f-g) = 0$, is denoted by $[f]_\rho$. The set $\{[f]_\rho : f \in \mathcal{L}(\rho,\mathcal{K})\}$ of all ρ-equivalence classes of functions from $\mathcal{L}(\rho,\mathcal{K})$ is denoted by $L(\rho,\mathcal{K})$. Then $L(\rho,\mathcal{K})$ is a normed space with respect to the linear operations induced by those of $\mathcal{L}(\rho,\mathcal{K})$ and the norm induced by the seminorm q_ρ. This norm is still denoted by q_ρ.

It is sometimes useful, even necessary, to indicate the domain, $\mathcal{K}$, of the gauge ρ not only in the symbol of the space $\mathcal{L}(\rho,\mathcal{K})$ but also in the symbol for its seminorm. Then, instead of q_ρ, we write more precisely $q_{\rho,\mathcal{K}}$. In fact, it is customary not to distinguish in the notation between a gauge ρ on $\mathcal{K}$ and its restriction to a nontrivial subfamily, $\mathcal{J}$, of $\mathcal{K}$. But then $\mathcal{L}(\rho,\mathcal{J}) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_{\rho,\mathcal{K}}(f) \le q_{\rho,\mathcal{J}}(f)$ for every $f \in \mathcal{L}(\rho,\mathcal{J})$. What is more, the inclusion may be strict and, for some functions $f \in \mathcal{L}(\rho,\mathcal{J})$, the inequality may be strict too.

PROPOSITION 2.1. *Let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ be functions such that*

$$\text{(A.4)} \qquad \sum_{j=1}^{\infty} q_\rho(f_j) < \infty$$

and let f be a function on Ω such that

$$\text{(A.5)} \qquad f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\text{(A.6)} \qquad \sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then $f \in \mathcal{L}(\rho,\mathcal{K})$ *and*

(A.7) $$\lim_{n\to\infty} q_\rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Proof. For every $j = 1,2,\ldots,$ let c_{jk} be numbers and $f_{jk} \in \mathcal{K}$ functions, $k = 1,2,\ldots,$ such that

$$\sum_{k=1}^{\infty} |c_{jk}|\rho(f_{jk}) < q_\rho(f_j) + 2^{-j}$$

and

$$f_j(\omega) = \sum_{k=1}^{\infty} c_{jk}f_{jk}(\omega)$$

for every $\omega \in \Omega$ such that

$$\sum_{k=1}^{\infty} |c_{jk}f_{jk}(\omega)| < \infty .$$

Then, for any $n = 0,1,2,\ldots,$

$$\sum_{j=n+1}^{\infty}\sum_{k=1}^{\infty} |c_{jk}|\rho(f_{jk}) < \sum_{j=n+1}^{\infty} q_\rho(f_j) + 2^{-n} < \infty$$

and

$$f(\omega) - \sum_{j=1}^{n} f_j(\omega) = \sum_{j=n+1}^{\infty}\sum_{k=1}^{\infty} c_{jk}f_{jk}(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=n+1}^{\infty}\sum_{k=1}^{\infty} |c_{jk}f_{jk}(\omega)| < \infty .$$

Therefore, the function

$$f - \sum_{j=1}^{n} f_j$$

belongs to $\mathcal{L}(\rho,\mathcal{K})$ and

$$q_\rho\left[f - \sum_{j=1}^{n} f_j\right] < \sum_{j=n+1}^{\infty} q_\rho(f_j) + 2^{-n}$$

for every $n = 0,1,2,\ldots$.

The most important implication of this proposition is, of course, that the space $\mathcal{L}(\rho,\mathcal{K})$ is q_ρ-complete so that $L(\rho,\mathcal{K})$ is a Banach space.

B. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω and let ρ be a gauge on $\mathcal{K}$.

A function f on Ω is said to be ρ-null if $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$. A set $X \subset \Omega$ is said to be ρ-null if its characteristic function is ρ-null. The family of all ρ-null sets is denoted by $\mathcal{Z}_\rho$. We shall use the customary jargon related to null sets. So, for example, we refer to a ρ-null set by saying that ρ-almost all points of Ω belong to its complement.

The next proposition says, among other things, that $\mathcal{Z}_\rho$ is a σ-ideal in the space Ω. (See Section 1D.)

PROPOSITION 2.2. *A function f is ρ-null if and only if the set $\{\omega \in \Omega : f(\omega) \neq 0\}$ is ρ-null.*

If the function f is ρ-null, then there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.1), *such that*

(B.1)
$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| = \infty$$

for every $\omega \in \Omega$ for which $f(\omega) \neq 0$.

Conversely, if there exist functions $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ satisfying condition (A.4), *such that*

(B.2)
$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every $\omega \in \Omega$ for which $f(\omega) \neq 0$, then the function f is ρ-null.

If X_j, $j = 1,2,\ldots,$ are ρ-null sets and

$$X \subset \bigcup_{j=1}^{\infty} X_j,$$

then the set X too is ρ-null.

Proof. Let X be a ρ-null set. Then, by the definition of $\mathcal{L}(\rho,\mathcal{K})$ and q_ρ, for every $k = 1,2,\ldots,$ there exist numbers c_{kn} and functions $f_{kn} \in \mathcal{K}$, $n = 1,2,\ldots,$ such that

$$\sum_{n=1}^{\infty} |c_{kn}|\rho(f_{kn}) < 2^{-k}$$

and

$$\sum_{n=1}^{\infty} |c_{kn}f_{kn}(\omega)| \geq 1 ,$$

for every $\omega \in X$. Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |c_{kn}|\rho(f_{kn}) < \infty$$

and

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |c_{kn}f_{kn}(\omega)| = \infty$$

for every $\omega \in X$. So there exist numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,...$, satisfying condition (A.1), such that (B.1) holds for every $\omega \in X$.

Let function g be ρ-null. Let f be the characteristic function of the set $\{\omega : g(\omega) \neq 0\}$. Then the function $f_j = jg$ is ρ-null and $q_\rho(f_j) = 0$, for every $j = 1,2,...$. Hence, condition (A.4) is satisfied and the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Therefore, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$.

Let f be a function such that the set $\{\omega : f(\omega) \neq 0\}$ is ρ-null. Let f_1 be the characteristic function of this set and let $f_j = jf_1$, for every $j = 1,2,3,...$. Then $q_\rho(f_j) = 0$, for every $j = 1,2,...$, and so, condition (A.4) is satisfied. Furthermore, the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. So, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$.

Now, let f be a function on Ω and let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,...$, be functions satisfying condition (A.4), such that the equality (B.2) holds for every $\omega \in \Omega$ for which $f(\omega) \neq 0$. Then, for every $n = 1,2,...$,

$$\sum_{j=n}^{\infty} q_\rho(f_j) + \sum_{j=n}^{\infty} q_\rho(-f_j) < \infty$$

and

$$f(\omega) = \sum_{j=n}^{\infty} f_j(\omega) + \sum_{j=n}^{\infty} (-f_j(\omega)) = 0$$

for every $\omega \in \Omega$ for which

$$\sum_{j=n}^{\infty} |f_j(\omega)| + \sum_{j=n}^{\infty} |-f_j(\omega)| < \infty .$$

Then, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$ so that f is ρ-null.

C. The following theorem is the Beppo Levi theorem stated in terms of absolute summability rather than monotone convergence.

THEOREM 2.3. *A function* f *on* Ω *belongs to* $\mathcal{L}(\rho,\mathcal{K})$ *if and only if there exist numbers* c_j *and functions* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *satisfying condition* (A.1), *such that the equality* (A.2) *holds for* ρ-*almost every* $\omega \in \Omega$.

Let $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ *be functions satisfying condition* (A.4). *Then the inequality* (A.6) *holds for* ρ-*almost every* $\omega \in \Omega$. *If, moreover,* f *is a function on* Ω *such that the equality* (A.5) *holds for* ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K})$ *and the equality* (A.7) *holds.*

Proof. It is a direct consequence of Proposition 2.1 and Proposition 2.2.

In the terminology of N. Aronszajn and K.T. Smith, [1], the following theorem says that $\mathcal{L}(\rho,\mathcal{K})$ is a complete normed functional space, in fact, it is a functional completion of $\operatorname{sim}(\mathcal{K})$.

THEOREM 2.4. *A function* f *on* Ω *belongs to* $\mathcal{L}(\rho,\mathcal{K})$ *if and only if there exists a* q_ρ-*Cauchy sequence of functions* $h_n \in \operatorname{sim}(\mathcal{K})$, $n = 1,2,\ldots,$ *such that*

(C.1)
$$f(\omega) = \lim_{n\to\infty} h_n(\omega)$$

for ρ-*almost every* $\omega \in \Omega$.

Every q_ρ-*Cauchy sequence of functions* $g_n \in \mathcal{L}(\rho,\mathcal{K})$, $n = 1,2,\ldots,$ *has a subsequence,* $\{h_n\}_{n=1}^\infty$, *such that the sequence of numbers* $\{h_n(\omega)\}_{n=1}^\infty$ *is convergent for* ρ-*almost every* $\omega \in \Omega$. *Moreover, if* $\{h_n\}_{n=1}^\infty$ *is such a subsequence of* $\{g_n\}_{n=1}^\infty$ *and* f *is a function on* Ω *such that the equality* (C.1) *holds for* ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K})$ *and*

(C.2)
$$\lim_{n\to\infty} q_\rho(f-g_n) = 0 .$$

Proof. If the sequence, $\{g_n\}_{n=1}^{\infty}$, of functions from $\mathcal{L}(\rho,\mathcal{K})$ is q_ρ-Cauchy, we can select a subsequence $\{h_n\}_{n=1}^{\infty}$ such that

$$\text{(C.3)} \qquad \sum_{j=1}^{\infty} q_\rho(h_{j+1}-h_j) < \infty .$$

Then Theorem 2.3, applied to the functions f_j such that $f_1 = h_1$ and $f_{j+1} = h_{j+1} - h_j$ for $j = 1,2,\ldots$, implies that the sequence $\{h_n(\omega)\}_{n=1}^{\infty}$ is convergent for ρ-almost every $\omega \in \Omega$.

Now, if $\{h_n\}_{n=1}^{\infty}$ is a subsequence of $\{g_n\}_{n=1}^{\infty}$ such that the sequence $\{h_n\}_{n=1}^{\infty}$ is convergent for ρ-almost every $\omega \in \Omega$, we can achieve, by passing to a subsequence of $\{h_n\}_{n=1}^{\infty}$, if necessary, that (C.3) holds. Then, if (C.1) holds for ρ-almost every $\omega \in \Omega$, by Theorem 2.3, $f \in \mathcal{L}(\rho,\mathcal{K})$ and

$$\lim_{n\to\infty} q_\rho(f-h_n) = 0 .$$

Because $\{h_n\}_{n=1}^{\infty}$ is a subsequence of the q_ρ-Cauchy sequence $\{g_n\}_{n=1}^{\infty}$, (C.2) holds.

COROLLARY 2.5. *Let $\mathcal{J}$ be a q_ρ-complete vector space, containing every ρ-null function, such that $\mathcal{K} \subset \mathcal{J} \subset \mathcal{L}(\rho,\mathcal{K})$. Then $\mathcal{J} = \mathcal{L}(\rho,\mathcal{K})$.*

D. Theorems 2.3 and 2.4 demonstrate the usefulness of the space $\mathcal{L}(\rho,\mathcal{K})$ and its seminorm q_ρ. But this usefulness could be limited by the fact that, in general, we can only say that $q_\rho(f) \le \rho(f)$, for every $f \in \mathcal{K}$, and the inequality may be sharp for some f even if $\mathcal{K}$ is a vector space and ρ is a seminorm on it.

EXAMPLE 2.6. Let $\Omega = (0,1]$, $\mathcal{Q} = \{(u,v] : 0 \le u \le v \le 1\}$, $\mathcal{K} = \mathrm{sim}(\mathcal{Q})$ and

$$\rho(f) = \lim_{t\to 0+} |f(t)| ,$$

for every $f \in \mathcal{K}$. Then every function on Ω belongs to $\mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = 0$ for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

So, of particular interest are the gauges singled out in the following definition.

We shall call the gauge ρ integrating if $q_\rho(f) = \rho(f)$ for every function f belonging to its domain, $\mathcal{K}$.

Obviously, if a gauge on a vector space is integrating, then it is a seminorm. A seminorm which is an integrating gauge will of course be called an integrating seminorm.

PROPOSITION 2.7. *The gauge ρ is integrating if and only if*

$$\rho(f) \leq \sum_{j=1}^{\infty} |c_j| \rho(f_j)$$

for any function $f \in \mathcal{K}$, numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that the equality (A.2) *holds for every $\omega \in \Omega$ for which the inequality* (A.3) *does.*

Let ρ be an integrating gauge and let $\mathcal{J}$ be a nontrivial subfamily of its domain, $\mathcal{K}$. Then the restriction, σ, of ρ to $\mathcal{J}$ is an integrating gauge, $\mathcal{L}(\sigma,\mathcal{J}) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) \leq q_\sigma(f)$, for every $f \in \mathcal{L}(\sigma,\mathcal{J})$.

If ρ is any gauge on a nontrivial family of functions, $\mathcal{K}$, then the functional q_ρ is an integrating seminorm on $\mathcal{L}(\rho,\mathcal{K})$ such that $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$ and $q_{q_\rho}(f) = q_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

Proof. The first statement is a direct consequence of the definitions and the second one follows from it. The third statement is a corollary to Proposition 2.1.

PROPOSITION 2.8. *Let $\mathcal{K}$ be a vector space of functions on Ω and let ρ be a seminorm on $\mathcal{K}$. Then ρ is integrating if and only if*

(D.1) $$\lim_{n\to\infty} \rho\left[\sum_{j=1}^{n} f_j\right] = 0$$

for any functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, satisfying the inequality

(D.2) $$\sum_{j=1}^{\infty} \rho(f_j) < \infty ,$$

such that

(D.2)
$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ *for which the inequality* (A.6) *holds.*

Proof. By Proposition 2.1, the stated condition is necessary for ρ to be integrating. Conversely, assume that (D.1) holds for any functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying (D.2) such that (D.3) holds for every $\omega \in \Omega$ for which (A.6) does. Let $f \in \mathcal{K}$ and let $\epsilon > 0$. Then there exist functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that (A.5) holds for every $\omega \in \Omega$ for which (A.6) does and

$$\sum_{j=1}^{\infty} \rho(f_j) < q_\rho(f) + \epsilon .$$

Then, by the assumption,

$$\lim_{n\to\infty} \rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Hence

$$\rho(f) \le \rho\left[\sum_{j=1}^{n} f_j\right] + \epsilon \le \sum_{j=1}^{n} \rho(f_j) + \epsilon$$

for a sufficiently large n. Consequently, $\rho(f) \le q_\rho(f) + 2\epsilon$.

PROPOSITION 2.9. *The seminorm* ρ *on a vector space* $\mathcal{K}$ *is integrating if and only if*

(D.4)
$$\lim_{n\to\infty} \rho(g_n) = 0$$

for every ρ*-Cauchy sequence* $\{g_n\}_{n=1}^{\infty}$ *of functions from* $\mathcal{K}$ *such that*

(D.5)
$$\lim_{n\to\infty} g_n(\omega) = 0$$

for ρ*-almost every* $\omega \in \Omega$.

Proof. Assume that (D.4) holds for every ρ-Cauchy sequence $\{g_n\}_{n=1}^{\infty}$ of functions from $\mathcal{K}$ which converges ρ-almost everywhere to 0. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions satisfying condition (D.2) such that the equality (D.3) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Let

(D.6) $$g_n = \sum_{j=1}^{n} f_j ,$$

for every $n = 1,2,\ldots$. Then, by Proposition 2.2, the equality (D.5) holds for ρ-almost every $\omega \in \Omega$. Hence, by the assumption, (D.4) holds, which means that (D.3) does. So, by Proposition 2.8, the seminorm ρ is integrating.

Conversely, assume that the seminorm ρ is integrating. Assume that $\{g_n\}_{n=1}^{\infty}$ is a ρ-Cauchy sequence of functions from $\mathcal{K}$ such that (D.5) holds for ρ-almost every $\omega \in \Omega$. To prove (D.4), it suffices to show that $\rho(h_n) \to 0$, as $n \to \infty$, for a subsequence $\{h_n\}_{n=1}^{\infty}$ of the sequence $\{g_n\}_{n=1}^{\infty}$. Therefore, assume that, if $f_1 = g_1$ and $f_j = g_j - g_{j-1}$, for $j = 2,3,\ldots$, then (D.2) holds. Because (D.5) holds for ρ-almost every $\omega \in \Omega$, we have (D.3) for ρ-almost every $\omega \in \Omega$. Then, by Theorem 2.3,

$$\lim_{n\to\infty} \rho(g_n) = \lim_{n\to\infty} \rho\left[\sum_{j=1}^{n} f_j\right] = 0 .$$

PROPOSITION 2.10. *Let* ρ *be a gauge on a nontrivial family of functions,* $\mathcal{K}$ *. For every* $f \in \operatorname{sim}(\mathcal{K})$ *, let*

$$\sigma(f) = \inf \sum_{j=1}^{n} |c_j| \rho(f_j) ,$$

where the infimum is taken over all expressions of the function f *in the form*

$$f = \sum_{j=1}^{n} c_j f_j$$

with arbitrary $n = 1,2,\ldots,$ *numbers* c_j *and functions* $f_j \in \mathcal{K}$ *,* $j = 1,2,\ldots,n$ *.*

Then $\mathcal{L}(\sigma,\operatorname{sim}(\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$ *and* $q_\sigma(f) = q_\rho(f)$ *, for every* $f \in \mathcal{L})\sigma,\operatorname{sim}(\mathcal{K}))$ *. The equality* $\sigma(f) = q_\rho(f)$ *holds for every* $f \in \operatorname{sim}(\mathcal{K})$ *if and only if the seminorm* σ *is integrating.*

Proof. Obviously, $\mathcal{L}(\rho,\mathcal{K}) \subset \mathcal{L}(\sigma,\operatorname{sim}(\mathcal{K}))$ and $q_\sigma(f) \le q_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$. On the other hand, $q_\rho(f) \le \sigma(f)$, for every $f \in \operatorname{sim}(\mathcal{K})$ and, therefore, $\mathcal{L}(\sigma,\operatorname{sim}(\mathcal{K})) \subset \mathcal{L}(q_\rho,\operatorname{sim}(\mathcal{K})) \subset \mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}))$. Because, by Proposition 2.7,

$\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K})) = \mathcal{L}(\rho,\mathcal{K})$, we have $\mathcal{L}(\sigma,\mathrm{sim}(\mathcal{K})) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) \le q_\sigma(f)$, for every $f \in \mathcal{L}(\sigma,\mathrm{sim}(\mathcal{K}))$.

Now, because $q_\sigma = q_\rho$, if $\sigma(f) = q_\rho(f)$, for every $f \in \mathrm{sim}(\mathcal{K})$, then the seminorm σ is integrating. Conversely, if σ is integrating, then $q_\sigma(f) = \sigma(f)$, for every $f \in \mathrm{sim}(\mathcal{K})$ and, hence, $\sigma(f) = q_\rho(f)$, for every $f \in \mathrm{sim}(\mathcal{K})$.

EXAMPLES 2.11. (i) Let $\mathcal{K}$ be a vector space of bounded functions on a space Ω. Let $\rho(f) = \sup\{|f(\omega)| : \omega \in \Omega\}$, for every $f \in \mathcal{K}$. Then ρ is an integrating seminorm on $\mathcal{K}$.

(ii) Let $-\infty < a < b < \infty$ and $\Omega = [a,b]$. Let $\mathcal{K}$ be the space of all functions on $[a,b]$ of bounded variation and let $\rho(f) = |f(a)| + \mathrm{var}(f)$, for every $f \in \mathcal{K}$. Then ρ is an integrating seminorm on $\mathcal{K}$.

E. It can be easily deduced from the general theory of measure and integral that (positive) measures are integrating gauges. However, we wish to show that the classical measure and integration theory is an instance of the theory presented here. Therefore, we prove first that a measure is an integrating gauge. Actually, we prove two slightly more general results. It is convenient to start with a re-statement of Stone's condition, [62], for a positive linear functional to be a Daniell integral.

Let $\mathcal{K}$ be a vector space consisting of real valued functions on a space Ω. A real valued linear functional, ι, on $\mathcal{K}$ is said to be positive if $\iota(f) \ge 0$, for every function $f \in \mathcal{K}$ such that $f(\omega) \ge 0$ for every $\omega \in \Omega$.

In this definition, it is not assumed that $\mathcal{K}$ is a vector lattice (see Section 1D), but, in the following proposition, such an assumption is made.

PROPOSITION 2.12. *Let $\mathcal{K}$ be a vector lattice of real valued functions on Ω and let ι be a positive linear functional on $\mathcal{K}$. Let $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$.*

Then ρ is a seminorm on $\mathcal{K}$ which is integrating if and only if

(E.1) $$\iota(|f|) \le \sum_{j=1}^{\infty} \iota(|f_j|)$$

for any functions $f \in \mathcal{K}$ *and* $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ *such that*

$$\text{(E.2)} \qquad |f(\omega)| \le \sum_{j=1}^{\infty} |f_j(\omega)|$$

for every $\omega \in \Omega$.

Proof. If this condition is satisfied then, by Proposition 2.7, the seminorm ρ is integrating.

Conversely, let us assume that the seminorm ρ is integrating. Let $f \in \mathcal{K}$ and $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions such that the inequality (E.2) holds for every $\omega \in \Omega$. Using the fact that $\mathcal{K}$ is a vector lattice, we construct inductively functions $g_j \in \mathcal{K}$ such that $|g_j| \le |f_j|$, $j = 1,2,\ldots,$ and

$$f(\omega) = \sum_{j=1}^{\infty} g_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |g_j(\omega)| < \infty .$$

Then

$$\iota(|f|) = \rho(f) \le \sum_{j=1}^{\infty} \rho(g_j) = \sum_{j=1}^{\infty} \iota(|g_j|) \le \sum_{j=1}^{\infty} \iota(|f_j|) ,$$

because ρ is integrating.

The following proposition says slightly more than that a non-negative σ-additive set function is an integrating gauge, even if we do not assume that its domain is rich. If we wanted to prove merely that a non-negative σ-additive set function on a quasiring (see Section 1D) is an integrating gauge, then the proof could be slightly simplified. (See Example 4.28(i) in Section 4G.)

PROPOSITION 2.13. *Let* ι *be a nonnegative real valued additive set function on a quasiring of sets* $\mathcal{Q}$ *in a space* Ω . *Then* ι *is an integrating gauge on* $\mathcal{Q}$ *if and only if it is* σ*-additive. Moreover, if* ι *is* σ*-additive and* $\rho(f) = \iota(|f|)$, *for every* $f \in \mathrm{sim}(\mathcal{Q})$, *then* ρ *is an integrating seminorm on* $\mathrm{sim}(\mathcal{Q})$, $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q})) = \mathcal{L}(\iota,\mathcal{Q})$ *and* $q_\rho(f) = q_\iota(f)$ *for every* $f \in \mathcal{L}(\iota,\mathcal{Q})$.

Proof. If ι is not σ-additive, then, obviously, ι is not an integrating gauge. So, let us assume that ι is σ-additive. Let $\mathcal{K} = \mathrm{sim}(\mathcal{Q})$ and let $\rho(f) = \iota(|f|)$ for every $f \in \mathcal{K}$. If we show that the seminorm ρ is integrating, it will follow, by Proposition 2.7, that ι is an integrating gauge. To do that, by Proposition 2.12, it suffices to show that (E.1) holds for any functions $f \in \mathcal{K}$ and $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that (E.2) holds for every $\omega \in \Omega$. But this follows from a result of F. Riesz ([59], Lemma A and Lemma B in no. 16). For completeness we include the proof.

Let m be a positive integer, $d_j \geq 0$ numbers and $Y_j \in \mathcal{Q}$ pair-wise disjoint sets, $j = 1,2,\ldots,m,$ such that

$$|f| = \sum_{j=1}^{m} d_j Y_j .$$

Let Y be the union of the sets Y_j and d the largest of the numbers d_j, $j = 1,2,\ldots,m$. Let $\epsilon > 0$ and,

$$g_n = \sum_{j=1}^{n} |f_j| ,$$

$$Z_n = \{\omega \in Y : g_n(\omega) > |f(\omega)| - \epsilon\} ,$$

for every $n = 1,2,\ldots$. Then $\iota(Y \backslash Z_n) \to 0$, as $n \to \infty$, because the sets $Y \backslash Z_n$ decrease monotonically to $\emptyset$, they belong to the ring generated by $\mathcal{Q}$ and the extension of ι to this ring is σ-additive. Furthermore,

$$\begin{aligned} \iota(g_n) \geq \iota(Yg_n) = \iota(Z_n g_n) + \iota((Y\backslash Z_n)g_n) &\geq \iota(Z_n(|f| - \epsilon)) + \iota((Y\backslash Z_n)(g_n - |f|)) = \\ &= \iota(Z_n|f|) - \epsilon\iota(Z_n) + \iota((Y\backslash Z_n)g_n) - \iota((Y/Z_n)|f|) \geq \\ &\geq \iota(|f|) - \epsilon\iota(Z_n) - 2\iota((Y\backslash Z_n)|f|) \geq \iota(|f|) - \epsilon\iota(Y) - 2d\iota(Y\backslash Z_n) , \end{aligned}$$

for every $n = 1,2,\ldots$. Therefore,

$$\sum_{j=1}^{\infty} \iota(|f_j|) = \lim_{n\to\infty} \iota(g_n) \geq \iota(|f|) - \epsilon\iota(Y) ,$$

and (E.1) follows.

Now, by Proposition 2.7, $\mathcal{L}(\iota,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) \leq q_\iota(f)$. On the other hand, $\mathcal{K} \subset \mathcal{L}(\iota,\mathcal{Q})$ and $\rho(f) = q_\iota(f)$, for every $f \in \mathcal{K}$, because, obviously, $q_\iota(f) \leq \rho(f)$

and, since the seminorm ρ is integrating, $\rho(f) = q_\rho(f) \le q_\iota(f)$. So, if $f \in \mathcal{L}(\rho,\mathcal{K})$, let $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions, satisfying condition (D.2), such that the equality (A.5) holds for every $\omega \in \Omega$ for which the inequality (A.6) does. Then, by Proposition 2.1, $f \in \mathcal{L}(\iota,\mathcal{Q})$ and

$$q_\iota(f) \le \sum_{j=1}^{\infty} q_\iota(f_j) = \sum_{j=1}^{\infty} \rho(f_j) \,.$$

Consequently, $q_\iota(f) \le q_\rho(f)$.

F. In this section, we present some methods of producing new integrating gauges if some are already given.

PROPOSITION 2.14. *Let* $\mathcal{K}$ *be a nontrivial family of functions on a space* Ω . *Let* P *be a collection of integrating gauges on* $\mathcal{K}$ *such that*

$$\sigma(f) = \sup\{\rho(f) : \rho \in \mathrm{P}\} < \infty \,,$$

for every $f \in \mathcal{K}$. *Then* σ *is an integrating gauge on* $\mathcal{K}$.

Proof. Let $f \in \mathcal{K}$ and $\epsilon > 0$. Let $\rho \in \mathrm{P}$ be a gauge such that $\sigma(f) - \epsilon < \rho(f)$. Then

$$\sigma(f) - \epsilon < \rho(f) = q_\rho(f) \le \sum_{j=1}^{\infty} |c_j| \rho(f_j) \le \sum_{j=1}^{\infty} |c_j| \sigma(f_j) \,,$$

for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty \,,$$

and (A.5) holds for every $\omega \in \Omega$ for which (A.6) does. Hence, $\sigma(f) \le q_\sigma(f)$, which means that σ is integrating.

EXAMPLE 2.15. Let Ω be any space. Let W be a real valued function on Ω such that $W(\omega) > 0$, for every $\omega \in \Omega$. Let $1 \le p \le \infty$. If $J \subset \Omega$ is a finite set, let

$$\rho_J(f) = \Big[\sum_{\omega \in J} W(\omega) |f(\omega)|^p\Big]^{1/p} \,,$$

if $1 \le p < \infty$, and

$$\rho_J(f) = \max\{ W(\omega)\,|f(\omega)| : \omega \in J\} ,$$

if $p = \infty$, for any scalar valued function f on Ω. Let $\mathcal{K}$ be the family of all functions f on Ω such that

$$\rho(f) = \sup \rho_J(f) < \infty ,$$

where the supremum is taken over all finite subsets, J, of Ω. By Proposition 2.14, ρ is an integrating gauge on $\mathcal{K}$.

It is straightforward that $\mathcal{K}$ is a vector space and that ρ is a seminorm on $\mathcal{K}$. Actually, ρ is a norm because the only ρ-null set is the empty set. Then it is not difficult to ascertain that $\mathcal{K}$ is ρ-complete. Hence, by Corollary 2.5, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $q_\rho = \rho$. Of, course, $\mathcal{K}$ is the classical weighted l^p space on Ω with the weight W and ρ in its norm;

$$\rho(f) = \left[\sum_{\omega\in\Omega} W(\omega)\,|f(\omega)|^p \right]^{1/p} ,$$

for $1 \le p < \infty$, and $\rho(f) = \sup\{ W(\omega)\,|f(\omega)| : \omega \in \Omega\}$ for $p = \infty$, $f \in \mathcal{K}$.

PROPOSITION 2.16 *Let $\mathcal{L}$ be a vector space of scalar valued functions on a space Ω and let σ be an integrating seminorm on $\mathcal{L}$. Let $\mathcal{K}$ be a vector subspace of $\mathcal{L}$ and let ρ be a seminorm on $\mathcal{K}$ such that*

(i) *$\sigma(f) \le \rho(f)$ for every $f \in \mathcal{K}$:*

(ii) *every σ-null function f is ρ-null, belongs to $\mathcal{K}$ and $\rho(f) = 0$; and*

(iii) *the space $\mathcal{K}$ is ρ-complete.*

Then $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $q_\rho = \rho$, so that the seminorm ρ is integrating.

Proof. Let $\{g_n\}_{n=1}^\infty$ be a ρ-Cauchy sequence of functions from $\mathcal{K}$ such that (D.5) holds for ρ-almost every $\omega \in \Omega$. Let $f \in \mathcal{K}$ be a function, existing by (iii), such that

$$\lim_{n\to\infty} \rho(f-g_n) = 0 .$$

The requirement (i) implies that every ρ-null function is σ-null. Hence, (D.5) holds for σ-almost every $\omega \in \Omega$. Furthermore, by (i) the sequence $\{g_n\}_{n=1}^{\infty}$ is σ-Cauchy. Hence, by Theorem 2.4, the function f is σ-null. Therefore, by (ii), $\rho(f) = 0$. Consequently, the equality (D.4) holds and, by Proposition 2.9, the seminorm ρ is integrating. By Corollary 2.5, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and so, $q_\rho = \rho$.

PROPOSITION 2.17. *Let $\mathcal{L}$ be a vector space of scalar valued functions on a space Ω and let σ be an integrating seminorm on $\mathcal{L}$ such that $\mathcal{L}(\sigma,\mathcal{L}) = \mathcal{L}$. Let $\mathcal{K}$ be a vector subspace of $\mathcal{L}$, let E be a Banach space and let $\mu : \mathcal{K} \to E$ be a closed linear map. Let*

$$\rho(f) = \sigma(f) + |\mu(f)|$$

for every $f \in \mathcal{K}$.

Then ρ is an integrating seminorm on $\mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$.

Proof. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, be functions satisfying condition (D.2) and let f be a function on Ω such that the equality (A.5) holds for each $\omega \in \Omega$ for which the inequality (A.6) does. Then

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty \text{ and } \sum_{j=1}^{\infty} |\mu(f_j)| < \infty .$$

Let the functions g_n be given by (D.6) for every $n = 1,2,\ldots$. Then, by Proposition 2.1, $f \in \mathcal{L}$ and $\sigma(g_n - f) \to 0$, as $n \to \infty$. Furthermore, there exists an element x of E such that $|\mu(g_n) - x| \to 0$. Therefore, $f \in \mathcal{K}$ and $\mu(f) = x$, because the map μ is closed. But then $\rho(g_n - f) \to 0$, as $n \to \infty$. Consequently, by Proposition 2.8, the seminorm ρ is integrating and, by the definition of ρ-integrable functions, $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$.

G. The space $\mathcal{L}(\rho,\mathcal{K})$ is not necessarily a vector lattice. (See Section 1D.)

EXAMPLE 2.18. Let $\mathcal{K}$ be the family of all functions continuous in the closed unit disc, $\Omega = \{(x,y) : x^2 + y^2 \le 1\}$, and harmonic in its interior. Let $\rho(f) = \sup\{|f(\omega)| : \omega \in \partial\Omega\}$, for every $f \in \mathcal{K}$. Then the seminorm ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$, but the space $\mathcal{K}$ is of course not a vector lattice.

We are going to give a sufficient condition for $\mathcal{L}(\rho,\mathcal{K})$ to be a vector lattice. The formulation of the following definition and propositions is slightly more general; it allows also for complex valued functions to belong to $\mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{K})$.

A gauge, ρ, on a nontrivial family of functions, $\mathcal{K}$, will be called monotonic if $\rho(f) \le \rho(g)$ for any functions $f \in \mathcal{K}$ and $g \in \mathcal{K}$ such that $|f| \le |g|$.

The seminorm ρ in Example 2.18 is obviously monotonic.

PROPOSITION 2.19. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω. Let ρ be an integrating seminorm on $\mathcal{K}$. Assume that $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{K}$, and that $q_\rho(|f|-|g|) \le \rho(f-g)$, for every $f \in \mathcal{K}$ and $g \in \mathcal{K}$. Then $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(|f|-|g|) \le q_\rho(f-g)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and $g \in \mathcal{L}(\rho,\mathcal{K})$.*

Proof. It is a matter of routine application of Theorem 2.4, say.

PROPOSITION 2.20. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω such that $|f| \in \mathcal{K}$ for every $f \in \mathcal{K}$. Let ρ be a monotonic integrating seminorm on $\mathcal{K}$. Then $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$ and the seminorm q_ρ is monotonic.*

Proof. The monotonicity of ρ implies that $\rho(|f|) = \rho(f)$, for every $f \in \mathcal{K}$. Moreover, $\rho(|f|-|g|) = \rho(||f|-|g||) \le \rho(|f-g|) = \rho(f-g)$, for every $f \in \mathcal{K}$ and $g \in \mathcal{K}$. Hence, the assumptions of Proposition 2.19 are satisfied and so, $|f| \in \mathcal{L}(\rho,\mathcal{K})$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$. Then it is again a matter of routine to deduce that $q_\rho(|f|) = q_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

Now, let $f \in \mathcal{L}(\rho,\mathcal{K})$ and $g \in \mathcal{L}(\rho,\mathcal{K})$ be functions such that $|f| \le |g|$. Let $\{f_n\}_{n=1}^\infty$ and $\{g_n\}_{n=1}^\infty$ be ρ-Cauchy sequences of functions from $\mathcal{K}$, converging ρ-almost everywhere to the functions f and g, respectively. Let $h_n = \frac{1}{2}(f_n+g_n-||f_n|-|g_n||)$, for every $n = 1,2,\ldots$. Then $|h_n-h_m| \le |f_n-f_m| + |g_n-g_m|$, for any integers $n \ge 1$ and $m \ge 1$, so that the sequence $\{h_n\}_{n=1}^\infty$ is ρ-Cauchy. Moreover, the sequence $\{h_n\}_{n=1}^\infty$ converges ρ-almost everywhere to the function $|f|$.

Because $\rho(h_n) \leq \rho(g_n)$ for every $n = 1,2,\ldots$, by Theorem 2.4,

$$q_\rho(f) = q_\rho(|f|) = \lim_{n\to\infty} \rho(h_n) \leq \lim_{n\to\infty} \rho(g_n) = q_\rho(g) .$$

PROPOSITION 2.21. *Let $\mathcal{K}$ be a vector space of scalar valued functions on a space Ω such that $|f| \in \mathcal{K}$ for every $f \in \mathcal{K}$. Let ρ be a monotonic integrating seminorm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$. Assume that each ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$ is ρ-Cauchy.*

(i) *Let $\{f_n\}_{n=1}^\infty$ be a ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$. Then the sequence $\{f_n(\omega)\}_{n=1}^\infty$ is convergent for ρ-almost every $\omega \in \Omega$. If, moreover, f is a function on Ω such that*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega) \tag{G.1}$$

for ρ-almost every $\omega \in \Omega$, then $f \in \mathcal{K}$ and

$$\lim_{n\to\infty} \rho(f-f_n) = 0 . \tag{G.2}$$

(ii) *Let $g \in \mathcal{K}$ be a real valued function and $f_n \in \mathcal{K}$, $n = 1,2,\ldots$, arbitrary functions such that $|f_n| \leq g$ for every $n = 1,2,\ldots$ and the sequence $\{f_n(\omega)\}_{n=1}^\infty$ is convergent for ρ-almost every $\omega \in \Omega$. Let f be a function on Ω such that* (G.1) *holds for ρ-almost every $\omega \in \Omega$. Then $f \in \mathcal{K}$ and* (G.2) *holds.*

Proof. (i) By Theorem 2.4, the sequence $\{f_n\}_{n=1}^\infty$ has a ρ-almost everywhere convergent subsequence. Hence, because of its monotonicity, the sequence $\{f_n(\omega)\}_{n=1}^\infty$ converges for ρ-almost every $\omega \in \Omega$. By Theorem 2.4, if f is a function on Ω such that (G.1) holds for ρ-almost every $\omega \in \Omega$, then $f \in \mathcal{K}$ and (G.2) holds.

(ii) Let the function f_n, $n = 1,2,\ldots$, be real valued. Let

$$g_n(\omega) = \lim_{m\to\infty}(\sup\{f_j(\omega) : n \leq j \leq m\}) , \quad h_n(\omega) = \lim_{m\to\infty}(\inf\{f_j(\omega) : n \leq j \leq m\}) ,$$

for every $n = 1,2,\ldots$ and $\omega \in \Omega$. Because the seminorm ρ is monotonic, by (i) we have $g_n \in \mathcal{K}$ and $h_n \in \mathcal{K}$, for every $n = 1,2,\ldots$. Also, the sequences $\{g_n\}_{n=1}^\infty$,

$\{h_n\}_{n=1}^{\infty}$ and $\{g_n - h_n\}_{n=1}^{\infty}$ are monotonic and ρ-bounded, $h_n(\omega) \le f(\omega) \le g_n(\omega)$ for every $n = 1,2,\ldots$ and ρ-almost every $\omega \in \Omega$ and

$$\lim_{n\to\infty} h_n(\omega) = f(\omega) = \lim_{n\to\infty} g_n(\omega)$$

for ρ-almost every $\omega \in \Omega$. Therefore, by (i), $f \in \mathcal{K}$ and $\rho(g_n - h_n) \to 0$, as $n \to \infty$. Because $\rho(f-h_n) \le \rho(g_n - h_n)$ and $\rho(f_n - h_n) \le \rho(g_n - h_n)$, for every $n = 1,2,\ldots,$ we have (G.2).

EXAMPLE 2.22. Let $\Omega = [0,1]$, $\mathcal{K} = C([0,1])$ and $\rho(f) = \sup\{|f(\omega)| : \omega \in \Omega\}$, for every $f \in \mathcal{K}$. Then ρ is a monotonic integrating norm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$. However, not every ρ-bounded monotonic sequence of real valued functions from $\mathcal{K}$ is ρ-Cauchy.

If the space $\mathcal{K}$ and the seminorm ρ satisfy the assumptions of Proposition 2.21, we say that they have the Lebesgue property.

H. Let B be a set of integrating gauges; each gauge $\beta \in \mathrm{B}$ is defined on a nontrivial family, $\mathcal{K}_\beta$, of functions on a space Ω_β. The spaces Ω_β, $\beta \in \mathrm{B}$, are assumed to be pair-wise disjoint.

Let $\mathcal{J}$ be a vector lattice or real valued functions on B and α a monotonic integrating seminorm on $\mathcal{J}$.

Let

$$\Omega = \bigcup_{\beta\in\mathrm{B}} \Omega_\beta .$$

Let $\mathcal{K}$ be the family of all functions f on Ω such that, for every $\beta \in \mathrm{B}$, the restriction, $f_\beta = f|\Omega_\beta$, of f to Ω_β belongs to $\mathcal{K}_\beta$ and the function φ_f on B, such that $\varphi_f(\beta) = \beta(f_\beta)$ for every $\beta \in \mathrm{B}$, belongs to $\mathcal{J}$.

Let

$$\rho(f) = \alpha(\varphi_f)$$

for every $f \in \mathcal{K}$.

PROPOSITION 2.23. *The functional ρ is an integrating gauge on $\mathcal{K}$.*

Proof. Let $f \in \mathcal{K}$ and let c_j be numbers and $f_j \in \mathcal{K}$ functions, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j| \rho(f_j) = \sum_{j=1}^{\infty} |c_j| \alpha(\varphi_{f_j}) < \infty$$

and the equality (A.2) holds for each $\omega \in \Omega$ for which the inequality (A.3) does. By Proposition 2.7,

$$\varphi_f(\beta) = \beta(f_\beta) \le \sum_{j=1}^{\infty} |c_j| \beta((f_j)_\beta) = \sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta)$$

for each $\beta \in \mathrm{B}$, because these gauges are integrating. Let

$$\psi(\beta) = \sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) ,$$

for every $\beta \in \mathrm{B}$ such that

$$\sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) < \infty ,$$

and let $\psi(\beta) = \varphi_f(\beta)$, for every $\beta \in \mathrm{B}$ such that

$$\sum_{j=1}^{\infty} |c_j| \varphi_{f_j}(\beta) = \infty .$$

Then $\psi \in \mathcal{L}(\alpha,\mathcal{J})$ and $0 \le \varphi_f \le \psi$. Therefore, by Proposition 2.19,

$$\rho(f) = \alpha(\varphi_f) = q_\alpha(\alpha_f) \le q_\alpha(\psi) \le \sum_{j=1}^{\infty} |c_j| \alpha(\varphi_{f_j}) = \sum_{j=1}^{\infty} |c_j| \rho(f_j) .$$

So, by Proposition 2.7, the gauge ρ is integrating.

In practice, the most useful choice of $\mathcal{J}$ is perhaps the space $l^1(\mathrm{B})$, or the space $l^\infty(\mathrm{B})$, with its natural norm (Example 2.15 with weight $W(\beta) = 1$ for each $\beta \in \mathrm{B}$).

J. The basic way of showing that a positive additive set function is in fact σ-additive is to exploit compactness and regularity of some sort or another, that is, to use the Alexandrov theorem or some of its generalizations. (See Section 1F.) In this section a similar means for showing that a gauge is integrating is presented.

Let $\mathcal{Q}$ be a quasiring of sets in a space Ω. (See Section 1D.) Let ρ be a gauge on $\mathcal{Q}$.

Let us call the gauge ρ very sub-additive if the inequality

$$\rho(X) \leq \sum_{j=1}^{n} c_j \rho(X_j)$$

holds for any set $X \in \mathcal{Q}$, any $n = 1,2,\ldots$, and any sets $X_j \in \mathcal{Q}$ and numbers $c_j \geq 0$, $j = 1,2,\ldots,n$, such that

$$X(\omega) \leq \sum_{j=1}^{n} c_j X_j(\omega)$$

for every $\omega \in \Omega$.

The use of the adverb "very" in this definition is dictated by a certain caution: it is a warning that a gauge may rather unexpectedly fail to be very sub-additive.

EXAMPLE 2.24. Let $\Omega = \mathbb{R}^2$ and let $\mathcal{Q}$ be the family of all intervals $X = (u_1,v_1]\times(u_2,v_2]$ with $u_1 \leq v_1$ and $u_2 \leq v_2$. Let $f = 2(0,3]\times(0,3] - 3(1,2]\times(1,2]$ and

$$\mu(X) = \int_X f \, d\iota ,$$

for every $X \in \mathcal{Q}$, where ι is the two-dimensional Lebesgue measure. The gauge ρ, defined by

$$\rho(X) = \sup\{\, |\mu(X \cap Z)| : Z \in \mathcal{Q}\}$$

for every $X \in \mathcal{Q}$, is not very sub-additive. In fact, the interval $X = (0,3]\times(0,3]$ is equal to the union of the intervals $X_1 = (1,2]\times(0,3]$, $X_2 = (0,3]\times(1,2]$, $X_3 = (0,1]\times(0,1]$, $X_4 = (2,3]\times(0,1]$, $X_5 = (2,3]\times(2,3]$ and $X_6 = (0,1]\times(2,3]$, but $\rho(X) = 15$, $\rho(X_1) = \rho(X_2) = 3$ and $\rho(X_3) = \rho(X_4) = \rho(X_5) = \rho(X_6) = 2$.

The property of being very sub-additive is rather advantageous though, because it allows us to use the following property of regularity to prove that a gauge is integrating.

Assuming that Ω is a topological space, the gauge ρ is said to be regular if, for every set $X \in \mathcal{Q}$ and $\epsilon > 0$, there is

(i) an open set $U \supset X$ and a set $Y \in \mathcal{Q}$ such that $U \subset Y$ and $\rho(Y) - \rho(X) < \epsilon$; and

(ii) a compact set $K \subset X$ and a set $Z \in \mathcal{Q}$ such that $Z \subset K$ and $\rho(X) - \rho(Z) < \epsilon$.

PROPOSITION 2.25. *A very sub-additive and regular gauge on a quasiring of sets in a topological space is integrating.*

Proof. Let $\mathcal{Q}$ be a quasiring of sets in a topological space Ω. Let ρ be a very sub-additive and regular gauge on $\mathcal{Q}$.

Let $X \in \mathcal{Q}$ and, for every $j = 1,2,\ldots$, let $X_j \in \mathcal{Q}$ be a set and c_j a number such that

(J.1) $$X(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

(J.2) $$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Our aim is to show that

(J.3) $$\rho(X) \leq \sum_{j=1}^{\infty} |c_j| \rho(X_j) .$$

Let $0 < \epsilon < 1$. Let K be a compact set and Z a set in $\mathcal{Q}$ such that $Z \subset K \subset X$ and $\rho(X) - \epsilon < \rho(Z)$. For every $j = 1,2,\ldots$, let U_j be an open set and Y_j a set in $\mathcal{Q}$ such that $X_j \subset U_j \subset Y_j$ and $|c_j| \rho(Y_j) < |c_j| \rho(X_j) + \epsilon 2^{-j}$. Let $n \geq 1$ be an integer such that

$$\sum_{j=1}^{n} |c_j| Y_j(\omega) \geq (1-\epsilon) Z(\omega)$$

for every $\omega \in \Omega$. Then

$$(1-\epsilon)(\rho(X)-\epsilon) < (1-\epsilon)\rho(Z) \le \sum_{j=1}^{n} |c_j|\rho(Y_j) <$$

$$< \sum_{j=1}^{n} (|c_j|\rho(X_j) + \epsilon 2^{-j}) < \sum_{j=1}^{\infty} |c_j|\rho(X_j) + \epsilon .$$

Hence, (J.3) holds and, by Proposition 2.7, the gauge ρ is integrating.

K. The first proposition of this section represents a method of producing new integrating gauges from ones already guaranteed to be integrating. Recall that a quasiring, $\mathcal{Q}$, is said to be multiplicative if $X \cap Y \in \mathcal{Q}$ for any sets $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$. (See Section 1D.)

Clearly, a gauge, σ, on a quasiring of sets, $\mathcal{Q}$, is monotonic if and only if $\sigma(X) \le \sigma(Y)$ for any sets $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$ such that $X \subset Y$. (See Section G.)

PROPOSITION 2.26. *Let σ be an integrating monotonic gauge on a multiplicative quasiring, $\mathcal{Q}$, of sets in a space Ω. Let φ be a real valued, continuous, strictly increasing and concave function on the interval $[0,\infty)$ such that $\varphi(0) = 0$. Let $\rho(X) = \varphi(\sigma(X))$ for every $X \in \mathcal{Q}$.*

Then ρ is an integrating gauge on $\mathcal{Q}$.

Proof. Let $X \in \mathcal{Q}$ be a set, c_j numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, such that the equality (J.1) holds for every $\omega \in \Omega$ for which the inequality (J.2) does. Our aim is to show that (J.3) holds.

Without loss of generality, we will assume that $X_j \subset X$, for every $j = 1,2,\ldots$, because, if the sets X_j are replaced by $X_j \cap X$, then the equality (J.1) remains valid for every $\omega \in \Omega$ satisfying (J.2) and $\rho(X_j \cap X) = \varphi(\sigma(X_j \cap X)) \le \varphi(\sigma(X_j)) = \rho(X_j)$, by the monotonicity of φ and σ. We will also assume that

$$\text{(K.1)} \qquad \sum_{j=1}^{\infty} |c_j|\rho(X_j) < \infty$$

and that $\sigma(X_j) \neq 0$, for some $j = 1,2,\ldots$.

Let $s = \sup\{\sigma(X_j) : j = 1,2,\ldots\}$. By the assumption just made and the monotonicity of σ, we have $0 < s \leq \sigma(X)$. Let $k = \varphi(s)/s$. Then $k\sigma(X_j) \leq \varphi(\sigma(X_j)) = \rho(X_j)$, for every $j = 1,2,\ldots,$ because the function φ is concave. Therefore

$$t = \sum_{j=1}^{\infty} |c_j| \sigma(X_j) \leq k^{-1} \sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty .$$

By Proposition 2.7, $\sigma(X) \leq t$, because the gauge σ is integrating, and so, $s \leq t$. Consequently, by the monotonicity and concavity of φ, we have

$$\rho(X) = \varphi(\sigma(X)) \leq \varphi(t) \leq kt = k \sum_{j=1}^{\infty} |c_j| \sigma(X_j) \leq \sum_{j=1}^{\infty} |c_j| \rho(X_j) .$$

So, (J.3) holds. But, if (K.1) does not hold, then (J.3) is trivially true. Moreover, if $\sigma(X_j) = 0$ for every $j = 1,2,\ldots,$ then, by Proposition 2.7, $\sigma(X) = 0$, because the gauge σ is integrating. Hence, (J.3) holds also in this case, and, by Proposition 2.7, the gauge ρ is integrating.

Typically, a non-negative σ-additive set function is used in the role of the gauge σ in Proposition 2.26.

The second proposition of this section says that if ρ is an integrating gauge on a quasiring of sets, then the assumptions of Proposition 2.10 are satisfied, that is, the seminorm, σ, defined in that proposition is integrating.

PROPOSITION 2.27. *Let $\mathcal{Q}$ be a quasiring of sets in a space Ω and let ρ be an integrating gauge on $\mathcal{Q}$. Then, for every real valued function $f \in \operatorname{sim}(\mathcal{Q})$,*

$$q_\rho(f) = \inf \sum_{j=1}^{n} |c_j| \rho(X_j) ,$$

where the infimum is taken over all expressions of f in the form

$$f = \sum_{j=1}^{n} c_j X_j ,$$

with arbitrary $n = 1,2,\ldots,$ real numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$.

Proof. Let $\mathcal{R}$ be the ring of sets generated by $\mathcal{Q}$. (See Section 1D.) For every set $Y \in \mathcal{R}$, let

$$\sigma(Y) = \min \sum_{j=1}^{n} \rho(X_j) ,$$

where the minimum is taken over all expressions of the set Y in the form

$$Y = \sum_{j=1}^{n} e_j X_j ,$$

with arbitrary $n = 1,2,\ldots,$ arbitrary choices of $e_j = \pm 1$ and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$. Then, for every real valued function $f \in \mathrm{sim}(\mathcal{Q})$, there exist unique integers $k \geq 0$ and $\ell \geq 0$, sets $Y_i \in \mathcal{R}$, $Z_j \in \mathcal{R}$ and real numbers $c_i > 0$, $d_j > 0$, $i = 1,2,\ldots,k$, $j = 1,2,\ldots,\ell$, such that $Y_1 \cap Z_1 = \emptyset$, $Y_{i-1} \subset Y_i$, $Z_{j-1} \subset Z_j$, for $i = 2,\ldots,k$ and $j = 2,\ldots,\ell$, and

$$f = \sum_{i=1}^{k} c_i Y_i - \sum_{j=1}^{\ell} d_j Z_j .$$

For any function so expressed, let

$$\sigma(f) = \sum_{i=1}^{k} c_i \sigma(Y_i) + \sum_{i=1}^{\ell} d_j \sigma(Z_j) .$$

Now, $\mathcal{L}(\sigma,\mathrm{sim}(\mathcal{Q})) = \mathcal{L}(\rho,\mathcal{Q})$ and $q_\sigma(f) = q_\rho(f)$, for every $f \in \mathcal{L}(\sigma,\mathrm{sim}(\mathcal{Q}))$. In fact, $\mathcal{Q} \subset \mathrm{sim}(\mathcal{Q})$ and $\rho(X) = \sigma(X)$, for every $X \in \mathcal{Q}$. On the other hand, if $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,\ldots,$ are functions such that

$$\sum_{j=1}^{\infty} \sigma(f_j) < \infty ,$$

then there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty , \quad \sum_{j=1}^{\infty} c_j X_j(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

and also

$$\sum_{j=1}^{\infty} |c_j| X_j(\omega) = \sum_{j=1}^{\infty} |f_j(\omega)| ,$$

for every $\omega \in \Omega$. It follows that σ is a seminorm on (real) $\mathrm{sim}(\mathcal{Q})$ and that $L(\rho,\mathcal{Q})$ can be identified with the completion of $\mathrm{sim}(\mathcal{Q})$ in (the norm induced by) σ . Consequently, $q_\rho(f) = \sigma(f)$, for every $f \in \mathrm{sim}(\mathcal{Q})$.

3. INTEGRALS

Besides an integrating gauge, ρ, on a family of functions, $\mathcal{K}$, we consider a functional, μ, on $\mathcal{K}$ which can be extended to a continuous linear functional, μ_ρ, defined on the whole of $\mathcal{L} = \mathcal{L}(\rho,\mathcal{K})$. The continuity is understood with respect to the seminorm, q_ρ, induced by ρ on $\mathcal{L}$, defined in the previous chapter. More generally, we consider a map, μ, from $\mathcal{K}$ into an arbitrary Banach space, E, and a continuous linear map, μ_ρ, from $\mathcal{L}$ into E, generated by μ. Given a function $f \in \mathcal{L}$, the number, or vector, $\mu_\rho(f)$ is looked upon as the integral of f with respect to μ.

The classical case of integration with respect to a (positive) measure, ι, is obtained by taking for $\mathcal{K}$ a sufficiently rich family of (characteristic functions of) sets of finite measure and putting both ρ and μ equal to (the restriction to $\mathcal{K}$ of) ι. If μ is an additive set function having finite and σ-additive variation, then integration with respect to μ can be introduced by choosing ρ equal to the variation of μ. Of course, this choice is not available in general, and so, given an additive set function, μ, the problem of integration with respect to μ is reduced to that of finding a suitable ρ. This problem will be treated more systematically in Chapter 4.

Here we show how the integration with respect to Banach space valued measures, due to R.G. Bartle, N. Dunford and J.T. Schwartz, [2], fits into the presented scheme. Also in this chapter, the definitions of the Orlicz, the Sobolev and the Hardy spaces are shown to be special cases of the construction of the space $\mathcal{L}(\rho,\mathcal{K})$ for suitable choices of $\mathcal{K}$ and ρ.

A. Let $\mathcal{K}$ be a nontrivial family of functions on a space Ω. Let E be a Banach space. Let $\mu : \mathcal{K} \to E$ be a linear map. Recall that the domain of a linear map, or a linear functional, is not necessarily a vector space. (See Section 1E.)

We shall say that a gauge, ρ, on $\mathcal{K}$ integrates for the map μ if it is integrating (see Section 2D) and $|\mu(f)| \le cq_\rho(f)$, for some number $c \ge 0$ and every function $f \in \mathrm{sim}(\mathcal{K})$.

If the gauge ρ integrates for the linear map $\mu : \mathcal{K} \to E$, then there exists a unique linear map $\mu_\rho : \mathcal{L}(\rho,\mathcal{K}) \to E$ such that $\mu_\rho(f) = \mu(f)$, for every $f \in \mathcal{K}$, and $|\mu_\rho(f)| \le cq_\rho(f)$, for some number $c \ge 0$ and every $f \in \mathcal{L}(\rho,\mathcal{K})$. In fact, μ has a unique linear extension on $\mathrm{sim}(\mathcal{K})$. In fact, μ has a unique linear extension on $\mathrm{sim}(\mathcal{K})$ (see Section 1B) which, by the assumption, is continuous with respect to q_ρ and $\mathrm{sim}(\mathcal{K})$ is q_ρ-dense in $\mathcal{L}(\rho,\mathcal{K})$.

We shall also use the conventional notation

$$\int_\Omega f \mathrm{d}_\rho \mu = \int_\Omega f(\omega)\mu(\mathrm{d}_\rho\omega) = \mu_\rho(f) \tag{A.1}$$

for every $f \in \mathcal{L}(\rho,\mathcal{K})$. The subscript is omitted when ρ is understood or immaterial.

If $\mathcal{K}$ happens to be a vector space, then an integrating gauge ρ on $\mathcal{K}$ integrates for the additive map $\mu : \mathcal{K} \to E$ if and only if there exists a constant $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$. In fact, in this case, $\mathrm{sim}(\mathcal{K}) = \mathcal{K}$ and $q_\rho(f) = \rho(f)$ for every $f \in \mathcal{K}$. For an arbitrary nontrivial family of functions $\mathcal{K}$, we have the following

PROPOSITION 3.1. *An integrating gauge ρ on $\mathcal{K}$ integrates for the additive map $\mu : \mathcal{K} \to E$ if and only if there exists a constant $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$, and*

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} c_j \mu(f) \right| = 0 \tag{A.2}$$

for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty \tag{A.3}$$

and

$$\sum_{j=1}^{\infty} c_j f_j(\omega) = 0 \tag{A.4}$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j f_j(\omega)| < \infty . \tag{A.5}$$

Proof. Let the gauge ρ integrate for μ. Let c_j be numbers and $f_j \in \mathcal{K}$ functions, $j = 1,2,\ldots,$ satisfying condition (A.3), such that (A.4) holds for every $\omega \in \Omega$ for which (A.5) does. Then, by Proposition 2.1,

$$\lim_{n\to\infty} q_\rho\left[\sum_{j=1}^{n} c_j f_j\right] = 0 .$$

Because, by the assumption,

$$\left|\sum_{j=1}^{n} c_j\,\mu(f_j)\right| \le c q_\rho\left[\sum_{j=1}^{n} c_j f_j\right] ,$$

for some $c \ge 0$ and every $n = 1,2,\ldots,$ the equality (A.2) follows.

Conversely, assume that ρ is an integrating gauge on $\mathcal{K}$, that there exists a number $c \ge 0$ such that $|\mu(f)| \le c\rho(f)$, for every $f \in \mathcal{K}$, and that (A.2) holds for any numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ satisfying (A.3), such that (A.4) holds for every $\omega \in \Omega$ for which (A.5) does. Then, for any function $f \in \mathcal{L}(\gamma,\mathcal{K})$, let $\tilde{\mu}(f)$ be the element of the space E such that

$$\tilde{\mu}(f) = \sum_{j=1}^{\infty} c_j\mu(f_j) ,$$

where the c_j are some numbers and the f_j some functions from $\mathcal{K}$, $j = 1,2,\ldots,$ satisfying condition (A.3), such that

$$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega)$$

for every $\omega \in \Omega$ for which the inequality (A.5) holds. By the assumption, the vector $\tilde{\mu}(f)$ depends on the function f alone and not on a particular choice of the numbers c_j and the functions f_j, $j = 1,2,\ldots$. Consequently, $\tilde{\mu}(f) = \mu(f)$ for every $f \in \mathrm{sim}(\mathcal{K})$. Furthermore, for every $\epsilon > 0$, we can choose these numbers and functions so that

$$\sum_{j=1}^{\infty} |c_j|\rho(f_j) < q_\rho(f) + \epsilon .$$

Hence, $|\tilde{\mu}(f)| \le cq_\rho(f) + \epsilon$, because $|\mu(f_j)| \le c\rho(f_j)$ for every $j = 1,2,\ldots$. So, $|\tilde{\mu}(f)| \le cq_\rho(f)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

Whenever applicable the following proposition is of course easier to use. By Proposition 2.27, it can be used, in particular, when $\mathcal{K}$ is a quasiring of sets.

PROPOSITION 3.2. *Let ρ be an integrating gauge on $\mathcal{K}$ such that, for every function $f \in \operatorname{sim}(\mathcal{K})$,*

$$q_\rho(f) = \inf \sum_{j=1}^{n} |c_j| \rho(f_j),$$

where the infimum is taken over all expressions of f in the form

$$f = \sum_{j=1}^{n} c_j f_j,$$

with arbitrary $n = 1,2,\ldots,$ numbers c_j and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots,n$. Let $\mu : \mathcal{K} \to E$ be an additive map such that $|\mu(f)| \le c\rho(f)$, for some $c \ge 0$ and every $f \in \mathcal{K}$.

Then the gauge ρ integrates for the map μ.

Proof. The assumptions imply that $|\mu(f)| \le cq_\rho(f)$, for every $f \in \operatorname{sim}(\mathcal{K})$.

B. Let $\mathcal{Q}$ be a quasiring of sets in a space Ω. Let ι be a σ-additive non-negative real valued set function on $\mathcal{Q}$. (See Sections 1D and 1F.)

Because $|\iota(f)| \le \iota(|f|)$, for every $f \in \operatorname{sim}(\mathcal{Q})$, by Proposition 2.13, ι is a gauge which integrates for itself. So, there exists a unique linear functional, ι_ι, on $\mathcal{L}(\iota,\mathcal{Q})$ such that $\iota_\iota(X) = \iota(X)$ for every $X \in \mathcal{Q}$, and the inequality $|\iota_\iota(f)| \le q_\iota(f)$ holds for every function $f \in \mathcal{L}(\iota,\mathcal{Q})$. Conforming to standard notation, we shall of course write

$$\iota(f) = \int_\Omega f d\iota = \int_\Omega f(\omega)\iota(d\omega) = \iota_\iota(f)$$

for every function $f \in \mathcal{L}(\iota,\mathcal{Q})$.

PROPOSITION 3.3. *If* $f \in \mathcal{L}(\iota,\mathcal{Q})$, *then also* $|f| \in \mathcal{L}(\iota,\mathcal{Q})$ *and* $q_\iota(f) = \iota(|f|)$ *for every function* $f \in \mathcal{L}(\iota,\mathcal{Q})$.

Proof. Let $\rho(f) = \iota(|f|)$, for every $f \in \mathrm{sim}(\mathcal{Q})$. Then the seminorm ρ is monotonic and, by Proposition 2.13, $\mathcal{L}(\iota,\mathcal{Q}) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$. Hence, by Proposition 2.20, if $f \in \mathcal{L}(\iota,\mathcal{Q})$, then also $|f| \in \mathcal{L}(\iota,\mathcal{Q})$. Now, the seminorms $f \mapsto \iota(|f|)$ and $f \mapsto q_\iota(f)$, $f \in \mathcal{L}(\iota,\mathcal{Q})$, are both q_ι-continuous and they agree on a q_ι-dense subspace, $\mathrm{sim}(\mathcal{Q})$, of $\mathcal{L}(\iota,\mathcal{Q})$. Therefore, they agree on the whole of $\mathcal{L}(\iota,\mathcal{Q})$.

The Beppo Levi monotone convergence theorem and the Lebesgue dominated convergence theorem are now special cases of the two respective statements of Proposition 2.21. The Fatou lemma can then be deduced in the well-known manner. (See e.g. [59], no. 20.)

Let $\mathcal{R}(\iota)$ be the family of all ι-integrable sets, that is, sets with characteristic function belonging to $\mathcal{L}(\iota,\mathcal{Q})$. Then $\mathcal{R}(\iota)$ is a δ-ring of sets in Ω . The existence of a (finite) non-negative σ-additive extension of ι onto the whole of $\mathcal{R}(\iota)$ is now obvious. Moreover, by Proposition 2.7, $\mathcal{L}(\iota,\mathcal{R}(\iota)) = \mathcal{L}(\iota,\mathcal{Q})$. Therefore, we may suppress the domain, $\mathcal{Q}$, of ι in the symbol for the space of ι-integrable functions and write simply $\mathcal{L}(\iota) = \mathcal{L}(\iota,\mathcal{Q})$.

There are now several possibilities of defining ι-measurable sets and functions. We may call a set ι-measurable if it belongs to the σ-algebra or just the σ-ring of sets generated by $\mathcal{R}(\iota)$. A larger family of ι-measurable sets is obtained if we call ι-measurable any set $X \subset \Omega$ such that $X \cap \Omega$ such that $X \cap Z \in \mathcal{R}(\iota)$ for every $Z \in \mathcal{R}(\iota)$. The choice of the definition depends of course on the purpose to which it is to be used. But in either case, it is customary to put $\iota(X) = \infty$ for every ι-measurable set X which is not ι-integrable.

So, the set function ι determines a measure in the space Ω which is of course denoted still by ι .

It should be noted perhaps that the term "measure" is not used in the same fashion throughout the literature. It often designates a non-negative extended real valued (∞ is allowed as a value) set function on a σ-ring of sets covering the whole space or a σ-algebra. Other authors designate by this term the corresponding integral, that is, the linear functional whose value at an integrable function, f, is equal to the integral of f with respect to the measure in question, or even its restriction to a linear subspace dense in the L^1-seminorm in the space of all integrable functions.

This lack of uniformity will not cause any inconvenience in the sequel, because, however the term "measure" is interpreted, specifying a measure, ι, entails the specification of the following objects: a vector lattice, $\mathcal{L}(\iota)$, of functions on Ω and a positive linear functional, ι, on $\mathcal{L}(\iota)$ such that $\mathcal{L}(\rho,\mathcal{L}(\iota)) = \mathcal{L}(\iota)$, where $\rho(f) = \iota(|f|)$ for every $f \in \mathcal{L}(\iota)$, and if $\mathcal{R}(\iota)$ is the family of sets (with characteristic functions) belonging to $\mathcal{L}(\iota)$, then $\mathcal{L}(\rho,\mathcal{R}(\iota)) = \mathcal{L}(\iota)$. The functions belonging to $\mathcal{L}(\iota)$ and sets belonging to $\mathcal{R}(\iota)$ are then called integrable with respect to the measure ι or ι-integrable.

Now, returning to the the measure, ι, determined by its values on the quasiring $\mathcal{Q}$, let us note that, in view of Proposition 2.13 and Proposition 3.1, the definitions adopted in Sections 2A, 2D and 3A, give us a direct and economical representation of integrable functions circumventing the Carathéodory theory of extension of ι onto all measurable sets. Namely, a function f is ι-integrable if and only if there exist numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,...,$ such that

$$\sum_{j=1}^{\infty} |c_j| \iota(X_j) < \infty$$

and

$$f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

The integral of such a function f is then given by the formula

$$\int_{\Omega} f d\iota = \sum_{j=1}^{\infty} c_j \iota(X_j) .$$

It is striking how close this characterization of integrability and integral is to the ideas of Archimedes, especially to one of his calculations of the area of a parabolic section; see e.g. [21]. As noted by J. Mikusiński in the Preface to his book [50], it makes the presentation of the Lebesgue integral at elementary level more viable than that of the Riemann integral. For further elementary comments, see [33].

An approach to integration along similar lines was suggested by J.L. Kelley and T.P. Srinivasan, [28]; see also [29].

As suggested, a measure in the space Ω is sometimes specified by specifying the values of the corresponding integral on a sufficiently rich vector subspace of the space of all integrable functions. It is done by invoking a theory of the Daniell integral or its generalization. Such theories too are instances of the general scheme presented in Section A. To describe the main points, let us recall some notation.

For a real valued function, f, on Ω, we write $f^+ = \frac{1}{2}(|f|+f)$, $f^- = \frac{1}{2}(|f|-f)$ and $f\wedge 1 = g$, where $g(\omega) = \frac{1}{2}(f(\omega)+1-|f(\omega)-1|)$ for every $\omega \in \Omega$. For a nontrivial family, $\mathcal{K}$, of real valued functions on Ω, we write $\mathcal{K}^+ = \{f \in \mathcal{K} : f \geq 0\}$ and $\mathcal{K}^+ - \mathcal{K}^+ = \{f-g : f \in \mathcal{K}^+, g \in \mathcal{K}^+\}$.

Let $\mathcal{K}$ be a vector lattice of real valued functions on the space Ω. A positive linear functional, ι, on $\mathcal{K}$ (see Section 2E) is called a Daniell integral, if $\iota(f_n) \to 0$, as $n \to \infty$, for any functions $f_n \in \mathcal{K}$ such that $f_n(\omega) \geq f_{n+1}(\omega)$, $n = 1,2,\ldots,$ and $f_n(\omega) \to 0$, for every $\omega \in \Omega$, as $n \to \infty$.

It is easy to show using Proposition 2.12, say, that a positive linear functional, ι, on $\mathcal{K}$ is a Daniell integral, if and only if, the seminorm, ρ, defined by $\rho(f) = \iota(|f|)$, for every $f \in \mathcal{K}$, is integrating.

Assume now that ι is a Daniell integral on $\mathcal{K}$. It is then obvious that the seminorm ρ integrates for the functional ι. Let us write $\mathcal{L}(\iota) = \mathcal{L}(\rho,\mathcal{K})$ and denote by $\mathcal{R}(\iota)$ the family of sets (with characteristic functions) belonging to $\mathcal{L}(\iota)$.

We say that the Daniell integral ι satisfies the Stone condition if the function $f\wedge 1$ belongs to $\mathcal{L}(\iota)$ whenever the function f does. It is well-known that, if $f\wedge 1$ belongs to $\mathcal{K}$ whenever f does, then ι satisfies the Stone condition.

Let ι_ρ be the unique continuous linear functional on $\mathcal{L}(\iota)$ that extends ι. (See Section A.) Its restriction to $\mathcal{R}(\iota)$ is a non-negative σ-additive set function. M.H. Stone has shown, [62], that $\mathcal{L}(\iota) = \mathcal{L}(\iota_\rho, \mathcal{R}(\iota))$ if and only if ι satisfies the Stone condition.

M. Leinert, [40], and H. König, [36], have generalized the notion of a Daniell integral by requiring that $\mathcal{K}$ be merely a vector space and not necessarily a vector lattice. Such generalization is interesting because it represents the abstract core of situations not infrequently occurring in analysis; see [37], [41].

So, let $\mathcal{K}$ be a vector space of real valued functions on Ω and let ι be a positive linear functional on $\mathcal{K}$. For any real valued function f on Ω, let

$$\iota^+(f) = \inf \sum_{j=1}^{\infty} \iota(f_j) ,$$

where the infimum is taken over all choices of functions $f_j \in \mathcal{K}^+$, $j = 1,2,\ldots$, such that

$$f(\omega) \leq \sum_{j=1}^{\infty} f_j(\omega) \tag{B.1}$$

for every $\omega \in \Omega$. The possibility that $\rho(f) = \infty$ is of course admitted. Let, further,

$$\bar{\iota}(f) = \inf \sum_{j=1}^{\infty} \iota(f_j) ,$$

where the infimum is taken over all choices of functions $f_1 \in \mathcal{K}$ and $f_j \in \mathcal{K}^+$, $j = 2,3,\ldots$, such that (B.1) holds for every $\omega \in \Omega$.

We say that the functional ι satisfies the König continuity condition, if $\iota^+(f^+) = \iota(f) + \iota^+(f^-)$, for every function $f \in \mathcal{K}$.

We say that the functional ι satisfies the Leinert continuity condition, if $\iota^+(f^+) \geq \iota(f)$, for every $f \in \mathcal{K}$.

Clearly, if ι satisfies the König continuity condition then it satisfies the Leinert continuity condition. Moreover, if $\mathcal{K}$ happens to be a vector lattice, then ι satisfies the König continuity condition if and only if it is a Daniell integral, and also it satisfies the Leinert continuity condition if and only if it is a Daniell integral.

Now, assume that $\mathcal{K} = \mathcal{K}^+ - \mathcal{K}^+$. Then we can define gauges, ρ_1 and ρ_2, on $\mathcal{K}$ by letting $\rho_1(f) = \bar{\iota}(|f|)$ and $\rho_2(f) = \iota^+(|f|)$, respectively, for every $f \in \mathcal{K}$. Let, further, ρ_3 be the gauge on $\mathcal{K}^+$ such that $\rho_3(f) = \iota(f)$, for every $f \in \mathcal{K}^+$.

If ι satisfies the Leinert continuity condition, then the gauge ρ_3 is integrating. The gauge ρ_2, which is a seminorm, is automatically integrating. If ι satisfies the König continuity condition, then $\rho_1 = \rho_2$, the gauge ρ_1 is integrating and $\mathcal{L}(\rho_1,\mathcal{K}) = \mathcal{L}(\rho_3,\mathcal{K}^+)$. We may note that, while the König condition is sufficient, it is not necessary for the gauge ρ_1 to be integrating. However, the König condition is convenient to use without loss in the context of uniform algebras.

For a more complete consolidated exposition we refer to [37].

C. Natural seminorms in the classical function spaces defined in terms of a measure usually turn out to be integrating.

Let ι be a measure in a space Ω and p a real number such that $1 \leq p < \infty$. The family of all functions f on Ω such that $f|f|^{p-1} \in \mathcal{L}(\iota)$ is denoted by $\mathcal{L}^p(\iota)$. So, in particular, $\mathcal{L}^1(\iota) = \mathcal{L}(\iota)$. It is well-known that $\mathcal{L}^p(\iota)$ is a vector space. Moreover, if

$$\|f\|_{p,\iota} = \left[\int_\Omega |f|^p \mathrm{d}\iota\right]^{1/p},$$

for every $f \in \mathcal{L}^p(\iota)$, then $\|\cdot\|_{p,\iota}$ is an integrating seminorm on $\mathcal{L}^p(\iota)$ such that $\mathcal{L}(\|\cdot\|_{p,\iota},\mathcal{L}^p(\iota)) = \mathcal{L}^p(\iota)$. This fact is implicit in the standard proof of the completeness of $\mathcal{L}^p(\iota)$ which avoids the notion of convergence in measure. The induced normed space is of course denoted by $L^p(\iota)$. M.H. Stone, [62], introduced the L^p-spaces along these lines in the context of Daniell integrals instead of measures.

These spaces (based on a measure rather than a Daniell integral) are special cases of the general Banach function spaces studied systematically by W.A.J. Luxemburg and A.C. Zaanen in a series of papers of which the first one, [47], contains an introduction to the subject with the relevant historical background. See also [72], §§63-64.

Let $\mathcal{S}$ be a σ-algebra of sets in the space Ω and let $\mathcal{Z}$ be a σ-ideal in the space Ω (see Section 1D) such that $\mathcal{Z} \subset \mathcal{S}$. Let $\mathcal{M} = \mathcal{M}(\mathcal{S})$ be the family of all complex valued $\mathcal{S}$-measurable functions and $\mathcal{M}^+$ the family of non-negative real-valued functions belonging to $\mathcal{M}$. Let $\mathcal{N} = \mathcal{M}(\mathcal{Z})$ be the family of all functions f on Ω such that the set $\{\omega : f(\omega) \neq 0\}$ belongs to $\mathcal{Z}$. Clearly, $\mathcal{N} \subset \mathcal{M}$ because $\mathcal{Z} \subset \mathcal{S}$.

Following [48], Definition 3.1, a functional, ρ, from $\mathcal{M}$ into $[0,\infty]$ (the value ∞ is allowed) will be called a function norm (with respect to $\mathcal{S}$ and $\mathcal{Z}$) if it has the following properties:

(i) $\rho(f) = 0$ if and only if $f \in \mathcal{N}$;

(ii) $\rho(f) = \rho(|f|)$ for every $f \in \mathcal{M}$;

(iii) $\rho(\alpha f) = |\alpha| \rho(f)$ for every number α and every function $f \in \mathcal{M}$;

(iv) $\rho(f + g) \leq \rho(f) + \rho(g)$ for every $f \in \mathcal{M}^+$ and $g \in \mathcal{M}^+$; and

(v) if $f \in \mathcal{M}^+$, $g \in \mathcal{M}^+$ and $f \leq g$, then $\rho(f) \leq \rho(g)$.

Given a function norm, ρ, let $\mathcal{K}_\rho = \{f \in \mathcal{M} : \rho(f) < \infty\}$. Then the restriction of ρ to $\mathcal{K}_\rho$ is a seminorm; it will be called the seminorm induced by the function norm ρ and still denoted by ρ. Our aim is to characterize those function norms which induce in this manner integrating seminorms such that $\mathcal{K}_\rho = \mathcal{L}(\rho,\mathcal{K}_\rho)$ and the family of ρ-null functions coincides with $\mathcal{N}$.

The function norm, ρ, is said to have the Riesz-Fischer property, see [48], Definition 4.1, if, for any functions $f_j \in \mathcal{M}^+$, $j = 1,2,\ldots$, such that

(C.1) $$\sum_{j=1}^{\infty} \rho(f_j) < \infty,$$

the set, Y, of all points $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} f_j(\omega) = \infty$$

belongs to $\mathcal{Z}$, and, if f is a function on Ω such that

(C.2) $$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ not belonging to Y, then $f \in \mathcal{K}_\rho$.

For the sake of unity of style, our formulation differs from that of W.A.J. Luxemburg and A.C. Zaanen, but the difference is merely technical. Luxemburg and Zaanen achieve some simplicity of the formulation by admitting into $\mathcal{M}$ also functions with infinite values. However, the resulting theories are equivalent because then, for every function $f \in \mathcal{M}$ such that $\rho(f) < \infty$, the set $Y = \{\omega : |f(\omega)| = \infty\}$ belongs to $\mathcal{Z}$. Indeed, $Y \leq n^{-1}|f|$, and, by (iii) and (v), $\rho(Y) \leq n^{-1}\rho(f)$, for every $n = 1,2,\dots$. So, $\rho(Y) = 0$, and, by (i), $Y \in \mathcal{Z}$.

The following lemma and proposition are due to I. Halperin and W.A.J. Luxemburg, [20].

LEMMA 3.4. *If* ρ *has the Riesz-Fischer property, then*

$$\rho(f) \leq \sum_{j=1}^{\infty} \rho(f_j),$$

whenever $f_j \in \mathcal{M}$, $j = 1,2,\dots,$ *are functions satisfying condition* (C.1) *and* $f \in \mathcal{M}$ *is a function such that*

$$|f(\omega)| \leq \sum_{j=1}^{\infty} |f_j(\omega)|$$

for every $\omega \in \Omega$.

Proof. If not, there exist such f_j, $j = 1,2,\dots,$ and f as in the statement of the lemma, but

$$\rho(f) > \alpha + \sum_{j=1}^{\infty} \rho(f_j)$$

with some $\alpha > 0$. Consequently, for each $k = 1,2,\dots,$ there exist functions $f_{kj} \in \mathcal{M}^+$, $j = 1,2,\dots,$ and a function $f_k \in \mathcal{M}^+$ such that

$$f_k(\omega) = \sum_{j=1}^{\infty} f_{kj}(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} f_{kj}(\omega) < \infty$$

and

(C.3)
$$\infty > \rho(f_k) > k + \sum_{j=1}^{\infty} \rho(f_{kj}) .$$

Because

$$\rho\left[\sum_{j=1}^{r} f_{kj}\right] \le \sum_{j=1}^{r} \rho(f_{kj})$$

for every $r = 1,2,\ldots,$ we can assume that, besides (C.3),

$$\sum_{j=1}^{\infty} \rho(f_{kj}) < k^{-2}$$

for every $k = 1,2,\ldots$. Let us arrange the functions f_{kj} , $j = 1,2,\ldots,$ $k = 1,2,\ldots,$ into a single sequence g_n , $n = 1,2,\ldots$. Then

$$\sum_{n=1}^{\infty} \rho(g_n) < \sum_{k=1}^{\infty} k^{-2} < \infty .$$

Let g be a function such that

$$g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{n=1}^{\infty} g_n(\omega) < \infty .$$

Then, for every $k = 1,2,\ldots,$ there is a function $h_k \in \mathcal{N}$ such that $g(\omega) \ge f_k(\omega) + h_k(\omega)$, for every $\omega \in \Omega$, and, hence,

$$\rho(g) \ge \rho(f_k) \ge k .$$

So, $\rho(g) = \infty$, contrary to ρ having the Riesz-Fischer property.

PROPOSITION 3.5. *The function norm* ρ *has the Riesz-Fischer property if and only if the induced seminorm on* $\mathcal{K}_\rho$ *is integrating and* $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$.

Proof. If ρ has the Riesz-Fischer property, let $f_j \in \mathcal{K}_\rho$, $j = 1,2,\ldots$, be functions satisfying (3.2) and let f be a function on Ω such that (C.2) holds for every $\omega \in \Omega$ such that

$$\text{(C.4)} \qquad \sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then $f \in \mathcal{M}$ and $\rho(f) < \infty$, that is, $f \in \mathcal{K}_\rho$, because $\mathcal{Z} \subset \mathcal{S}$ and ρ has the Riesz-Fischer property. Furthermore,

$$f(\omega) - \sum_{j=1}^{n} f_j(\omega) = \sum_{j=n+1}^{\infty} f_j(\omega)$$

for every $\omega \in \Omega$ for which (C.4) holds, and so, by Lemma 3.4,

$$\rho\left[f - \sum_{j=1}^{n} f_j\right] \leq \sum_{j=n+1}^{\infty} \rho(f_j) ,$$

for every $n = 1,2,\ldots$. Therefore, ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$.

Conversely, if the seminorm induced by ρ is integrating and $\mathcal{L}(\rho,\mathcal{K}_\rho) = \mathcal{K}_\rho$, then, obviously, ρ has the Riesz-Fischer property.

Besides the L^p-spaces, the classical spaces which are covered by this proposition include notably the Orlicz spaces.

Let ι be a σ-finite measure in the space Ω ; that is, the space Ω is equal to the union of a sequence of ι-integrable sets. Let $\mathcal{M}$ be the family of all ι-measurable functions; the assumption of σ-finiteness implies that the definitions of measurability mentioned in the previous section are equivalent. $\mathcal{N}$ is the family of ι-null functions. Let Φ be a Young function. (See Section 1G.)

For any function $f \in \mathcal{M}$, let

$$M_\Phi(f) = \int_\Omega \Phi(|f|)\mathrm{d}\iota .$$

We are using the convention that, if the function $\omega \mapsto \Phi(|f(\omega)|)$, $\omega \in \Omega$, is not ι-integrable, then $M_\Phi(f) = \infty$. Let, further,

$$\rho_\Phi(f) = \inf\{k : k > 0 \, , \, M_\Phi(k^{-1}f) \le 1\}$$

for every $f \in \mathcal{M}$. Now we are using the convention that the infimum of the empty set is ∞.

PROPOSITION 3.6. *The functional* ρ_Φ *is a function norm having the Riesz-Fischer property.*

Proof. For brevity, we write $\rho = \rho_\Phi$.

If the set $\{\omega : |f(\omega)| > 0\}$ has non-zero measure, then, for some $\epsilon > 0$, the set $X = \{\omega : |f(\omega)| \ge \epsilon\}$ has non-zero measure and, hence, $M_\Phi(f) \ge \Phi(\epsilon)\iota(X) > 0$. Consequently, $0 < \rho(f) \le \infty$. Conversely, if $\iota(\{\omega : |f(\omega)| > 0\}) = 0$, then $M_\Phi(\alpha f) = 0$, for every $\alpha > 0$, and, hence, $\rho(f) = 0$.

Assume that $0 < \rho(g) < \infty$. Choosing a decreasing sequence of numbers k_n, $n = 1,2,\ldots,$ tending to $\rho(g)$ and applying the sequential form of the Beppo Levi theorem on the functions $k_n^{-1}|g|$, $n = 1,2,\ldots,$ tending point-wise monotonically to $(\rho(g))^{-1}|g|$, we deduce that $M_\Phi((\rho(g))^{-1}g) \le 1$. From this observation we deduce further that, if $|f| \le |g|$, then $\rho(f) \le \rho(g)$. For, if $|f| \le |g|$, then $M_\Phi((\rho(g))^{-1}f) \le M_\Phi((\rho(g))^{-1}g) \le 1$.

Now, assuming that $f \ge 0$, $g \ge 0$, $\rho(f) + \rho(g) = \gamma > 0$, let $\rho(f) = \alpha\gamma$ and $\rho(g) = \beta\gamma$, so that $\alpha + \beta = 1$. Then, by the Jensen inequality,

$$M_\Phi((f+g)/\gamma) = M_\Phi(\alpha f/\alpha\gamma + \beta g/\beta\gamma) \le \alpha M_\Phi(f/\rho(f)) + \beta M_\Phi(g/\rho(g)) \le \alpha + \beta = 1 \, ,$$

and so, $\rho(f+g) \le \gamma$.

From these remarks and from the definition of ρ, it follows easily that ρ satisfies all the requirements (i) - (v), which means that it is a function norm on $\mathcal{M}$.

To prove that ρ has the Riesz-Fischer property, let $g_n \in \mathcal{M}^+$, $n = 1,2,\ldots,$ be functions forming a non-decreasing sequence such that $0 < \alpha = \sup\{\rho(g_n) : n = 1,2,\ldots\} < \infty$. Because $M_\Phi(\alpha^{-1}g_n) \le 1$, for every $n = 1,2,\ldots,$ by the Beppo Levi theorem, $M_\Phi(\alpha^{-1}g) \le 1$. Hence, $\rho(g) \le \alpha$. In particular $g \in \mathcal{K}_\rho$. It is now evident that ρ has the Riesz-Fischer property.

The proof of Proposition 3.6 gives slightly more than the Riesz–Fischer property of the function norm ρ_{Φ} . For more details we refer to [72].

Let us note that the space $\mathcal{K}_\rho$, for $\rho = \rho_\Phi$, consists of all functions $f \in \mathcal{M}$ for which there exists a number $k > 0$ such that $M_\Phi(kf) < \infty$. Furthermore, if the Young function Φ satisfies condition (Δ_2) , then $f \in \mathcal{K}_\rho$ if and only if $M_\Phi(f) < \infty$. If $\iota(\Omega) < \infty$ and Φ satisfies condition (Δ_2) for large values of the argument, then $f \in \mathcal{K}_\rho$ if and only if $M_\Phi(f) < \infty$.

The space $\mathcal{K}_\rho$ is conventionally denoted by $\mathcal{L}^\Phi(\iota)$ and the corresponding normed space by $L^\Phi(\iota)$. These spaces are known as Orlicz spaces. One writes $\|f\|_{\Phi,\iota} = \rho_\Phi(f)$, that is,

$$\|f\|_{\Phi,\iota} = \inf\{k>0 : \int_\Omega \Phi(k^{-1}|f(\omega)|)\iota(d\omega) \le 1\} ,$$

for any function $f \in \mathcal{L}^\Phi(\iota)$. The seminorm $\|\cdot\|_{\Phi,\iota}$ and the induced norm on $L^\Phi(\iota)$ are called the Luxemburg seminorm and the Luxemburg norm, respectively. Another seminorm on $\mathcal{L}^\Phi(\iota)$ is defined by the formula

$$\|f\|^0_{\Phi,\iota} = \sup\{|\int_\Omega fg\, d\iota| : \int_\Omega \Psi(|g|)d\iota \le 1\} ,$$

for every $f \in \mathcal{L}^\Phi(\iota)$, where Ψ is the Young function complementary to Φ . (See Section 1G.) The so-defined seminorm and the corresponding norm on $L^\Phi(\iota)$ are called the Orlicz seminorm and the Orlicz norm, respectively. The inequalities

$$\|f\|_{\Phi,\iota} \le \|f\|^0_{\Phi,\iota} \le 2\|f\|_{\Phi,\iota}$$

hold for every $f \in \mathcal{L}^\Phi(\iota)$, so that the Luxemburg and the Orlicz norms are equivalent.

The classical reference about Orlicz spaces is [38]. Useful information can also be found in [39], especially Sections 3.1 – 3.9, and of course elsewhere.

For the definition of the class $\mathcal{L}^\Phi(\iota)$, the assumption that the measure ι be σ-finite is of course not necessary. Explicitly, $\mathcal{L}^\Phi(\iota)$ consists of the ι-measurable functions f on Ω for which there exists a number $k > 0$ (depending on f) such that

$$\int_{\Omega} \Phi(k^{-1}|f(\omega)|)\iota(d\omega) < \infty .$$

D. Another class of important and extensively studied integrating seminorms is constituted by the seminorms inducing the natural norms of the Sobolev spaces. Following A. Kufner, O. John and S. Fučík, [39], Section 5.1, we present a general scheme for introducing these spaces which may be useful also in other contexts.

Let $\mathcal{K}$ be a vector space of functions on a space Ω. Let ρ_0 be an integrating seminorm on $\mathcal{K}$. Let J be an index set. For every $\alpha \in J$, let ρ_α be an integrating seminorm on a vector space, $\mathcal{L}_\alpha$, of functions on a space Ω_α such that $\mathcal{L}_\alpha = \mathcal{L}(\rho_\alpha,\mathcal{L}_\alpha)$ and let $S_\alpha : \mathcal{K} \to \mathcal{L}_\alpha$ be a linear map.

The maps S_α, $\alpha \in J$, will be called collectively closable if $\rho_\alpha(h_\alpha) = 0$ for any functions $h_\alpha \in \mathcal{L}_\alpha$ for which there exist functions $g_n \in \mathcal{K}$, $n = 1,2,\ldots$, such that

(D.1) $$\lim_{n\to\infty} \rho_0(g_n) = 0 \text{ and } \lim_{n\to\infty} \rho_\alpha(S_\alpha g_n - h_\alpha) = 0 ,$$

for every $\alpha \in J$.

PROPOSITION 3.7. *If the set J is finite and the maps S_α, $\alpha \in J$, are collectively closable, then the functional, ρ, defined by*

(D.2) $$\rho(f) = \rho_0(f) + \sum_{\alpha\in J} \rho_\alpha(S_\alpha f) ,$$

for every $f \in \mathcal{K}$, is an integrating seminorm on $\mathcal{K}$.

Proof. Clearly, ρ is a seminorm. Let $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, be functions such that

$$\sum_{j=1}^{\infty} \rho(f_j) < \infty$$

and

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

Then

$$\sum_{j=1}^{\infty} \rho_0(f_j) < \infty \text{ and } \sum_{j=1}^{\infty} \rho_\alpha(f_j) < \infty$$

for every $\alpha \in J$. Let

$$g_n = \sum_{j=1}^{n} f_j ,$$

for $n = 1,2,\ldots$. Then, by Proposition 2.8, $\rho_0(g_n) \to 0$ as $n \to \infty$, because the seminorm ρ_0 is integrating. Furthermore, by Theorem 2.4, for every $\alpha \in J$, there exists a function $h_\alpha \in \mathcal{L}_\alpha$ such that $\rho_\alpha(S_\alpha g_n - h_\alpha) \to 0$ as $n \to \infty$. Then $\rho_\alpha(h_\alpha) = 0$, for every $\alpha \in J$, because the maps S_α, $\alpha \in J$, are collectively closable. By Proposition 2.1, $\rho_\alpha(S_\alpha g_n) \to 0$, for every $\alpha \in J$, and, hence, $\rho(g_n) \to 0$ as $n \to \infty$, because the set J is finite. By Proposition 2.8, the seminorm ρ is integrating.

To describe the most important particular cases, let $n \geq 1$ be an integer. Let ι be the Lebesgue measure in $\mathbb{R}^n$. Let Ω be a non-empty bounded open set in $\mathbb{R}^n$. Let $k \geq 0$ be an integer and $1 \leq p \leq \infty$. For J, we take the set of all n-tuples $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n)$ of integers $\alpha_1 \geq 0$, $\alpha_2 \geq 0$,..., $\alpha_n \geq 0$ such that

$$0 < |\alpha| = \sum_{j=1}^{n} \alpha_j \leq k .$$

For any such $\alpha \in J$, let $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$, where $D_1,D_2,\ldots,D_n$ are the operators of partial differentiation on $\mathbb{R}^n$ with respect to the first, second, ..., n-th variable, respectively.

Now, for $\mathcal{K}$, we take the space of all restrictions to Ω of C^∞-functions on $\mathbb{R}^n$ and let

$$(\rho_0(f))^p = \int_\Omega |f(\omega)|^p \iota(d\omega) ,$$

for every $f \in \mathcal{K}$. For every $\alpha \in J$, we take $\mathcal{L}_\alpha$ to be the space of all ι-measurable functions on Ω such that $(\rho_0(f))^p < \infty$ and put $\rho_\alpha(f) = \rho_0(f)$, for every $f \in \mathcal{L}_\alpha$. Finally, we put $S_\alpha f = \mathrm{D}^\alpha f$, for every $f \in \mathcal{K}$ and every $\alpha \in J$.

LEMMA 3.8. *For every* $\alpha \in J$, *the map* $S_\alpha : \mathcal{K} \to \mathcal{L}_\alpha$ *is closable.*

Proof. Let $g_n \in \mathcal{K}$, $n = 1,2,\ldots$, and $h_\alpha \in \mathcal{L}_\alpha$, be functions such that (D.1) holds. Then, by the Green formula and the Hölder inequality,

$$\int_\Omega h_\alpha \varphi \mathrm{d}\iota = \lim_{n\to\infty} \int_\Omega \mathrm{D}^\alpha g_n \varphi \mathrm{d}\iota = (-1)^{|\alpha|} \lim_{n\to\infty} \int_\Omega g_n \mathrm{D}^\alpha \varphi \mathrm{d}\iota = 0 ,$$

for every C^∞-function φ whose support is contained in Ω. Consequently, $\rho_\alpha(h_\alpha) = 0$.

This lemma obviously implies that the maps S_α, $\alpha \in J$, are collectively closable. So, by Proposition 3.7, the seminorm, ρ, defined by (D.2) for every $f \in \mathcal{K}$, is integrating. The corresponding Banach space $L(\rho,\mathcal{L})$ is usually denoted by $W^{k,p}(\Omega)$.

Let $\mathcal{K}_0$ be the space of all C^∞ functions with supports contained in Ω. Then $\mathcal{K}_0 \subset \mathcal{K}$, but the restriction of ρ to $\mathcal{K}_0$ is still denoted by ρ. The corresponding space $L(\rho,\mathcal{K}_0)$ is denoted by $W_0^{k,p}(\Omega)$. The spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ do not coincide, in general.

For further discussion, examples and ramification along these lines we refer to [39], Chapters 5, 7 and 8. The literature on the Sobolev space is of course very large.

E. Both the classical and the real-variable definitions of the Hardy spaces can be viewed as the special cases of the construction of the space $L(\rho,\mathcal{K})$ with a suitably chosen integrating gauge ρ on a family of functions $\mathcal{K}$. Let us start with the classical definition.

Let $1 \le p \le \infty$. Let $\mathcal{K} = \mathcal{K}_\delta$ be the family of complex functions on the closed unit disc $\bar{D}_1 = \{z \in \mathbb{C} : |z| \le 1\}$ for which there exists a δ such that $0 < \delta \le \infty$ and $f \in \mathcal{K}$ if and only if f has an analytic continuation on the disc $D_{1+\delta} = \{z \in \mathbb{C} : |z| < 1 + \delta\}$. In particular, $\mathcal{K}_\infty$ consists of the restrictions to $\bar{D}_1$ of the entire functions. Given an r such that $0 < r < 1$, let

$$\rho_r(f) = \left[\frac{1}{2\pi}\int_0^{2\pi} |f(r\exp(i\theta))|^p d\theta\right]^{1/p}$$

for every $f \in \mathcal{K}$. Finally, let

$$\rho(f) = \sup\{\rho_r(f) : 0 < r < 1\}$$

for every $f \in \mathcal{K}$.

PROPOSITION 3.9. *The functional ρ is an integrating seminorm on $\mathcal{K}$.*

Proof. We have $\rho(f) < \infty$ for every $f \in \mathcal{K}$ because every function belonging to $\mathcal{K}$ is bounded on $\bar{\mathrm{D}}_1$. Using the analyticity of the functions in $\mathcal{K}$, it is easy to deduce that each seminorm ρ_r, $0 < r < 1$, is integrating. Then, by Proposition 2.14, the seminorm ρ is integrating too.

The space $L(\rho,\mathcal{K})$ is usually denoted by H^p.

It may be noted that, for $\mathcal{K}$, the space of all complex polynomials could be taken, which is even smaller than $\mathcal{K}_\infty$, or, on the other hand, the space of all functions continuous in $\bar{D}_1$ and analytic in the open disc D_1, which is larger than all the spaces $\mathcal{K}_\delta$, $\delta > 0$.

The given definition of the space H^p can of course be adapted to the case of the space $\mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y \geq 0\}$, or even $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y \geq 0\}$ for any $n = 1,2,\ldots$, replacing the disc $\bar{\mathrm{D}}_1$.

Let us turn now to the real variable definition. We will consider only the H^1 spaces on $\mathbb{R}^n$. That will suffice for our purposes; any attempt to treat systematically the Hardy spaces, or even just their connection with the theory of integrating gauges, is out of place here anyway. We may refer, however, to the survey [6] in which the history and the richness of the subject are elegantly presented.

Let $n \geq 1$ be an integer. Let ι be the Lebesgue measure in $\mathbb{R}^n$.

By an H^1-atom in $\Omega = \mathbb{R}^n$ is understood any function, f, for which there exists a (solid) ball B such that $|f(\omega)| \leq (\iota(\mathrm{B}))^{-1}B(\omega)$, for ι-almost every $\omega \in \Omega$, and $\iota(f) = 0$. We say that the atom f is supported by the ball B. Let $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$ be the family of all H^1-atoms in Ω.

For every $f \in \mathcal{K}$, let

$$\rho_{\mathrm{a}}(f) = \inf \sum_{j=1}^{\infty} |c_j| ,$$

where the infimum is taken over all choices of the numbers c_j, $j = 1,2,\dots,$ such that

$$\text{(E.1)} \qquad \sum_{j=1}^{\infty} |c_j| < \infty ,$$

and there exist functions $f_j \in \mathcal{K}$, $j = 1,2,\dots,$ such that

$$\text{(E.2)} \qquad f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

$$\text{(E.3)} \qquad \sum_{j=1}^{\infty} |c_j| \, |f_j(\omega)| < \infty .$$

If c is a number and $f \in \mathcal{K}$, then, clearly, $\iota(|cf|) \le |c|$. Therefore, condition (E.1) implies that the inequality (E.3) is satisfied ι-almost everywhere. So, by the Beppo Levi theorem, $\iota(|f|) \le \rho_{\mathrm{a}}(f) \le 1$, for every $f \in \mathcal{K}$.

PROPOSITION 3.10. *The functional ρ_{a} is an integrating gauge on the family of functions $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$. A function f belongs to the space $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K})$ if and only if there exist numbers c_j, $j = 1,2,\dots,$ satisfying condition* (E.1), *and functions $f_j \in \mathcal{K}$, $j = 1,2,\dots,$ such that the equality* (E.2) *holds for every $\omega \in \Omega$ for which the inequality* (E.3) *does.*

Proof. Let $\sigma(f) = 1$, for every function $f \in \mathcal{K}$ which does not vanish ι-almost everywhere, and $\sigma(f) = 0$, if $f(\omega) = 0$ for almost every $\omega \in \Omega$. Then $q_\sigma(f) = \rho_{\mathrm{a}}(f)$, for every $f \in \mathcal{K}$, by the definition of q_σ (Section 2A) and that of ρ_{a}. Because, by Proposition 2.7, q_σ is an integrating seminorm on $\mathcal{L}(\sigma,\mathcal{K})$, its restriction, ρ_{a}, to the family $\mathcal{K}$ too is integrating.

Now, by Proposition 2.7, $\mathcal{L}(q_\sigma,\mathcal{L}(\sigma,\mathcal{K})) = \mathcal{L}(\sigma,\mathcal{K})$ and, by Corollary 2.5, $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K}) = \mathcal{L}(q_\sigma,\mathcal{L}(\sigma,\mathcal{K}))$. So, $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K}) = \mathcal{L}(\sigma,\mathcal{K})$ which means that the space $\mathcal{L}(\rho_{\mathrm{a}},\mathcal{K})$ consists precisely of those functions f for which there exist numbers c_j and atoms $f_j \in \mathcal{K}$, $j = 1,2,\dots$, such that (E.1) holds and the equality (E.2) holds for all $\omega \in \Omega$ for which the inequality (E.3) does.

In view of the atomic representation of $H^1(\mathbb{R}^n)$, this proposition says that the spaces $L(\rho_{\mathrm{a}},\mathcal{K})$ and $H^1(\mathbb{R}^n)$ are identical and their respective norms are equivalent. This fact can also be deduced from the consideration of their duals; cf. the discussion in [6]. We will only identify the dual of the space $L(\rho_{\mathrm{a}},\mathcal{K})$ by showing that the continuous linear functionals on it are generated by functions of bounded mean oscillation. Let us recall the definition.

A function F on $\Omega = \mathbb{R}^n$ is said to have bounded mean oscillation if it is locally integrable and there is a constant M such that

$$(\iota(B))^{-1}\int_B \left|F(\omega) - (\iota(B))^{-1}\int_B F\mathrm{d}\iota\right|\iota(\mathrm{d}\omega) \le M, \tag{E.4}$$

for every ball $B \subset \Omega$. The infimum of all the constants M for which (E.4) holds is denoted by $\|F\|_{\mathrm{BMO}}$. Let us note that $\|F\|_{\mathrm{BMO}} = 0$ if and only if the function F is ι-almost everywhere equal to a constant.

PROPOSITION 3.11. *If F is a function of bounded mean oscillation, then there exists a unique continuous linear functional, ℓ, on the space $L(\rho,\mathcal{K})$, $\mathcal{K} = \mathcal{K}_{\mathrm{a}}(H^1(\mathbb{R}^n))$, $\rho = \rho_{\mathrm{a}}$, such that*

$$\ell([f]_\rho) = \int_\Omega f(\omega)F(\omega)\iota(\mathrm{d}\omega), \tag{E.5}$$

for every $f \in \mathcal{K}$; the norm of ℓ is equal to $\|F\|_{\mathrm{BMO}}$. Conversely, for every continuous linear functional, ℓ, on $L(\rho,\mathcal{K})$, there is a function F of bounded mean oscillation such that (E.5) *holds for every $f \in \mathcal{K}$.*

Proof. Let the function F have bounded mean oscillation. Then the formula (E.5) determines the number $\ell([f]_\rho)$ unambiguously for every atom f. Moreover, if $f \in \mathcal{K}$, let c_j be numbers satisfying (E.1) and $f_j \in \mathcal{K}$ atoms, $j = 1,2,\ldots$, such that (E.2) holds for every $\omega \in \Omega$ for which (E.3) does; let the atom f_j be supported by the ball B_j, $j = 1,2,\ldots$. Then, by definitions of atoms and of $\|F\|_{\mathrm{BMO}}$,

$$\left|\int_\Omega f_j F \mathrm{d}\iota\right| = \left|\int_{B_j} f_j(\omega)\left[F(\omega) - (\iota(B_j))^{-1}\int_{B_j} F\mathrm{d}\iota\right]\iota(\mathrm{d}\omega)\right| \le \|F\|_{\mathrm{BMO}},$$

for every $j = 1,2,\ldots$. So, by the series version of the Beppo Levi theorem,

$$\ell([f]_\rho) = \int_\Omega fF\mathrm{d}\iota = \sum_{j=1}^{\infty} c_j \int_\Omega f_j F\mathrm{d}\iota .$$

Consequently, $|\ell([f]_\rho)| \le \rho(f)\|F\|_{\mathrm{BMO}}$, because the numbers c_j, $j = 1,2,\ldots$, can be chosen so that the sum of their absolute values is arbitrarily close to $\rho(f)$. This argument can obviously be applied to any function $f \in \mathcal{L}(\rho,\mathcal{K})$. Alternatively, Proposition 3.2 implies that there is a unique continuous linear functional ℓ on $L(\rho,\mathcal{K})$, satisfying (E.5) for every $f \in \mathcal{K}$, whose norm is not larger than $\|F\|_{\mathrm{BMO}}$. Because, however, there are atoms, f, such that $\rho(f) = 1$ and $\iota(fF)$ is as close to $\|F\|_{\mathrm{BMO}}$ as we please, the norm of ℓ is actually equal to $\|F\|_{\mathrm{BMO}}$.

Conversely, assume that ℓ is a continuous linear functional on $L(\rho,\mathcal{K})$. For $n = 1,2,\ldots$, let $\mathcal{L}_n$ be the subspace of $\mathcal{L}(\rho,\mathcal{K})$ consisting of the (equivalence classes of) functions, f, represented in the form (E.2), where the numbers c_j satisfy (E.1) and the atoms f_j are supported by balls wholly contained in $B_n = \{\omega : |\omega| \le n\}$, $j = 1,2,\ldots$. Then $\mathcal{L}_n$ contains all essentially bounded functions supported by B_n with integral equal to 0. Because the dual space of L^∞ (on a space of finite measure) is equal to L^1, there is a function F_n, determined uniquely ι-almost everywhere on B_n, such that $\ell([f]_\rho) = \iota(fF_n)$, for every $f \in \mathcal{L}_n$, and

$$\int_{B_n} F_n \mathrm{d}\iota = 0 ,$$

$n = 1,2,\ldots$. Consequently, there is a locally integrable function F which coincides ι-almost everywhere on B_n with the function F_n, for every $n = 1,2,\ldots$. Then, using a similar argument as in the first part of the proof, it is straightforward to deduce that

$\ell([f])_\rho) = \iota(fF)$, for every $f \in \mathcal{L}(\rho,\mathcal{K})$, or just for every $f \in \mathcal{K}$. By what we have proved already, the function F has bounded mean oscillation.

F. Let E be a Banach space. Let $\mathcal{Q}$ be a quasiring of sets in a space Ω and $\mu : \mathcal{Q} \to E$ an additive set function. (See Sections 1D and 1E.)

For every set $X \in \mathcal{Q}$, let

$$\iota(X) = \sup \sum_{j=1}^{n} |\mu(X_j)| ,$$

where the supremum is taken over all integers $n = 1,2,...$ and all choices of pair-wise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,...,n$, whose union is equal to X . Then ι is an extended real valued additive set function on $\mathcal{Q}$ such that

(i) $|m(X)| \leq \iota(X)$, for every $X \in \mathcal{Q}$; and

(ii) if κ is any extended real valued additive set function on $\mathcal{Q}$ such that $|\mu(X)| \leq \kappa(X)$, for every $X \in \mathcal{Q}$, then $\iota(X) \leq \kappa(X)$, for every $X \in \mathcal{Q}$.

The set function ι is called the variation of μ . We write $v(\mu) = \iota$, $v(\mu,X) = \iota(X)$ for $X \in \mathcal{Q}$ and even $v(\mu,f) = \iota(f)$ for any function f such that $\iota(f)$ is defined. Alternatively, we write $|\mu| = \iota$.

The set function μ is said to have finite variation if $v(\mu,X) < \infty$ for every $X \in \mathcal{Q}$.

It is well-known that the variation of a σ-additive set function is σ-additive. Also, if the space E is finite-dimensional, $\mathcal{Q}$ is a δ-ring and the set function $\mu : \mathcal{Q} \to E$ is σ-additive, then μ has finite variation.

The conventions about the integration 'with respect to μ ' are not fixed even if the set function $\mu : \mathcal{Q} \to E$ is σ-additive. The reason being that there may exist several gauges on $\mathcal{Q}$, or sim($\mathcal{Q}$) , integrating for μ but generating different spaces of integrable functions, all considered 'natural' from alternative points of view.

If μ has finite variation which is σ-additive then we can let $v(\mu)$ integrate for μ . That is to say, we let $\iota = v(\mu)$ and note that there exists a unique linear map $\mu_\iota : \mathcal{L}(\iota) \to E$ such that

(i) $\mu_\iota(f) = \mu(X)$, whenever f is the characteristic function of a set $X \in \mathcal{Q}$;

and

(ii) $|\mu_\iota(f)| \leq \iota(|f|)$, for every $f \in \mathcal{L}(\iota)$.

Then we write $\mathcal{L}(\mu) = \mathcal{L}(\iota)$ and

$$\mu(f) = \int_\Omega f\,d\mu = \int_\Omega f(\omega)\mu(d\omega) = \mu_\iota(f)$$

for every $f \in \mathcal{L}(\mu)$. It is often assumed that $\mathcal{Q}$ is a σ-algebra, or at least a δ-ring, [10], but this assumption has no significant bearing on the theory.

Another possibility arises when, for every $x' \in E'$, the set function $x' \circ \mu$ has finite and σ-additive variation and

(F.1) $$\sup\{v(x' \circ \mu, X) : x' \in E' , |x'| \leq 1\} < \infty$$

for every $X \in \mathcal{Q}$. In that case, let

(F.2) $$\rho(f) = \sup\{v(x' \circ \mu, |f|) : x' \in E' , |x'| \leq 1\} ,$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. By Proposition 3.3 and Proposition 2.14, ρ is an integrating seminorm on $\mathrm{sim}(\mathcal{Q})$. Obviously, the seminorm ρ integrates for μ . So, one can define $\mathcal{L}(\mu) = \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$ and

$$\mu(f) = \int_\Omega f\,d\mu = \int_\Omega f\,d_\rho\mu$$

for every $f \in \mathcal{L}(\mu)$.

Condition (F.1) is surely satisfied and the seminorm (F.2) integrates for μ if μ has finite σ-additive variation. In that case, $\mathcal{L}(v(\mu)) \subset \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$ and the inclusion may be proper even when $\mathcal{Q}$ is a σ-algebra.

EXAMPLE 3.12. Let $\Omega = \{1,2,...\}$ be the set of all positive integers and let $\mathcal{Q}$ be the family of all subsets of Ω . Let $\{x_j\}_{j=1}^\infty$ be an absolutely summable sequence of elements of the space E . Let

$$\mu(X) = \sum_{\omega \in X} x_\omega$$

for every $X \in \mathcal{Q}$.

Then $\mathcal{L}(v(\mu))$ consists of all functions f on Ω such that the sequence $\{f(j)x_j\}_{j=1}^{\infty}$ is absolutely summable. The space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ consists of all functions f on Ω such that the sequence $\{f(j)x_j\}_{j=1}^{\infty}$ is unconditionally summable.

So, if μ has finite and σ-additive variation, then the symbol " $\mathcal{L}(\mu)$ " is ambiguous and would remain so even if the domain of μ were indicated. Though, if the space E is finite-dimensional, then $\mathcal{L}(\iota) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$, with $\iota = v(\mu)$ and ρ defined by (F.2), and the respective seminorms are equivalent.

Of course, it might be possible to form the space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$, with ρ defined by (F.2), also when μ does not have finite variation. By the following proposition, this space surely can be formed when $\mathcal{Q}$ is a δ-ring and μ is σ-additive.

PROPOSITION 3.13. *Let $\mathcal{Q}$ be a δ-ring of sets in the space Ω . Let $\mathcal{S}$ be the σ-algebra of all sets $X \subset \Omega$ such that $X \cap Z \in \mathcal{Q}$ for every $Z \in \mathcal{Q}$. Let $\mu : \mathcal{Q} \to E$ be a σ-additive set function.*

Then, for every $x' \in E'$, the set function $x' \circ \mu$ has finite variation and the inequality (F.1) *holds for every $X \in \mathcal{Q}$.*

Let the seminorm ρ be defined by (F.2) *for every $f \in \mathrm{sim}(\mathcal{Q})$. Then the seminorm ρ integrates for μ . The seminorm ρ is monotonic and the space $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ is a vector lattice. A function on Ω is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$. The seminorm ρ is equivalent to the seminorm σ defined by*

$$\sigma(f) = \sup\{\,|\mu(Xf)|\, : X \in \mathcal{Q}\}$$

for every $f \in \mathrm{sim}(\mathcal{Q})$.

Let f be a function on Ω . Then the following statements are equivalent:

(i) $f \in \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$.

(ii) *There exist $\mathcal{Q}$-simple functions f_n , $n = 1,2,\ldots,$ such that*

(F.3) $$f(\omega) = \lim_{n\to\infty} f_n(\omega)$$

for ρ-almost every $\omega \in \Omega$ and the sequence $\{\mu(Xf_n)\}_{n=1}^{\infty}$ converges to an element of the space E, for every $X \in \mathcal{S}$.

(iii) *For every $x' \in E'$, the function f is $v(x' \circ \mu)$-integrable and, for every $X \in \mathcal{S}$, there exists an element $\nu(X)$ of E such that*

$$x'(\nu(X)) = \int_{\Omega} Xf\,\mathrm{d}(x' \circ \mu)$$

for every $x' \in E'$.

Proof. Some of the statements were already proved. The equivalence of the seminorms ρ and σ was noted by R.G. Bartle, N. Dunford and J.T. Schwartz in [2]; see also [14], Lemma IV.10.4(b). They also noted that a set is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$. Hence, by Proposition 2.2, a function is ρ-null if and only if it is $v(x' \circ \mu)$-null for every $x' \in E'$.

Given a function f on Ω, the equivalence of the statements (i), (ii) and (iii) was essentially proved by D.R. Lewis in [44]. In fact, (i) obviously implies (ii) and (ii) implies (iii). Now, let $\mathcal{K}$ be the family of all functions f for which the statement (iii) holds. Define $\rho(f)$ by (F.2) for every $f \in \mathcal{K}$. Then $\rho(f) < \infty$, for every $f \in \mathcal{K}$ because, by the Orlicz-Pettis lemma, the set function ν is σ-additive and

$$\rho(f) = \sup\{v(x' \circ \nu, \Omega) : x' \in E' \,|\, \leq 1\}.$$

By Theorem 3.5 of [44], for every $f \in \mathcal{K}$ and $\epsilon > 0$, there exists a function $g \in \mathrm{sim}(\mathcal{Q})$ such that $\rho(f-g) < \epsilon$. Hence, by Theorem 2.4, for every $f \in \mathcal{K}$, one can produce a sequence, $\{f_n\}_{n=1}^{\infty}$, of $\mathcal{Q}$-simple functions such that

$$\lim_{n\to\infty} \rho(f-f_n) = 0$$

and (F.3) holds for ρ-almost every $\omega \in \Omega$. So, $\mathcal{K} \subset \mathcal{L}(\rho, \mathrm{sim}(\mathcal{Q}))$.

This proposition summarizes the main approaches to integration 'with respect to Banach valued measures' of not necessarily finite variation which appeared in the literature. R.G. Bartle, N. Dunford and J.T. Schwartz, [2], used condition (ii) to define integrability in the case when $\mathcal{Q}$ is a σ-algebra; see also [14], Section IV.10. Property (iii) was used by D.R. Lewis in [44] and [45]. Different approaches, leading to different spaces of integrable functions are of course possible. One of them will be described in Example 4.27 (Section 4F).

G. The structure described in Section 3A represents a possibility for defining, in a reasonably systematic manner, integrals of the form

$$\int_a^b f\,\mathrm{d}w\,,$$

where w is an arbitrary continuous function in the interval $[a,b]$. In this section, we present a way of doing so sketched in [32]. We shall return to this theme again in Sections 4C and 4D, where we impose on w some additional conditions, similar to but still much weaker than the finiteness of variation, and, on the other hand, extend the generality of the whole set-up.

Let w be a bounded continuous real or complex valued function on the real-line, $\Omega = (-\infty,\infty)$.

Let $C_0((-\infty,\infty])$ be the Banach space of all functions continuous on the two-point compactification, $[-\infty,\infty]$, of the space Ω and vanishing at $-\infty$, under the usual sup-norm, $\|\cdot\|_\infty$. Let E be the space of all bounded sequences of elements of $C_0((-\infty,\infty])$ equipped with the norm defined by

$$\|\varphi\| = \sup\{\|\varphi_n\|_\infty : n = 1,2,\ldots\}\,,$$

for every element, $\varphi = \{\varphi_n\}_{n=1}^\infty$, of E. Let F be the subspace of E consisting of those sequences of elements of $C_0((-\infty,\infty])$ which are convergent in $C_0((-\infty,\infty])$.

Let ι be the Lebesgue measure in the space Ω. As usual, this measure is not shown in integrals written down using a dummy variable. For the functions f and g

on Ω, we denote

$$(f*g)(t) = \int_{\Omega} f(t-s)g(s)\mathrm{d}s ,$$

for every $t \in \Omega$ for which this integral exists (in the sense described in Section B).

Now, let k_n, $n = 1,2,\ldots,$ be continuously differentiable functions on Ω, ι-integrable together with their derivatives such that $k_n*\varphi \to \varphi$, as $n \to \infty$, uniformly on Ω, for every continuous function φ on Ω with compact support, and $k_n'*\varphi \to \varphi'$, as $n \to \infty$, uniformly on Ω, for every continuously differentiable function φ on Ω with compact support (the dash denotes the derivative). For example, we can take

$$k_n(t) = \frac{n}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}nt^2) ,$$

for every $t \in \Omega$ and $n = 1,2,\ldots$.

Given a function $f \in \mathcal{L}(\iota)$, let

$$\nu_n(f)(t) = \int_{-\infty}^{t} f(s)(k_n'*w)(s)\mathrm{d}s ,$$

for every $t \in [-\infty,\infty]$ and $n = 1,2,\ldots$.

Let $\mathcal{K}$ be the vector space of all functions $f \in \mathcal{L}(\iota)$ such that the sequence $\nu(f) = \{\nu_n(f)\}_{n=1}^{\infty}$ belongs to E, and let

$$\rho(f) = \iota(|f|) + \|\nu(f)\| ,$$

for every $f \in \mathcal{K}$. Let $\mathcal{J}$ be the subspace of $\mathcal{K}$ consisting of the functions $f \in \mathcal{K}$ such that $\nu(f)$ belongs to F.

PROPOSITION 3.14. *The functional ρ is an integrating seminorm on $\mathcal{K}$ such that $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ and $\mathcal{L}(\rho,\mathcal{J}) = \mathcal{J}$.*

Proof. It is obvious that ρ is a seminorm. So, let $f_j \in \mathcal{K}$, $j = 1,2,\ldots,$ be functions such that

$$\sum_{j=1}^{\infty} \rho(f_j) < \infty$$

and let f be a function on Ω such that

$$f(t) = \sum_{j=1}^{\infty} f_j(t) ,$$

for every $t \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(t)| < \infty .$$

Then

$$\sum_{j=1}^{\infty} \iota(|f_j|) < \infty ,$$

and, by the Beppo Levi theorem (or Proposition 2.1 applicable by Proposition 2.13), $f \in \mathcal{L}(\iota)$. Also

$$\sum_{j=1}^{\infty} \|\nu(f_j)\| < \infty ,$$

and, because the space E is complete, there exists an element, $\varphi = \{\varphi_n\}_{n=1}^{\infty}$, of E such that

$$\sum_{j=1}^{\infty} \nu(f_j) = \varphi$$

in the sense of convergence in the space E. It follows that $\nu_n(f) = \varphi_n$, because the continuity of the map $\nu_n : \mathcal{L}(\iota) \to C_0((0,\infty])$ implies that

$$\nu_n(f) = \sum_{j=1}^{\infty} \nu_n(f_j) ,$$

for every $n = 1,2,\ldots$. So, by the definition of $\mathcal{K}$, the function f belongs to it, by Proposition 2.8, the seminorm ρ is integrating and, by the definition of the space $\mathcal{L}(\rho,\mathcal{K})$, the equality $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{K}$ holds. Then also the restriction of ρ to $\mathcal{J}$ is integrating and the same argument shows that $\mathcal{L}(\rho,\mathcal{J}) = \mathcal{J}$.

Let LIM be a Banach limit. That is, LIM is a continuous linear functional on the space of all bounded sequences of scalars equipped with the sup-norm, independent on any finite number of coordinates, such that

$$\mathrm{LIM}\{\alpha_n\}_{n=1}^{\infty} = \lim_{n\to\infty} \alpha_n$$

whenever the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is convergent.

Given a function $f \in \mathcal{K} = \mathcal{L}(\rho,\mathcal{K})$, we define

$$\int_{-\infty}^{t} f\,\mathrm{d}w = \mathrm{LIM}\ \{\nu_n(f)(t)\}_{n=1}^{\infty}$$

for every $t \in [-\infty,\infty]$. Then

$$f \mapsto \mu_{\mathrm{LIM}}(f) = \int_{-\infty}^{\infty} f\,\mathrm{d}w\,,\ f \in \mathcal{K}\,,$$

is a continuous linear functional on the complete space $\mathcal{L}(\rho,\mathcal{K})$ such that

$$\int_{-\infty}^{\infty} f\,\mathrm{d}w = -\int_{-\infty}^{\infty} f'(t)w(t)\mathrm{d}t\,,$$

for every continuously differentiable function f on Ω with compact support. This functional depends of course on the choice of the Banach limit LIM. However, its values on the functions belonging to $\mathcal{J}$ are determined uniquely.

4. SET FUNCTIONS

Given an additive set function, μ, on a semiring of sets, $\mathcal{Q}$, the problem arises naturally of finding a gauge which integrates for μ. (See Section 3A.) If there exists a finite non-negative σ-additive set function, ι, on $\mathcal{Q}$ such that $|\mu(X)| \leq \iota(X)$, for every $X \in \mathcal{Q}$, then μ is said to have finite variation. In that case, ι is a gauge integrating for μ. This situation is classical.

The point of this chapter is that, even when μ does not have finite variation, there may exist gauges integrating for μ. For, there may exist a continuous, convex and increasing function, Φ, on $[0,\infty)$ such that $\Phi(0) = 0$ and a σ-additive set function $\iota : \mathcal{Q} \to [0,\infty)$ such that $\Phi(|\mu(X)|) \leq \iota(X)$, for every $X \in \mathcal{Q}$. Then $|\mu(X)| \leq \rho(X)$, where $\rho(X) = \varphi(\iota(X))$, for every $X \in \mathcal{Q}$, and φ is the inverse function to Φ. By Proposition 2.26, the gauge ρ is integrating.

So, we are led to the consideration of higher variations introduced by N. Wiener and L.C. Young. (See Example 4.1 in Section A below.)

A. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Recall that, by $\Sigma = \Sigma(\mathcal{Q})$ is denoted the set of all families of pair-wise disjoint sets belonging to $\mathcal{Q}$. (See Section 1D.) An element, $\mathcal{P}$, of Σ such that its union is equal to Ω and, for every $X \in \mathcal{Q}$, the sub-family $\{Y \in \mathcal{P} : Y \cap X \neq \emptyset\}$ of $\mathcal{P}$ is finite, is called a partition. The set of all partitions is denoted by $\Pi = \Pi(\mathcal{Q})$.

Let E be a Banach space and $\mu : \mathcal{Q} \to E$ an additive set function.

Given a Young function Φ (see Section 1G), a set X from $\mathcal{Q}$ and a partition $\mathcal{P}$, let

(A.1) $$v_{\Phi}(\mu,\mathcal{P};X) = \sum_{Y \in \mathcal{P}} \Phi(|\mu(X \cap Y)|) .$$

Then, for the given Φ, X and a set of partitions $\Delta \subset \Pi$, let

(A.2) $$v_{\Phi}(\mu,\Delta;X) = \sup\{ v_{\Phi}(\mu,\mathcal{P};X) : \mathcal{P} \in \Delta\} .$$

The possibility of $v_\Phi(\mu,\Delta;X) = \infty$ is admitted. We write $v_\Phi(\mu;X) = v_\Phi(\mu,\Pi;X)$, for every $X \in \mathcal{Q}$.

The set function $v_\Phi(\mu,\Delta)$, that is, $X \mapsto v_\Phi(\mu,\Delta;X)$, $X \in \mathcal{Q}$, is called the Φ-variation of the set function μ with respect to the family of partitions Δ . The set function $v_\Phi(\mu) = v_\Phi(\mu,\Pi)$ is called simply the Φ-variation of μ . If $v_\Phi(\mu,\Delta;X) < \infty$ for every $X \in \mathcal{Q}$, the set function μ is said to have finite Φ-variation with respect to the set of partitions Δ .

In the case when $\Phi(s) = s^p$, or even when $\Phi(s) = cs^p$, for some constants $c > 0$ and $p \geq 1$ and every $s \in [0,\infty)$, we shall write simply $v_p(\mu,\Delta)$ instead of $v_\Phi(\mu,\Delta)$ and speak of the p-variation instead of the Φ-variation. Similar conventions are used without explicit mention in other symbols denoting objects depending on Φ , and in the corresponding terminology. The 1-variation, $v_1(\mu,\Delta)$, of the set function μ with respect to the family of partitions Δ is called simply the variation of μ with respect to Δ and denoted by $v(\mu,\Delta)$.

Formulas (A.1) and (A.2) have meaning as they stand for arbitrary quasirings, not only multiplicative ones. For, $X \cap Z = XZ \in \mathrm{sim}(\mathcal{Q})$, whenever $X \in \mathcal{Q}$ and $Z \in \mathcal{Q}$, and so, by the convention introduced in Section 1B, $\mu(X \cap Z)$ is well-defined. However, in such wider context, useful pronouncements would require more complicated formulations and the gained generality would be of little value.

On the other hand, it is sometimes advantageous to define $v_\Phi(\mu,\mathcal{P};X)$ and $v_\Phi(\mu,\Delta;X)$ by (A.1) and (A.2), respectively, for any set X belonging to the ring, $\mathcal{R} = \mathcal{R}(\mathcal{Q})$, generated by $\mathcal{Q}$, not only for $X \in \mathcal{Q}$. This represents no difficulty because every set belonging to $\mathcal{R}$ is equal to the union of a finite family of pair-wise disjoint sets belonging to $\mathcal{Q}$.

EXAMPLE 4.1. Let a and b be real numbers such that $a < b$. Let $\Omega = (a,b]$ and $\mathcal{Q} = \{(s,t] : a \leq s \leq t \leq b\}$. Let d be a function on the interval $[a,b]$ and let

$$\mu((s,t]) = d(t) - d(s) ,$$

for any s and t such that $a \leq s \leq t \leq b$.

Although not much attention seems to have been paid to Φ-variation of additive set functions in general, there is already considerable literature devoted to this case. To be sure, the Φ-variation of the set function μ is discussed in terms of the function d. In fact, if the partition $\mathcal{P}$ is determined by the points $a = s_0 < s_1 < s_2 < \dots < s_{n-1} < s_n = b$, that is, $\mathcal{P} = \{(s_{j-1}, s_j] : j = 1,2,\dots,n\}$, then

$$v_{\Phi}(\mu,\mathcal{P};\Omega) = \sum_{j=1}^{n} \Phi(|d(s_j) - d(s_{j-1})|) .$$

Actually, often the function d itself is the centre of interest, because some convergence properties of the Fourier series of d can be studied using the notion of the Φ-variation; see e.g. [66].

Besides $\Delta = \Pi$, the set of all dyadic partitions is often taken for Δ, especially when $a = 0$ and $b = 1$.

The variation (that is, 1-variation) is a classical concept dating back to C. Jordan. The notion of the p-variation was introduced in this case by N. Wiener in [67]. It was subsequently studied by several authors, notably by L.C. Young, who considered, in [69], Stieltjes integration with respect to functions of finite p-variation and introduced, in [70], the notion of a function of finite Φ-variation. Spaces of functions of finite Φ-variation were studied by W. Orlicz and his collaborators, [51], [42], and by M. Bruneau, [4].

The notation and terminology are not firmly established in the literature although they seem to converge to similar ones to those adopted here.

The introduction of the set of partitions, Δ, as an additional parameter on which the Φ-variation, $v_{\Phi}(\mu,\Delta)$, depends, genuinely increases the generality of this notion. It is illustrated by the following classical

EXAMPLE 4.2. In the situation of Example 4.1, let $a = 0$ and $b = 1$. For every $m = 1,2,\dots,$ let $\mathcal{P}_m$ be a partition, determined by the points

$$0 = s_{m,0} < s_{m,1} < \dots < s_{m,n_m} = 1 ,$$

such that $\mathcal{P}_m \prec \mathcal{P}_{m+1}$, that is, every point $s_{m,\ell}$, $\ell = 0,1,\ldots,n_m$, is among the points determining the partition $\mathcal{P}_{m+1}$, and

$$\lim_{m\to\infty} \max\{s_{m,\ell} - s_{m,\ell-1} : \ell = 1,2,\ldots,n_m\} = 0 .$$

Let $\Delta = \{\mathcal{P}_m : m = 1,2,\ldots\}$. By a classical result of P. Lévy, [43], (see also [11], Theorem VIII.2.3) the limit

$$\lim_{m\to\infty} v_2(\mu,\mathcal{P}_m;\Omega)$$

exists for almost every, in the sense of the Wiener measure, continuous function d on $[0,1]$ and, hence, $v_2(\mu,\Delta;\Omega) < \infty$. However, $v_2(\mu,\Pi;\Omega) = \infty$. See, e.g., [64], §4.

EXAMPLE 4.3. Let $\Omega = \mathbb{R}$. Let $\mathcal{Q}$ be the family of all bounded Borel sets in Ω. Let ι be the Lebesgue measure on $\mathbb{R}$. Let $1 < p < \infty$ and let $E = L^p(\iota)$. If $X \in \mathcal{Q}$, let

$$(\mu(X))(t) = \lim_{u\to 0+} \frac{1}{\pi} \left[\int_{-\infty}^{t-u} + \int_{t+u}^{\infty} \right] \frac{X(s)}{t-s}\, ds ,$$

for every $t \in \mathbb{R}$ for which this limit exists. Then $\mu(X)$ represents an element of the space E. What is more, M. Riesz has proved, see [7], that there exists a constant, A, depending on p, such that

$$\|\mu(f)\|_{p,\iota}^p \le A \int_{-\infty}^{\infty} |f(s)|^p ds$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. Consequently, the resulting additive set function $\mu : \mathcal{Q} \to E$ has finite p-variation.

The Riesz estimate was extended to a wide class of kernels in Euclidean spaces of arbitrary dimension by A.P. Calderón and A. Zygmund, [7]. Accordingly, such kernels give rise to similar vector valued set functions of finite p-variation on bounded Borel sets in $\mathbb{R}^n$, $n = 1,2,\ldots$.

EXAMPLE 4.4. Let $\Omega = \mathbb{R}$ and let $\mathcal{Q}$ be the family of all bounded intervals (of all kinds) in Ω. Let

$$S_X f(s) = \int_X \hat{f}(\omega)\exp(2\pi i s\omega)\mathrm{d}\omega ,$$

for any $s \in \mathbb{R}$, $X \in \mathcal{Q}$ and any function f on $\mathbb{R}$ integrable with respect to the Lebesgue measure, where

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(s)\exp(2\pi i s\omega)\mathrm{d}s ,$$

for every $\omega \in \Omega$. J.L. Rubio de Francia, F.J. Ruiz and J.L. Torrea have proved, in [60], Corollary 2.4, that, for every $p \in [2,\infty)$, there exists a constant C_p such that

$$\sum_{X \in \mathcal{P}} \int_{\mathbb{R}} |S_X f(s)|^p \mathrm{d}s \leq C_p \int_{\mathbb{R}} |f(x)|^p \mathrm{d}s ,$$

for any such function f and every family of intervals $\mathcal{P} \in \Sigma(\mathcal{Q})$.

Consequently, if $E = L^p(\iota)$, $f \in \mathcal{L}^1 \cap \mathcal{L}^p(\iota)$, where ι is the Lebesgue measure in $\mathbb{R}$, and if, for every $X \in \mathcal{Q}$, we define $\mu(X)$ to be the element of the space E determined by the function $S_X f$, we obtain an additive set function $\mu : \mathcal{Q} \to E$ having finite p-variation.

PROPOSITION 4.5. *Let $\Delta \subset \Pi$, $\mathcal{P} \in \Delta$ and $X \in \mathcal{Q}$. Then*

$$\sum_{Y \in \mathcal{P}} v_\Phi(\mu,\Delta;X \cap Y) \leq v_\Phi(\mu,\Delta;X) ,$$

for any additive set function $\mu : \mathcal{Q} \to E$ and a Young function Φ.

Proof. It is obvious.

It is worth-while to note explicitly that, if the Young function Φ is not a multiple of the identity function on $[0,\infty)$, then the Φ-variation is not necessarily additive.

EXAMPLE 4.6. In the situation of Example 4.1, let $a = 0$, $b = 1$ and $d(s) = s$ for every $s \in [0,1]$. Then $v_2(\mu;(s,t]) = (t-s)^2$, for every s and t such that $0 \leq s \leq t \leq 1$.

B. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Let Φ be a Young function.

Recall that the set $\Pi = \Pi(\mathcal{Q})$ of all partitions is directed by the relation of refinement. (See Section 1D.) We refer to the same relation when we speak of directed subsets of Π.

To avoid some trivialities, we assume that, for every finite set $\mathcal{P}_0 \in \Sigma(\mathcal{Q})$, there exists a partition $\mathcal{P} \in \Pi$ such that $\mathcal{P}_0 \subset \mathcal{P}$.

Let E be a Banach space and $\mu : \mathcal{Q} \to E$ an additive set function.

PROPOSITION 4.7. *The Φ-variation, $v_\Phi(\mu)$, of the set function μ is additive if and only if*

(B.1) $$v_\Phi(\mu;X) = \sup\{v_\Phi(\mu,\mathcal{P};X) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\},$$

for every $X \in \mathcal{Q}$ and $\mathcal{P}_0 \in \Pi$.

Proof. For any $X \in \mathcal{Q}$ and $\mathcal{P}_0 \in \Pi$,

$$\sum_{Y\in\mathcal{P}_0} v_\Phi(\mu;X\cap Y) = \sum_{Y\in\mathcal{P}_0} \sup\{v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P} \in \Pi\} =$$
$$= \sum_{Y\in\mathcal{P}_0} \sup\{v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\} =$$
$$= \sup\left\{\sum_{Y\in\mathcal{P}_0} v_\Phi(\mu,\mathcal{P};X\cap Y) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\right\} = \sup\{v_\Phi(\mu,\mathcal{P};X) : \mathcal{P}_0 \prec \mathcal{P} \in \Pi\}.$$

Therefore,

$$v_\Phi(\mu;X) = \sum_{Y\in\mathcal{P}_0} v_\Phi(\mu;X\cap Y)$$

if and only if (B.1) holds.

Let ι be a non-atomic measure in the space Ω such that every set $X \in \mathcal{Q}$ is ι-integrable. (See Section 3B.)

For a partition $\mathcal{P} \in \Pi$, the ι-mesh, $\|\mathcal{P}\|_\iota$, of $\mathcal{P}$ is defined by

$$\|\mathcal{P}\|_\iota = \sup\{\iota(X) : X \in \mathcal{P}\}.$$

Because the cardinal number of $\mathcal{P}$ may be infinite, the possibility that $\|\mathcal{P}\|_\iota = \infty$ may occur.

A set of partitions $\Delta \subset \Pi$ will be called ι-fine if,

$$\inf\{\|\mathcal{P}\|_\iota : \mathcal{P} \in \Delta\} = 0 .$$

We say that the Φ-variation, $v_\Phi(\mu,\Delta)$, of μ with respect to a set of partitions, Δ, is ι-continuous if, for every $\epsilon > 0$, there is a $\delta > 0$ such that $v_\Phi(\mu,\Delta;X) < \epsilon$, for every set X in the ring, $\mathcal{R} = \mathcal{R}(\mathcal{Q})$, generated by $\mathcal{Q}$ such that $\iota(X) < \delta$. Recall that, by formula (A.2) in the previous section, $v_\Phi(\mu,\Delta;X)$ is indeed well-defined for any $X \in \mathcal{R}$.

Now, if $\Delta \subset \Pi$ is a directed set of partitions, then the family

$$\mathcal{Q}_\Delta = \{\emptyset\} \cup \bigcup_{\mathcal{P}\in\Delta} \mathcal{P} \tag{B.2}$$

of all sets, X, for which there exists a partition, $\mathcal{P} \in \Delta$, such that $X \in \mathcal{P}$, augmented by $\emptyset$, is a quasiring.

PROPOSITION 4.8. *Let Δ be a directed set of partitions. If*

$$v_\Phi(\mu,\Delta,X) = \lim_{\mathcal{P}\in\Delta} v_\Phi(\mu,\mathcal{P};X) , \tag{B.3}$$

for every $X \in \mathcal{Q}$, then the set function $v_\Phi(\mu,\Delta)$ is additive on the quasiring $\mathcal{Q}_\Delta$. If, moreover, $v_\Phi(\mu,\Delta)$ is ι-continuous then $v_\Phi(\mu,\Delta)$ is σ-additive on the whole of $\mathcal{R}$.

Proof. The first statement is obvious. The second one follows from the fact that, for every set $X \in \mathcal{R}$ and $\epsilon > 0$, there is a set Y, which is the union of a finite family of pair-wise disjoint sets from $\mathcal{Q}_\Delta$, such that $\iota(|X-Y|) < \epsilon$.

In some cases of great interest, instead of (B.3), the formula

$$v_\Phi(\mu,\Delta;X) = \lim_{r\to0+} \sup\{v_\Phi(\mu,\mathcal{P};X) : \|\mathcal{P}\|_\iota < r , \mathcal{P} \in \Delta\} \tag{B.4}$$

holds for every $X \in \mathcal{Q}$. It might be expected that this formula too would imply the additivity of $v_{\Phi}(\mu,\Delta)$. However, this is not necessarily the case.

EXAMPLE 4.9. Let the set-up be as in Example 4.1 with $a = 0$ and $b = 1$. By a result of S.J. Taylor, [64], Theorem 1, if Φ is a Young function such that

$$2s^{-2}\Phi(s) \log\log s^{-1} \to 1 ,$$

as $s \to 0+$, then, for almost every (in the sense of the Wiener measure) continuous function d on $[0,1]$, (B.4) holds with $\Delta = \Pi$ and with the Lebesgue measure in the role of ι. On the other hand, M. Bruneau proved, [5], Théorème 1, that the set of points $t \in [0,1]$ such that

$$v_{\Phi}(\mu,\Pi;(0,1]) = v_{\Phi}(\mu,\Pi;(0,t]) + v_{\Phi}(\mu,\Pi;(t,1]) ,$$

for almost every continuous function d, has empty interior.

Because $v_{\Phi}(\mu,\Delta)$ indeed, also in interesting cases, fails to be σ-additive, it is desirable to find a σ-additive set function $\sigma : \mathcal{Q} \to [0,\infty)$ such that $v_{\Phi}(\mu,\Delta;X) \le \sigma(X)$, for every $X \in \mathcal{Q}$. Such a set function σ can be used together with the inverse function, to Φ, to produce a gauge integrating for μ.

EXAMPLE 4.10. Let the set-up be as in Example 4.1 with arbitrary $a \in \mathbb{R}$ and $b \in \mathbb{R}$, $a \le b$. For some $\Delta \subset \Pi$, assume that $v_{\Phi}(\mu,\Delta;\Omega) < \infty$. Let

$$\sigma((s,t]) = v_{\Phi}(\mu,\Delta;(a,t]) - v_{\Phi}(\mu,\Delta;(a,s])$$

for any s and t such that $a \le s \le t \le b$.

Now, if Δ is a directed set of partitions, then σ is a non-negative and additive set function on the quasiring $\mathcal{Q}_{\Delta}$ such that $v_{\Phi}(\mu,\Delta;X) \le \sigma(X)$, for every $X \in \mathcal{Q}_{\Delta}$. If, moreover, Δ is ι-fine, where ι is the one-dimensional Lebesgue measure, and the function d is continuous, then σ is σ-additive on the whole of $\mathcal{Q}$ and the inequality $v_{\Phi}(\mu,\Delta;X) \le \sigma(X)$ holds for every $X \in \mathcal{Q}$.

If $\Delta = \Pi$, then σ is σ-additive on $\mathcal{Q}$ and $v_{\Phi}(\mu,\Delta;X) \leq \sigma(X)$, for every $X \in \mathcal{Q}$. This observation is due to L.C. Young, [71].

PROPOSITION 4.11. *Let $\mathcal{P}_n \in \Pi$ be a partition such that $\mathcal{P}_n \prec \mathcal{P}_{n+1}$, for every $n = 1,2,\ldots$, and*

$$\lim_{n\to\infty} \|\mathcal{P}_n\|_\iota = 0 .$$

Let $\Delta = \{\mathcal{P}_n : n = 1,2,\ldots\}$ and assume that $\iota(X) > 0$ for every non-empty set $X \in \mathcal{Q}_\Delta$.

Let Φ be a Young function such that μ has finite and ι-continuous Φ-variation with respect to the set of partitions Δ.

Then there exists a σ-additive set function $\sigma : \mathcal{Q} \to [0,\infty)$ such that

(B.5) $$v_{\Phi}(\mu,\Delta;X) \leq \sigma(X)$$

for every $X \in \mathcal{Q}_\Delta$.

Proof. Let

$$\sigma_1(X) = \sum_{Y\in\mathcal{P}_1} v_{\Phi}(\mu,\Delta;Y)(\iota(Y))^{-1}\iota(X\cap Y)$$

for every ι-measurable set X. Then σ_1 is a measure in Ω such that

$$v_{\Phi}(\mu,\Delta;X) = \sigma_1(X)$$

for every $X \in \mathcal{P}_1$.

Now, if $n \geq 1$ is an integer and σ_n a measure in Ω such that

(B.6) $$v_{\Phi}(\mu,\Delta;X) \leq \sigma_n(X)$$

for every $X \in \mathcal{P}_n$, for every set $Y \in \mathcal{P}_{n+1} \cup \{\emptyset\}$, let $w(Y)$ be a number such that $w(\emptyset) = 0$, $v_{\Phi}(\mu,\Delta;Y) \leq w(Y)$ and

$$\sum_{Y\in\mathcal{P}_{n+1}} w(X\cap Y) = \sigma_n(X)$$

for every $X \in \mathcal{P}_n$. By (B.6) and Proposition 4.5, such numbers $w(Y)$, $Y \in \mathcal{P}_{n+1}$, do exist. Then we put

$$\sigma_{n+1}(X) = \sum_{Y\in\mathcal{P}_{n+1}} w(Y)(\iota(Y))^{-1}\iota(X\cap Y)$$

for every ι-measurable set X. This defines a measure, σ_{n+1}, in Ω such that $\sigma_{n+1}(Y)$, for every $Y\in\mathcal{P}_{n+1}$, and $\sigma_{n+1}(X) = \sigma_n(X)$, for every $X\in\mathcal{P}_n$.

So, by induction, a sequence of measures, σ_n, $n = 1,2,\ldots$, is constructed such that, if we define

$$\sigma(X) = \lim_{n\to\infty} \sigma_n(X) ,$$

for every ι-measurable set X, we obtain a measure in Ω such that (B.5) holds for every $X\in\mathcal{Q}_\Delta$.

C. Let ι be a measure in a space Ω. Let $\mathcal{R}(\iota)$ be the family of all ι-integrable sets. (See Section 3B.) Let $\mathcal{Q}$ be a multiplicative quasiring of sets such that $\mathcal{Q}\subset\mathcal{R}(\iota)$. To avoid some trivialities, we assume that the measure ι is generated by its restriction to $\mathcal{Q}$. Let φ be a real valued, continuous, concave and strictly increasing function on $[0,\infty)$ such that $\varphi(0) = 0$. Let $\rho(X) = \varphi(\iota(X))$ for every $X\in\mathcal{Q}$. By Proposition 2.26, ρ is an integrating gauge on $\mathcal{Q}$.

The reason why we are interested in this situation is clear: If E is a Banach space, $\mu : \mathcal{Q}\to E$ an additive set function, Φ a Young function and $\Delta\subset\Pi(\mathcal{Q})$ a set of partitions such that $v_\Phi(\mu,\Delta;X) \le \iota(X)$, for every $X\in\mathcal{Q}$, then, assuming that φ is the inverse function to Φ, the gauge ρ integrates for the set function μ. (See Section 3A.)

The purpose of this and the next section is to provide some information about the space $\mathcal{L}(\rho,\mathcal{Q})$, namely to present workable sufficient conditions for a function to belong to $\mathcal{L}(\rho,\mathcal{Q})$. In this section, we discuss the relation of the spaces $\mathcal{L}(\rho,\mathcal{Q})$ and $\mathcal{L}^\Phi(\iota)$, where Φ is the inverse function to φ. (See Section 3C.)

PROPOSITION 4.12. *Let* $p\in[1,\infty)$ *and* $\varphi(t) = t^{1/p}$ *for every* $t\in[0,\infty)$. *Then* $\mathcal{L}(\rho,\mathcal{Q})\subset\mathcal{L}^p(\iota)$.

Proof. Let $f \in \mathcal{L}(\rho,\mathcal{Q})$. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, such that

(C.1) $$\sum_{j=1}^{\infty} |c_j| \rho(X_j) < \infty$$

and

(C.2) $$f(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every $\omega \in \Omega$ for which

(C.3) $$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Denote $f_j = c_j X_j$, for every $j = 1,2,\ldots$. Then $\|f_j\|_{p,\iota} = |c_j| (\iota(X_j))^{1/p} = |c_j| \rho(X_j)$, for every $j = 1,2,\ldots$. (See Section 3C.) So, by (C.1),

$$\sum_{j=1}^{\infty} \|f_j\|_{p,\iota} < \infty .$$

Consequently, $f \in \mathcal{L}^p(\iota)$.

The following proposition extends the above result to more general functions φ . (For the notion of a Young function, see Section 1G; for the definition of the class $\mathcal{L}^{\Phi}(\iota)$, see Section 3C.)

PROPOSITION 4.13. *Let φ be the inverse function to a Young function, Φ , and K a constant such that $0 < K < \varphi(t)\varphi(t^{-1})$ for every $t \in (0,\infty)$. Then $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}^{\Phi}(\iota)$.*

Proof. First, let c be a number, X a set belonging to $\mathcal{Q}$ and $g = cX$. Assume that $c \neq 0$ and $\iota(X) > 0$. Recall that the Luxemburg norm, $\|g\|_{\Phi,\iota}$, of the function g is defined by the formula

$$\|g\|_{\Phi,\iota} = \inf\{k : k > 0 , \int_{\Omega} \Phi(k^{-1}|g(\omega)|)\iota(\mathrm{d}\omega) \leq 1\} .$$

Hence, $\|g\|_{\Phi,\iota} = k$, where k is the number that satisfies the condition $\Phi(k^{-1}|c|)\iota(X) = 1$. It follows that $\|g\|_{\Phi,\iota} \leq K^{-1}|c|)\varphi(\iota(X)) = K^{-1}|c|\rho(X)$, where K is the constant mentioned in the statement of this proposition. This estimate is, obviously, true also if $c = 0$ or $\iota(x) = 0$.

The proof is now finished as that of Proposition 4.12. Namely, if $f \in \mathcal{L}(\rho,\mathcal{Q})$ and c_j are numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, satisfying (C.1), such that (C.2) holds for every $\omega \in \Omega$ for which (C.3) does, we denote $f_j = c_j X_j$, for every $j = 1,2,\ldots$. Then we use the obtained estimate of the Luxemburg norm to deduce from (C.1) that

$$\sum_{j=1}^{\infty} \|f_j\|_{\Phi,\iota} < \infty ,$$

which implies that $f \in \mathcal{L}^{\Phi}(\iota)$.

In the following proposition, no additional conditions are imposed on φ . (For the concepts used in its statement, see Section 1D.)

PROPOSITION 4.14. *If $\mathcal{Q}$ is an algebra of sets, then every bounded function measurable with respect to the σ-algebra generated by $\mathcal{Q}$ belongs to $\mathcal{L}(\rho,\mathcal{Q})$.*

Proof. Let $\mathcal{S}$ be the σ-algebra of sets generated by $\mathcal{Q}$. Because, for every set $Y \in \mathcal{S}$ and $\epsilon > 0$, there is a set $X \in \mathcal{Q}$ such that $\iota(|Y-X|) < \epsilon$ and the function φ is continuous, it is obvious that $\mathcal{S} \subset \mathcal{L}(\rho,\mathcal{Q})$. Then, by Proposition 2.7, $\mathcal{L}(q_\rho,\mathcal{S}) = \mathcal{L}(\rho,\mathcal{Q})$ and, by continuity, $q_\rho(Y) = \varphi(\iota(Y))$, for every $Y \in \mathcal{S}$. Hence, without a loss of generality, we can assume that $\mathcal{Q}$ is a σ-algebra.

Now, let f be a $\mathcal{Q}$-measurable function such that $0 \le f(\omega) \le 1$, for every $\omega \in \Omega$. Assuming that $k \ge 1$ is an integer and the sets X_j, $j = 1,2,\ldots,k-1$, are already constructed, let

$$X_k = \left\{\omega : f(\omega) - \sum_{j=1}^{k-1} 2^{-j} X_j(\omega) \ge 2^{-k}\right\} .$$

Then

$$\sum_{j=1}^{\infty} 2^{-j} \rho(X_j) \le \rho(\Omega) < \infty$$

and

$$f(\omega) = \sum_{j=1}^{\infty} 2^{-j} X_j(\omega)$$

for every $\omega \in \Omega$.

PROPOSITION 4.15. *Let* $\mathcal{Q}$ *be an algebra of sets. Let* $1 < p < q$ *and let* $\varphi(t) = t^{1/p}$, *for every* $t \geq 0$, *so that* $\rho(X) = (\iota(X))^{1/p}$, *for every* $X \in \mathcal{Q}$. *Then* $\mathcal{L}^q(\iota) \subset \mathcal{L}(\rho,\mathcal{Q})$.

Proof. Without loss of generality, we shall assume, as in the proof of Proposition 4.14, that $\mathcal{Q}$ is the family of all ι-measurable sets.

Let f be a non-negative function belonging to $\mathcal{L}^q(\iota)$. Let $X_j = \{\omega : f(\omega) \geq j\}$, for every $j = 1,2,\ldots$. Then

$$\sum_{j=1}^{\infty} q(j-1)^{q-1}\iota(X_j) \leq \sum_{j=1}^{\infty} (j^q-(j-1)^q)\iota(X_j) \leq \int_{\Omega} f^q d\iota < \infty ,$$

so that

$$\sum_{j=1}^{\infty} j^{q-1}\iota(X_j) < \infty .$$

By the Hölder inequality,

$$\sum_{j=1}^{\infty} \rho(X_j) = \sum_{j=1}^{\infty} (\iota(X_j))^{1/p} = \sum_{j=1}^{\infty} j^{(1-q)/p}(j^{q-1}\iota(X_j))^{1/p}$$

$$\leq \left[\sum_{j=1}^{\infty} j^{(1-q)/(p-1)}\right]^{(p-1)/p}\left[\sum_{j=1}^{\infty} j^{q-1}\iota(X_j)\right]^{1/p} < \infty ,$$

because $(q-1)/(p-1) > 1$. So, if we let

$$g(\omega) = \sum_{j=1}^{\infty} X_j(\omega) ,$$

for every $\omega \in \Omega$, then $g \in \mathcal{L}(\rho,\mathcal{Q})$.

Now, let $h = f - g$. Then $0 \leq h(\omega) \leq 1$, for every $\omega \in \Omega$. By Proposition 4.14, h belongs to $\mathcal{L}(\rho,\mathcal{Q})$ and, therefore, $f = g + h$ too belongs to $\mathcal{L}(\rho,\mathcal{Q})$.

The following examples settle some natural questions about the space $\mathcal{L}(\rho,\mathcal{Q})$. They were designed by Susumu Okada.

EXAMPLES 4.16. Let ι be the one-dimensional Lebesgue measure. Let $\Omega = (0,1]$, $\mathcal{Q} = \{(s,t] : 0 \leq s \leq t \leq 1\}$ and $\mathcal{R}$ be the algebra of sets generated by $\mathcal{Q}$. Let $1 < p$ and let $\rho(X) = (\iota(X))^{1/p}$, for every $X \in \mathcal{R}$. Then, obviously $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathcal{R})$ and, by

Proposition 4.12, $\mathcal{L}(\rho,\mathcal{R}) \subset \mathcal{L}^p(\iota)$. We wish to show that $\mathcal{L}(\rho,\mathcal{Q}) \neq \mathcal{L}(\rho,\mathcal{R})$ and $\mathcal{L}(\rho,\mathcal{R}) \neq \mathcal{L}^p(\iota)$. Let us denote, for short, $\alpha = p^{-1}$.

(i) Let us note first that there exists a constant $c_1 > 0$ such that

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}| \leq c_1 |t-s|^{\alpha},$$

for every $s \in \Omega$ and $t \in \Omega$. Indeed, let $0 < s < t < 1$. Let $n \geq 1$ be the integer such that $(n+1)^{-1} < t \leq n^{-1}$. Assume first that $(n+2)^{-1} \leq s$ and put $u = (n+2)^{-1}$ and $v = n^{-1}$, so that $v \leq 3u$. By the Lagrange theorem,

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}|\,|t-s|^{-\alpha} \leq 3|t-s|^{1-\alpha}s^{2\alpha-2} \leq 3(2uv)^{1-\alpha}u^{2\alpha-2} \leq 3.6^{1-\alpha}$$

If $s < (n+2)^{-1}$, then

$$|t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos s^{-1}|\,|t-s|^{-\alpha} \leq (n^{-2\alpha}+(n+2)^{-2\alpha})((n+1)^{-1} - (n+2)^{-1})^{-\alpha} \leq 2.2^{\alpha}.$$

Integrating by parts, we then obtain that

$$\left|\int_s^t u^{2\alpha-2}\sin u^{-1}du\right| \leq |t^{2\alpha}\cos t^{-1} - s^{2\alpha}\cos t^{-1}| + \int_s^t 2\alpha u^{2\alpha-1}du \leq c|t-s|^{\alpha},$$

for some $c > 0$ and every $s \in \Omega$ and $t \in \Omega$. So, if we put $d(0) = 0$ and

$$d(t) = \lim_{s\to 0+} \int_s^t u^{2\alpha-2}\sin u^{-1}du,$$

for every $t \in (0,1]$, then d is a well-defined continuous function on $[0,1]$.

Let $\mu((s,t]) = d(t) - d(s)$, for every s and t such that $0 \leq s \leq t \leq 1$. Furthermore, given a point $s \in \Omega$, let $\mu^s(X) = \mu(X \cap (s,1])$, for every $X \in \mathcal{Q}$. We have noted that $|\mu^s(X)| < c(\iota(X))^{\alpha} = c\rho(X)$, for some $c > 0$ and every $X \in \mathcal{Q}$. Therefore, by Proposition 3.1,

$$\left|\int_s^1 f(u)u^{2\alpha-2}\sin u^{-1}du\right| = |\mu^s_{\rho}(f)| \leq cq_{\rho}(f),$$

for every $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Let $g(t) = t^{1-2\alpha}\sin t^{-1}$, for every $t \in \Omega$. Then the function g does not belong to $\mathcal{L}(\rho,\mathcal{Q})$, because

$$\lim_{s\to 0+} \int_s^1 g(u)u^{2\alpha-2}\sin u^{-1}du = \infty .$$

None-the-less, g belongs to $\mathcal{L}(\rho,\mathcal{R})$. Indeed, if $p \geq 2$, that is, $\alpha \leq \frac{1}{2}$, this follows from Proposition 4.14. If $p < 2$, we choose a number $q \in (p,p/(2-p))$. Then $g \in \mathcal{L}^q(\iota)$ and, by Proposition 4.15, $g \in \mathcal{L}^q(\rho,\mathcal{R})$.

Consequently, $\mathcal{L}(\rho,\mathcal{Q}) \neq \mathcal{L}(\rho,\mathcal{R})$.

(ii) To show that $\mathcal{L}(\rho,\mathcal{R}) \neq \mathcal{L}^p(\iota)$, let $h(t) = t^{-\alpha}|\log t|^{-1}$, for $t \in (0,\frac{1}{2}]$, and $h(t) = 0$, for $t \in (\frac{1}{2},1]$. Then $h \in \mathcal{L}^p(\iota)$. However, the function h does not belong to $\mathcal{L}(\nu,\mathcal{R}) = \mathcal{L}(\nu)$, where

$$\nu(X) = \alpha \int_X u^{\alpha-1}du ,$$

for every $X \in \mathcal{Q}$. Using the fact that every set in $\mathcal{R}$ is the union of a finite collection of pair-wise disjoint intervals belonging to $\mathcal{Q}$, we can prove that $\nu(X) \leq \rho(X)$, for every $X \in \mathcal{R}$. Therefore, the function h does not belong to $\mathcal{L}(\rho,\mathcal{R})$ either.

D. We maintain the notation of Section C.

A function f on Ω will be called $\mathcal{Q}$-locally ι-integrable if it is integrable with respect to ι on every set belonging to $\mathcal{Q}$, that is, if $Xf \in \mathcal{L}(\iota)$ for every $X \in \mathcal{Q}$.

Now, assuming that f is a $\mathcal{Q}$-locally ι-integrable function, let

$$M_\iota(f,X) = \frac{1}{\iota(X)} \int_X f d\iota$$

for every set $X \in \mathcal{Q}$ such that $\iota(X) > 0$, and $M_\iota(f,X) = 0$ for every set X such that $\iota(X) = 0$. If $\iota(X) > 0$, then the number $M_\iota(f,X)$ is the mean value of the function f on the set X with respect to the measure ι .

Furthermore, if $\mathcal{P} \in \Pi(\mathcal{Q})$ is a partition, let

$$M_\iota(f,\mathcal{P}) = \sum_{X\in\mathcal{P}} M_\iota(f,X)X .$$

So, $M_\iota(f,\mathcal{P})$ is a function on Ω, constant on every set belonging to $\mathcal{P}$, having the same mean value as the function f on every set $X \in \mathcal{P}$ such that $\iota(X) > 0$.

Let ψ be a real valued, continuous, and strictly increasing function on $[0,\infty)$ such that $\psi(0) = 0$.

We shall say that a function f on Ω satisfies the ψ-Hölder condition with respect to the quasiring $\mathcal{Q}$ and the measure ι if

$$|f(\omega) - f(v)| \leq \psi(\iota(X)),$$

for every set $X \in \mathcal{Q}$ and any points $\omega \in X$ and $v \in X$.

PROPOSITION 4.17. *Let f be a ι-measurable function satisfying the ψ-Hölder condition with respect to $\mathcal{Q}$ and ι. Then f is $\mathcal{Q}$-locally ι-integrable.*

Let $\mathcal{P}_n \in \Pi(\mathcal{Q})$ be a partition such that $\mathcal{P}_n \prec \mathcal{P}_{n+1}$, for every $n = 0,1,2,\ldots$, and $\|\mathcal{P}_n\|_\iota \to 0$ as $n \to \infty$. If

$$\sum_{X\in\mathcal{P}_0} M_\iota(f,X)\varphi(\iota(X)) + \sum_{j=1}^{\infty} \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) < \infty,$$

then $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Proof. The first statement is clear, because the function f is bounded on every set belonging to $\mathcal{Q}$.

Let $f_0 = M_\iota(f,\mathcal{P}_0)$ and

$$f_j = M_\iota(f-M_\iota(f,\mathcal{P}_{j-1}),\mathcal{P}_j),$$

for every $j = 1,2,\ldots$. Then

$$\sum_{j=0}^{n} f_j = M_\iota(f,\mathcal{P}_n),$$

for $n = 0,1,2,\ldots$. Now,

$$q_\rho(f_0) \leq \sum_{X\in\mathcal{P}_0} M_\iota(f,X)\rho(X).$$

(See Section 2A.) Furthermore, for every $j = 1,2,\ldots,$

$$M_\iota(f-M_\iota(f,\mathcal{P}_{j-1}),Y) = M_\iota(f-M_\iota(f,Z),Y),$$

for any $Y \in \mathcal{P}_j$, where Z is the set belonging to $\mathcal{P}_{j-1}$ such that $Y \subset Z$. Then $|f(\omega) - \mathrm{M}_\iota(f,Z)|| \leq \psi(\iota(Z))$, for every $\omega \in Z$, and, hence,

$$|M_\iota(f-M_\iota(f,Z),Y)| \leq \psi(\iota(Z)) .$$

Consequently,

$$q_\rho(f_j) \leq \sum_{Y\in\mathcal{P}_j} |M_\iota(f-M_\iota(f,\mathcal{P}_{j-1}),Y)|\rho(Y) \leq \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) ,$$

for every $j = 1,2,\dots$. So, Proposition 2.1 applies.

COROLLARY 4.18. *Let $\Omega \in \mathcal{Q}$, $\mathcal{P}_0 = \{\Omega\}$, $\mathcal{P}_j \prec \mathcal{P}_{j+1}$, $\iota(X) = \|\mathcal{P}_j\|_\iota = (\frac{1}{2})^j\iota(\Omega)$, for every $X \in \mathcal{P}_j$ and $j = 0,1,2,\dots,$ and $\Delta = \{\mathcal{P}_j : j = 0,1,2,\dots\}$.*

If f is an ι-measurable function satisfying the ψ-Hölder condition with respect to the quasiring $\mathcal{Q}_\Delta$ and the measure ι, and

(D.1) $$\int_0^1 \frac{\varphi(t)\psi(t)}{t^2}\,dt < \infty ,$$

then $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Proof. Let $\alpha = \iota(\Omega)$. Because the functions φ and ψ are increasing,

$$\sum_{j=1}^{\infty} \sum_{Z\in\mathcal{P}_{j-1}} \psi(\iota(Z)) \sum_{Y\in\mathcal{P}_j} \varphi(\iota(Y\cap Z)) = \sum_{j=1}^{\infty} 2^j\psi(2^{1-j}\alpha)\varphi(2^{-j}\alpha) \leq$$

$$\leq \sum_{j=1}^{\infty} 2^{2j}\psi(2^{1-j}\alpha)\varphi(2^{1-j}\alpha)2^{-j} \leq 4\int_0^1 \frac{\varphi(\alpha t)\psi(\alpha t)}{t^2}\,dt = 4\alpha\int_0^\alpha \frac{\varphi(t)\psi(t)}{t^2}\,dt .$$

COROLLARY 4.19. *Let $\Omega = (a,b]$ with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a < b$. Let $\mathcal{Q} = \{(s,t] : a \leq s \leq t \leq b\}$. Let d be a function on $[a,b]$ such that*

$$|d(t) - d(s)| \leq \varphi(t-s) ,$$

and let

$$\mu((s,t]) = d(t) - d(s) \text{ and } \rho((s,t]) = \varphi(t-s) ,$$

for every s *and* t *such that* $a \leq s \leq t \leq b$. *Then* ρ *is a gauge integrating for the additive set function* μ .

If, moreover, f *is a function on* Ω *such that*

$$|f(t) - f(s)| \leq \psi(|t-s|) ,$$

for any $s \in \Omega$ *and* $t \in \Omega$, *and* (D.1) *holds, then* $f \in \mathcal{L}(\rho,\mathcal{Q})$.

Condition (D.1) is satisfied, in particular, when $\varphi(t) = c_1 t^{1/p}$ and $\psi(t) = c_2 t^{1/q}$, for every $t \geq 0$, where $c_1 > 0$, $c_2 > 0$ and $p^{-1} + q^{-1} > 1$.

E. In some sense the notion of an additive set function with finite p-variation is analogous to the notion of a (point) function locally belonging to an L^p space. The analogy reverses the extension of these notions though, because, if $p < q$, to have finite p-variation is a more restrictive condition than one to have finite q-variation. In this section, we introduce additive set functions which are analogous to functions locally belonging to an L^∞ space.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω . (See Section 1D.) Let E be a Banach space. Let $\mu : \mathcal{Q} \to E$ be an additive set function.

For any set $X \in \mathcal{Q}$, let

$$v_\infty(\mu;X) = \sup\{|\mu(X \cap Z)| : Z \in \mathcal{Q}\} . \tag{E.1}$$

The possibility $v_\infty(\mu;X) = \infty$ is admitted.

The set function μ will be called locally bounded if $v_\infty(\mu;X) < \infty$, for every $X \in \mathcal{Q}$.

A wealth of locally bounded additive set functions do not have finite Φ-variation for any Young function Φ is provided in Chapter 6. Here is a simple example of such a set function.

EXAMPLE 4.20. Let Ω and $\mathcal{Q}$ be as in Corollary 4.19. Let E be the Banach space of all bounded Borel measurable functions on Ω with the sup-norm. For every $X \in \mathcal{Q}$, let $\mu(X)$ be the characteristic function of X considered as an element of the space E. Then $\mu : \mathcal{Q} \to E$ is an additive set function such that $v_\infty(\mu;X) = 1$, but $v_\Phi(\mu;X) = \infty$, for every $X \in \mathcal{Q}$, $X \neq \emptyset$, no matter what the Young function Φ.

The set function μ will be called indeficient if it is locally bounded and the gauge, ρ, defined on $\mathcal{Q}$ by

(E.2) $$\rho(X) = v_\infty(\mu;X) ,$$

for every $X \in \mathcal{Q}$, is integrating. (See Section 2D.)

So, if the set function μ is indeficient then this gauge integrates for it. (See Section 3A.)

PROPOSITION 4.21. *The set function μ is indeficient if and only if it is locally bounded and*

(E.3) $$\sum_{j=1}^{\infty} c_j \mu(X_j) = 0 ,$$

for any numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,$ such that

(E.4) $$\sum_{j=1}^{\infty} |c_j| v_\infty(\mu;X_j) < \infty$$

and

(E.5) $$\sum_{j=1}^{\infty} c_j X_j(\omega) = 0$$

for every $\omega \in \Omega$ such that

(E.6) $$\sum_{j=1}^{\infty} |c_j| X_j(\omega) < \infty .$$

Proof. Let us show first that, if the condition is satisfied, then the gauge, ρ, defined by (E.2) is integrating. Let $X \in \mathcal{Q}$. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots,$ satisfying condition (E.4), such that

(E.7) $$X(\omega) = \sum_{j=1}^{\infty} c_j X_j(\omega)$$

for every ω satisfying the inequality (E.6). Let $\epsilon > 0$ and let $Z \in \mathcal{Q}$ be a set such that $\rho_\mu(X) < |\mu(X \cap Z)| + \epsilon$. Because

$$\lim_{n\to\infty} |\mu(X\cap Z) - \sum_{j=1}^{n} c_j\mu(X_j\cap Z)| = 0 ,$$

the inequality

$$\rho(X) - \epsilon < |\mu(X\cap Z)| \le \sum_{j=1}^{\infty} |c_j|\, |\mu(X_j\cap Z)| \le \sum_{j=1}^{\infty} |c_j|\rho(X_j)$$

holds. So, by Proposition 2.7, the gauge ρ is integrating.

Conversely, assume that μ is indeficient. That is, $v_\infty(\mu;X) < \infty$ for each $X \in Q$ and the gauge (E.2) integrates for μ. So, if c_j are numbers and $X_j \in Q$ sets, $j = 1,2,\ldots,$ satisfying (E.4), such that (E.5) holds for every $\omega \in \Omega$ for which (E.6) does, then, by Proposition 2.1,

$$\lim_{n\to\infty} q_\rho\left[\sum_{j=1}^{n} c_jX_j\right] = 0 .$$

Because

$$\left|\sum_{j=1}^{n} c_j\mu(X_j)\right| \le cq_\rho\left[\sum_{j=1}^{n} c_jX_j\right] ,$$

for some number $c \ge 0$ and every $n = 1,2,\ldots,$ (E.3) follows.

The following proposition is a simple means for producing examples: it helps us to prove the indeficiency of some additive set functions which arise in connection with classical improper integrals and are not σ-additive.

PROPOSITION 4.22. *Let the set function $\mu : Q \to E$ be locally bounded. Let $\Omega_n \in Q$ be sets such that $\Omega_n \subset \Omega_{n+1}$ and the restriction of μ to the quasiring $Q\cap\Omega_n$ is indeficient, for every $n = 1,2,\ldots,$ and that*

$$\lim_{n\to\infty} |\mu(X) - \mu(X\cap\Omega_n)| = 0 ,$$

for every $X \in Q$.

Then the set function μ is indeficient.

Proof. Let c_j be numbers and $X_j \in Q$ sets, $j = 1,2,\ldots,$ satisfying condition (E.4) such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does.

Let $\epsilon > 0$. Let J be a positive integer such that

$$\sum_{j=J+1}^{\infty} |c_j| v_\infty(\mu;X_j) < \epsilon .$$

Let m be a positive integer such that

$$\left| \sum_{j=1}^{J} c_j\mu(X_j) - \sum_{j=1}^{J} c_j\mu(X_j \cap \Omega_m) \right| < \epsilon .$$

Let N be a positive integer such that

$$\left| \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| < \epsilon$$

for every $n > N$. Such an integer N exists because, by the assumption, the restriction of μ to $\mathcal{Q} \cap \Omega_m$ is indeficient. Then

$$\left| \sum_{j=1}^{n} c_j\mu(X_j) \right| \le \left| \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| + \left| \sum_{j=1}^{n} c_j\mu(X_j) - \sum_{j=1}^{n} c_j\mu(X_j \cap \Omega_m) \right| \le$$

$$\le \epsilon + \left| \sum_{j=1}^{J} c_j\mu(X_j) - \sum_{j=1}^{J} c_j\mu(X_j \cap \Omega_m) \right| + \left| \sum_{j=J+1}^{n} c_j\mu(X_j) - \sum_{j=J+1}^{n} c_j\mu(X_j \cap \Omega_m) \right| \le$$

$$\le 2\epsilon + 2 \sum_{j=J+1}^{\infty} |c_j| v_\infty(\mu;X_j) < 4\epsilon ,$$

for every $n > \max\{J,N\}$. Hence, by Proposition 4.21, the set function μ is indeficient.

EXAMPLES 4.23. (i) A non-negative real valued additive set function on a quasiring of sets is indeficient if and only if it is σ-additive. This follows from Proposition 2.13 and Proposition 4.21. However, the argument establishing Proposition 2.13 can be simplified for the purpose of proving the indeficiency of such a set function directly.

So, let ι be a non-negative real valued additive set function on the quasiring $\mathcal{Q}$. Then $v_\infty(\iota,X) = \iota(X)$, for every $X \in \mathcal{Q}$.

If ι is not σ-additive, then, obviously, it is not an integrating gauge. Let us assume, therefore, that ι is σ-additive. We want to prove that

$$\text{(E.8)} \qquad \iota(X) \le \sum_{j=1}^{\infty} |c_j| \iota(X_j)$$

for any set $X \in \mathcal{Q}$, numbers c_j and sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots$, such that the equality (E.7) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Let $\epsilon > 0$ and, for every $n = 1,2,\ldots$, let Z_n be the set of those points $\omega \in X$ for which

$$\sum_{j=1}^{n} |c_j| X_j(\omega) > 1 - \epsilon .$$

Then $Z_n \in \mathrm{sim}(\mathcal{Q})$, $Z_n \subset Z_{n+1}$ and

$$\sum_{j=1}^{n} |c_j| \iota(X_j) \ge \sum_{j=1}^{n} |c_j| \iota(X_j \cap Z_n) \ge (1-\epsilon)\iota(Z_n) ,$$

for every $n = 1,2,\ldots$. Because ι is σ-additive on the ring of sets whose characteristic functions belong to $\mathrm{sim}(\mathcal{Q})$, and the union of the sets Z_n, $n = 1,2,\ldots$, is equal to X, there is an integer $n \ge 1$ such that $\iota(Z_n) > \iota(X) - \epsilon$. Hence,

$$\sum_{j=1}^{\infty} |c_j| \iota(X_j) \ge (1-\epsilon)(\iota(X)-\epsilon)$$

for every $\epsilon > 0$, and the inequality (E.8) follows. By Proposition 2.7, the gauge $X \mapsto v_\infty(\iota;X) = \iota(X)$ is integrating and, hence, ι is indeficient.

(ii) Let $\mathcal{Q}$ be a ring of sets and let μ be a locally bounded real valued σ-additive set function on $\mathcal{Q}$. Then μ is indeficient.

In fact, let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. So μ^+ and μ^- are non-negative σ-additive set functions on $\mathcal{Q}$ such that $\rho_\mu(X) \le \mu^+(X) + \mu^-(X)$ and $\mu^+(X) \le \rho_\mu(X)$, $\mu^-(X) \le \rho_\mu(X)$, for every $X \in \mathcal{Q}$. Hence, the indeficiency of μ follows from that of μ^+ and μ^- by Proposition 4.21.

(iii) Let $\mathcal{Q}$ be a ring of sets and let μ be a locally bounded complex valued σ-additive set function on $\mathcal{Q}$. Then μ is indeficient. This follows from (ii) by considering the real and imaginary parts of μ.

(iv) Let $\Omega = \{1,2,\ldots\}$ be the set of all positive integers. Let $\mathcal{Q}$ be the family of all intervals in Ω, that is, intersections of Ω with intervals of the real-line. Let E be a Banach space and let $\{a_j\}_{j=1}^{\infty}$ be a conditionally summable sequence of its elements. Let

$$\mu(X) = \lim_{n\to\infty} \sum_{j=1}^{n} X(j)a_j$$

for every $X \in \mathcal{Q}$.

If we choose $\Omega_n = \{1,2,\ldots,n\}$, for $n = 1,2,\ldots,$ in Proposition 4.22, we deduce easily that the set function μ is indeficient.

(v) Let $\Omega = \mathbb{R}$ and let $\mathcal{Q}$ be the family of all (bounded and unbounded) intervals of the real-line. Let $s \neq 0$ be a real number and let

$$\mu(X) = \lim_{u\to\infty} \int_{-u}^{u} X(t)\exp(ist^2)\mathrm{d}t$$

for every $X \in \mathcal{Q}$. Then μ is an indeficient additive set function on $\mathcal{Q}$.

In fact, let $\Omega_n = (-n,n)$, for every $n = 1,2,\ldots$. The restriction of μ to $\mathcal{Q}\cap\Omega_n$ is indeficient for every $n = 1,2,\ldots$. This can be seen by considering the real and imaginary parts of μ separately and noting that each Ω_n can be divided into a finite number of intervals such that in each of them $\mathrm{Re}\mu$ and $\mathrm{Im}\mu$ are of constant sign. Proposition 4.22 then applies.

If the set function $\mu : \mathcal{Q} \to E$ is indeficient then the gauge ρ, defined by (E.1) and (E.2), integrates for μ. However, this is not necessarily the only gauge which integrates for μ. For example, if μ has finite and σ-additive variation it might be convenient to let the variation integrate for μ. But the resulting spaces of integrable functions could be very different even if E is just the space of scalars.

EXAMPLE 4.24. Let Ω and $\mathcal{Q}$ be as in Example 4.23(iv). Let

$$\mu(X) = \sum_{j\in X} (-1)^j j^{-2}$$

for every $X \in \mathcal{Q}$. Then μ has finite and σ-additive variation, $v(\mu)$, and, by Example 4.23(ii), it is indeficient.

Let $e(\omega) = \omega$, for every $\omega \in \Omega$. Then

$$e(\omega) = \sum_{j=1}^{\infty} X_j(\omega)$$

for every $\omega \in \Omega$, where $X_j = \{j,j+1,...\}$ for every $j = 1,2,...$. Because

$$\rho(X_j) = v_\infty(\mu;X_j) = \sup\{|\mu(X_j \cap Z)| : Z \in \mathcal{Q}\} = j^{-2},$$

for every $j = 1,2,...$, the function e belongs to $\mathcal{L}(\rho,\mathcal{Q})$.

On the other hand, a function f belongs to $\mathcal{L}(v(\mu),\mathcal{Q})$ if and only if

$$\sum_{j=1}^{\infty} |f(j)| j^{-2} < \infty .$$

F. Roughly speaking, indeficiency is preserved by closed rather than continuous maps.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. Let E be a Banach space.

Let A be an index set and, for every $\alpha \in A$, let F_α be a Banach space and $T_\alpha : E \to F_\alpha$ a continuous linear map. We say that the family of maps $\{T_\alpha : \alpha \in A\}$ separates the points of the space E if the equality $T_\alpha(x) = 0$, for some $x \in E$ and every $\alpha \in A$, implies that $x = 0$.

For every $\alpha \in A$, let $\nu_\alpha : \mathcal{Q} \to F_\alpha$ be a locally bounded additive set function. The family of set functions $\{\nu_\alpha : \alpha \in A\}$ is said to be collectively indeficient if

$$\text{(F.1)} \qquad \sum_{j=1}^{\infty} c_j \nu_\alpha(X_j) = 0 ,$$

for every $\alpha \in A$, whenever c_j are numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,...$, such that

$$\text{(F.2)} \qquad \sum_{j=1}^{\infty} |c_j| v_\infty(\sigma_\alpha;X_j) < \infty ,$$

for every $\alpha \in A$, and the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does.

By Proposition 4.21, if each set function ν_α, $\alpha \in A$, is indeficient, then the family $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient.

PROPOSITION 4.25. *Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. Let $\nu_\alpha = T_\alpha \circ \mu$, for every $\alpha \in A$.*

If the family of maps $\{T_\alpha : \alpha \in A\}$ separates points of the space E and the family of set functions $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient, then the set function μ is indeficient.

Proof. Let us note first that the local boundedness of μ and the boundedness of T_α imply that each set function ν_α, $\alpha \in A$, is locally bounded.

Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\dots$, satisfying condition (E.4), such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Let

$$x = \sum_{j=1}^{\infty} c_j \mu(X_j) .$$

Condition (E.4) and the continuity of T_α imply that (F.2) holds for every $\alpha \in A$. Consequently, (F.1) holds for every $\alpha \in A$, because the family of set functions $\{\nu_\alpha : \alpha \in A\}$ is collectively indeficient. So, by the continuity of T_α, the equality

$$T_\alpha(x) = T_\alpha\left[\sum_{j=1}^{\infty} c_j \mu(X_j)\right] = \sum_{j=1}^{\infty} c_j \nu_\alpha(X_j) = 0$$

holds for every $\alpha \in A$. Then $x = 0$, that is, (E.3) holds, because the family of maps $\{T_\alpha : \alpha \in A\}$ separates points of the space E. So, by Proposition 4.21, the set function μ is indeficient.

COROLLARY 4.26. *Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function.*

If the family of functionals $x' \in E'$, such that the scalar valued set function $x' \circ \mu$ is indeficient, separates points of the space E, then the set function μ is indeficient.

EXAMPLE 4.27. Let E be a Banach space. Let $\mathcal{Q}$ be a ring of sets in a space Ω and let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. By Corollary 4.26 and Example 4.23(iii), if the set of functionals $x' \in E'$, such that the set function $x' \circ \mu$ is σ-additive, separates the space E, then the set function μ is indeficient. In particular, a locally bounded σ-additive set function $\mu : \mathcal{Q} \to E$ is indeficient. This fact opens another way to integration 'with respect to vector measures'.

So, let $\mu : \mathcal{Q} \to E$ be a locally bounded σ-additive set function. Let $\rho_\mu(X) = v_\infty(\mu;X)$, for every $X \in \mathcal{Q}$. Let ρ be the seminorm on $\mathrm{sim}(\mathcal{Q})$ defined by

$$\rho(f) = \sup\{v(x' \circ \mu, |f|) : x' \in E', |x'| \le 1\},$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. Then $\rho_\mu(X) \le \rho(X) \le C\rho_\mu(X)$, for some $C \ge 1$ and every $X \in \mathcal{Q}$. (See Proposition 3.13.) Therefore, $\mathcal{L}(\rho,\mathcal{Q}) = \mathcal{L}(\rho_\mu,\mathcal{Q})$. But of course $\mathcal{L}(\rho,\mathcal{Q}) \subset \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ and the inclusion may be strict.

In fact, let $\Omega = \{1,2,\ldots\}$ be the set of all positive integers and let $E = c_0$ be the space of all scalar valued sequences tending to 0 equipped with the usual sup norm. Let $\mathcal{Q}$ be the family of all subsets of Ω. For every $X \in \mathcal{Q}$, let

$$\mu(X) = \sum_{j \in X} j^{-1} e_j,$$

where e_j, $j = 1,2,\ldots,$ are the elements of the standard base of the space c_0. Let

$$f = \sum_{j=2}^{\infty} j(\log j)^{-1}\{j\}.$$

The function f is $v(x' \circ \mu)$-integrable, for every $x' \in E'$. (See Section 3F and/or Section A of this chapter.) Moreover, if

$$\nu(X) = \sum_{j \in X, j \ge 2} (\log j)^{-1} e_j,$$

for every $X \in \mathcal{Q}$, then

$$(x' \circ \nu)(X) = \int_\Omega fX \mathrm{d}(x' \circ \mu),$$

for every $x' \in E'$ and $X \in \mathcal{Q}$. Hence, by Proposition 3.13, the function f belongs to $\mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$.

On the other hand, the function f does not belong to $\mathcal{L}(\rho_\mu,\mathcal{Q})$. In fact, let

$$\lambda(X) = \sum_{j\in X}^{\infty} j^{-2},$$

for every $X \in \mathcal{Q}$. Then $\lambda(X) \le 2\rho_\mu(X)$, for every $X \in \mathcal{Q}$. Therefore, $\mathcal{L}(\rho_\mu,\mathcal{Q}) \subset \mathcal{L}(\lambda,\mathcal{Q})$. Because f does not belong to $\mathcal{L}(\lambda,\mathcal{Q})$, it does not belong to $\mathcal{L}(\rho_\mu,\mathcal{Q})$ either.

EXAMPLE 4.28. Let $\Omega = (0,1]$ and let $\mathcal{Q}$ be the semiring of all intervals $X = (s,t]$ such that $0 \le s \le t \le 1$. Let c be the space of all convergent sequences $x = \{x_n\}_{n=1}^{\infty}$ of scalars equipped with the standard sup norm. Let d be a continuous scalar valued function in the interval $[0,1]$ and let $\nu((s,t]) = d(t) - d(s)$ for every s and t such that $0 < s < t < 1$. Let ι be the one-dimensional Lebesgue measure. Given an integer $n \ge 1$, let $Z_j = ((j-1)n^{-1}, jn^{-1}]$ for every $j = 1,2,\ldots,n$, and let

$$\mu_n(X) = \sum_{j=1}^{n} n\iota(X\cap Z_j)\nu(Z_j)$$

for every $X \in \mathcal{Q}$. Finally, let $\mu(X) = \{\mu_n(X)\}_{n=1}^{\infty}$ for every $X \in \mathcal{Q}$. This defines an additive set function $\mu : \mathcal{Q} \to c$.

The set function μ is locally bounded. Furthermore, by Proposition 2.23, each component of μ is indeficient because it is the direct sum of a finite collection of multiples of the Lebesgue measure. Since the coordinate functionals separate the space c, by Corollary 4.26, the set function μ is indeficient.

If the set function $\mu : \mathcal{Q} \to E$ is indeficient, then the set functions $x' \circ \mu$, $x' \in E'$, are not necessarily all indeficient.

EXAMPLE 4.29. Let Ω and $\mathcal{Q}$ be as in Example 4.28. Let E be the closure of $\mathrm{sim}(\mathcal{Q})$ in the space of bounded functions on Ω equipped with the sup norm. For every $X \in \mathcal{Q}$, let $\mu(X) = X$, interpreted as an element of the space E.

To see that μ is indeficient, let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, satisfying condition (E.4), such that (E.5) holds for every $\omega \in \Omega$ for which (E.6) does. Then of course

$$\sum_{j=1}^{\infty} c_j \mu(X_j) = \sum_{j=1}^{\infty} c_j X_j = 0$$

in the space E.

On the other hand, let

$$x'(x) = \lim_{\omega \to 0+} x(\omega)$$

for every $X \in E$. Then $x' \in E'$ and $x' \circ \mu$ is scalar valued additive set function which is not indeficient.

G. Proposition 4.25 and its consequence, Corollary 4.26, are only effective when the space E is infinite-dimensional. However, we describe now a device which makes it possible, at least in principle, to use these propositions also on scalar valued set functions.

Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω. We assume that $\mathcal{Q}$ is directed upwards by inclusion. That is, the union of any finite collection of sets from $\mathcal{Q}$ is contained in a set belonging to $\mathcal{Q}$.

Let E be a Banach space. Let $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ be the set of all bounded additive set functions $\xi : \mathcal{Q} \to E$. Then $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ is a vector space with respect to the natural (set-wise) operations. Let

$$V_{\infty}(\xi) = \sup\{ |\xi(X)| : X \in \mathcal{Q}\}$$

for every $\xi \in \mathrm{BV}^{\infty}(\mathcal{Q},E)$. Then $\xi \mapsto V_{\infty}(\xi)$, $\xi \in \mathrm{BV}^{\infty}(\mathcal{Q},E)$, is a norm which makes of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ a Banach space.

Let $\mu : \mathcal{Q} \to E$ be a locally bounded additive set function. For every $f \in \mathrm{sim}(\mathcal{Q})$, let $f\mu$ be the element of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$ such that $(f\mu)(X) = \mu(fX)$, for every $X \in \mathcal{Q}$. It is straightforward that the set function $f\mu$ so defined is indeed an element of $\mathrm{BV}^{\infty}(\mathcal{Q},E)$.

PROPOSITION 4.30. *Let* $\mu : \mathcal{Q} \to E$ *be a locally bounded additive set function. Let* $\hat{\mu} : \mathcal{Q} \to \mathrm{BV}^\infty(\mathcal{Q},E)$ *be the set function defined by* $\hat{\mu}(X) = X\mu$, *for every* $X \in \mathcal{Q}$.

Then μ *is indeficient if and only if* $\hat{\mu}$ *is indeficient.*

Proof. The set function $\hat{\mu}$ is obviously additive and locally bounded.

Now, if $\hat{\mu}$ is indeficient then it follows easily from Proposition 4.21 that μ is indeficient because

$$v_\infty(\hat{\mu};X) = \sup\{ V_\infty(\hat{\mu}(X \cap Z)) : Z \in \mathcal{Q}\} = \sup\{ |\mu(X \cap Z)| : Z \in \mathcal{Q}\} = v_\infty(\mu;X) ,$$

for every $X \in \mathcal{Q}$. The multiplicativity of $\mathcal{Q}$ is used.

Conversely, let μ be indeficient. Again, Proposition 4.21 implies that $\hat{\mu}$ is indeficient. Indeed, let c_j be some numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots,$ such that

$$\text{(G.1)} \qquad \sum_{j=1}^{\infty} |c_j| v_\infty(\hat{\mu};X_j) < \infty$$

and the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Then

$$\lim_{n\to\infty} \left| \sum_{j=1}^{n} c_j \mu(X_j \cap Z) \right| = 0 ,$$

for every $Z \in \mathcal{Q}$, by the indeficiency of μ . But then

$$\lim_{n\to\infty} V_\infty\left[\sum_{j=1}^{n} c_j \hat{\mu}(X_j) \right] = 0 .$$

For a locally bounded additive set function $\mu : \mathcal{Q} \to E$, let $\mathrm{BV}^\infty(\mu,\mathcal{Q},E)$ be the closure of the space $\{f\mu : f \in \mathrm{sim}(\mathcal{Q})\}$ in $\mathrm{BV}^\infty(\mathcal{Q},E)$.

PROPOSITION 4.31. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be multiplicative quasirings of sets in the space* Ω *such that* $\mathcal{Q} \subset \mathcal{P}$. *Let* E *and* F *be Banach spaces and* $\mu : \mathcal{Q} \to E$ *and* $\nu : \mathcal{P} \to F$ *locally bounded additive set functions. Assume that* ν *is indeficient and that there exists an injective continuous linear map* $T : \mathrm{BV}^\infty(\mu,\mathcal{Q},E) \to \mathrm{BV}^\infty(\nu,\mathcal{P},F)$ *such that* $T(X\mu) = X\nu$, *for every* $X \in \mathcal{Q}$. *Then the set function* μ *is indeficient.*

Proof. Let c_j be numbers and $X_j \in \mathcal{Q}$ sets, $j = 1,2,\ldots$, satisfying condition (G.1), such that the equality (E.5) holds for every $\omega \in \Omega$ for which the inequality (E.6) does. Then the sequence $\{c_j X_j \mu\}_{j=1}^{\infty}$ is absolutely summable in the space $\mathrm{BV}^{\infty}(\mu,\mathcal{Q},E)$; let ξ be its sum. Because the map T is linear and continuous and $X_j\nu = T(X_j\mu)$, for every $j = 1,2,\ldots$, the sequence $\{c_j X_j \nu\}_{j=1}^{\infty}$ is absolutely summable in the space $\mathrm{BV}^{\infty}(\nu,\mathcal{P},E)$. By the indeficiency of ν and Proposition 4.30, the sum of the sequence $\{c_j X_j \nu\}_{j=1}^{\infty}$ is the zero-element of the space $\mathrm{BV}^{\infty}(\nu,\mathcal{P},E)$. Then $T(\xi) = 0$, because the map T is continuous, and then $\xi = 0$, because T is injective. Hence, by Proposition 4.30, the set function μ is indeficient.

The use of Proposition 4.31 is mainly in that it gives a sufficient condition for the preservation of indeficiency in passing to a sub-quasiring.

H. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω . Let E be a normed space and $\mu : \mathcal{Q} \to E$ an indeficient additive set function. Let the gauge ρ be defined by (E.1) and (E.2), for every $X \in \mathcal{Q}$. Then of course the gauge ρ integrates for the set function μ . But the usefulness of ρ is thereby not exhausted; the gauge ρ integrates possibly for many other, not necessarily indeficient, additive set functions on $\mathcal{Q}$. For instance, it does integrate for every set function of the form $T \circ \mu$, where T is a continuous map from E into another Banach space.

EXAMPLE 4.32. Let us adopt the notation of Example 4.28. Because

$$\nu(X) = \lim_{n \to \infty} \mu_n(X) ,$$

for every $X \in \mathcal{Q}$, and the limit is a continuous linear functional on the space c , the gauge ρ integrates for the scalar valued set function ν .

Such a gauge integrating for the set function ν is especially interesting if ν does not have finite variation in any interval.

EXAMPLE 4.33. Let $E = L^2(\mathbb{R})$. Let $S(0) = I$ be the identity operator on the space E . For $t \neq 0$, let $S(t)$ be the operator on E such that

$$(S(t)\varphi)(x) = \frac{1}{\sqrt{2\pi i t}} \int_{\mathbb{R}} \exp\left[-\frac{(x-y)^2}{2i\,t}\right] \varphi(y)\mathrm{d}y$$

for every $\varphi \in L^1 \cap L^2(\mathbb{R})$. It is well-known that by this a unitary operator $S(t) : E \to E$ is defined and that the resulting one-parameter family of operators $t \mapsto S(t)$, $t \in (-\infty,\infty)$, is a unitary group.

For a Borel set B in $\mathbb{R}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of B on the space E.

Let $t > 0$ be fixed and let Ω be the set of all continuous functions (paths) $\omega : [0,t] \to \mathbb{R}$. Let $\mathcal{Q}$ be the family of all sets

(H.1) $$X = \{\omega \in \Omega : \omega(t_j) \in B_j \,,\, j = 1,2,\ldots,n\} \,,$$

for arbitrary $n = 1,2,\ldots$, $0 \le t_1 < t_2 < \ldots < t_{n-1} < t_n \le t$ and Borel sets B_j in $\mathbb{R}$, $j = 1,2,\ldots,n$.

Let φ be a non-zero element of the space E. Let

$$\nu(X) = S(t-t_n)P(B_n)S(t_n-t_{n-1})P(B_{n-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)\varphi$$

for any set $X \in \mathcal{Q}$ written in the form (H.1).

Then $\nu : \mathcal{Q} \to E$ is an additive set function which has infinite variation on every set $X \in \mathcal{Q}$. A gauge integrating for ν can be constructed in a similar manner as a gauge for the set function of Example 4.28.

Indeed, let $\not{h}_n$ be partitions of the real-line into finite numbers of intervals such that $\not{h}_{n+1}$ is a refinement of $\not{h}_n$, $n = 1,2,\ldots$. For every $n = 1,2,\ldots$, let $\mathcal{P}_n$ be the family of all sets $X \in \mathcal{Q}$, which can be written in the form

$$X = \{\omega : \omega(j/2^n) \in B_j \,,\; j = 1,2,\ldots,2^n\} \,,$$

where the sets B_j, depending on X, belong to $\not{h}_n$, $j = 1,2,\ldots,2^n$. Then $\mathcal{P}_n \in \Pi(\mathcal{Q})$ are partitions such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, for every $n = 1,2,\ldots$. Let ι be the Wiener measure in Ω with unit variance per unit of time and with the standard normal initial distribution, say. That is, ι is the measure such that

$$\iota(X) = \left[(2\pi)^{n+1} \prod_{j=1}^{n} (t_j - t_{j-1})\right]^{-\frac{1}{2}} \times$$

$$\int_{B_n} \int_{B_{n-1}} \cdots \int_{B_1} \int_{\mathbb{R}} \exp\left[-\frac{x_0^2}{2} - \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}\right] dx_0 dx_1 \ldots dx_{n-1} dx_n ,$$

for every set, X, of the form (H.1), where we put $t_0 = 0$. Assume that the partitions $\mathcal{P}_n$ are so chosen that, for every $n = 1,2,\ldots,$ there is a number $m_n > 0$ such that $\iota(X) = m_n$ for every $X \in \mathcal{P}_n$ and that $m_n \to 0$ as $n \to \infty$. Partitions of Ω similar to $\mathcal{P}_n$ were used by N. Wiener in the first constructions of the measure named after him; see, for example, [68].

Now, given an integer $n \geq 1$, let

$$\mu_n(X) = m_n^{-1} \sum_{Z \in \mathcal{P}_n} \iota(Z \cap X)\nu(Z)$$

for every $X \in \mathcal{Q}$. Then $\mu_n : \mathcal{Q} \to E$ is an indeficient additive set function.

Let c_E be the space of all convergent sequences of elements of the space E equipped with the usual sup norm. Let $\mu : \mathcal{Q} \to c_E$ be the set functions such that $\mu(X) = \{\mu_n(X)\}_{n=1}^{\infty}$, for every $X \in \mathcal{Q}$. Let $F_n = E$ and let $T_n : c_E \to F_n$ be the n-th coordinate map, for every $n = 1,2,\ldots$. The set functions $T_n \circ \mu : \mathcal{Q} \to F_n$ are then indeficient because $T_n \circ \mu = \mu_n$, $n = 1,2,\ldots$. Therefore, by Proposition 4.25, the set function μ is indeficient.

Because

$$\nu(X) = \lim_{n \to \infty} \mu_n(X) ,$$

for every $X \in \mathcal{Q}$, and the limit is a continuous linear map from the space c_E onto E, the gauge ρ, defined by (E.1) and (E.2) for every $X \in \mathcal{Q}$, integrates for ν.

J. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω directed upward by inclusion. (See Section G.) Let $\Delta \subset \Pi(\mathcal{Q})$ be a set of partitions. Let E be a Banach space.

Given a Young function, Φ, the family of all additive set functions $\xi : \mathcal{Q} \to E$

such that

$$\sup\{v_{\Phi}(\xi,\Delta;X) : X \in \mathcal{Q}\} < \infty ,$$

will be denoted by $\mathrm{BV}^{\Phi}(\Delta,E)$. We shall write $\mathrm{BV}^{\Phi}(\mathcal{Q},E) = \mathrm{BV}^{\Phi}(\Pi,E)$.

These notions are useful mainly in the case when $\mathcal{Q}$ is a quasialgebra, that is, $\Omega \in \mathcal{Q}$. In that case, the definitions can be simplified somewhat.

PROPOSITION 4.34. *If Φ is a Young function, then $\mathrm{BV}^1(\Delta,E) \subset \mathrm{BV}^{\Phi}(\Delta,E) \subset \mathrm{BV}^{\infty}(\mathcal{Q},E)$, for any set of partitions $\Delta \subset \Pi$.*

If Φ and Ψ are Young functions for which there exist numbers $a > 0$ and $k > 0$ such that $\Psi(s) \le k\Phi(s)$, for every $s \in [0,a]$, then $\mathrm{BV}^{\Phi}(\Delta,E) \subset \mathrm{BV}^{\Psi}(\Delta,E)$, for any set of partitions $\Delta \subset \Pi$.

Proof. The first statement is obvious. The second one is analogous to the statement 1.15 in [51]. For its proof, let us note first that, if the condition is satisfied, then, for every $b > 0$, there is a constant $\ell > 0$ such that $\Psi(s) \le \ell\Phi(s)$, for every $s \in [0,b]$. In fact, if $a \le s \le b$, then

$$\Phi(s) \ge \frac{\Psi(a)}{k\Phi(a)}\,\Phi(s) = \frac{1}{k}\,\frac{\Psi(a)}{\Phi(a)}\,\frac{\Phi(s)}{\Psi(s)}\,\Psi(s) \ge \frac{1}{k}\,\frac{\Psi(a)}{\Psi(b)}\,\Psi(s) .$$

So, let us assume that $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$. Then there exists a $b > 0$ such that $|\xi(X \cap Y)| \le b$ for every set $X \in \mathcal{Q}$ and every set Y belonging to some $\mathcal{P} \in \Delta$. Consequently, $\Psi(|\xi(X\cap Y)| \le \ell\Phi(|\xi(X\cap Y)|)$ and $v_{\Psi}(\xi,\Delta;X) \le \ell v_{\Phi}(\xi,\Delta;X)$.

The second part of this proposition has a converse: If Ω and $\mathcal{Q}$ are as in Example 5.28 and $\mathrm{BV}^{\Phi}(\mathcal{Q},\mathbb{R}) \subset \mathrm{BV}^{\Psi}(\mathcal{Q},\mathbb{R})$, then there exist numbers $a > 0$ and $k > 0$ such that $\Psi(s) \le k\Phi(s)$ for every $s \in [0,a]$. Cf. statement 1.15 in [51].

The sets $\mathrm{BV}^1(\Delta,E)$ and $\mathrm{BV}^{\infty}(\Delta,E)$ are, obviously, vector spaces with respect to the natural operations. The following proposition says that, if the Young function, Φ , satisfies condition (Δ_2) for small values of the argument (see Section 1G), then also $\mathrm{BV}^{\Phi}(\Delta,E)$ is a vector space. It is analogous to statement 1.13 in [51] and so, its proof too is analogous.

PROPOSITION 4.35. *If the Young function, Φ, satisfies the condition (Δ_2) for small values of the argument, then $\mathrm{BV}^{\Phi}(\Delta,E)$ is a vector space under the natural operations.*

Proof. Assume that $k > 0$ and $a > 0$ are numbers such that $\Phi(2s) \le k\Phi(s)$ for every $s \in [0,a]$. Then, for every $b > 0$, there is an $\ell(b) \ge 1$ such that $\Phi(2s) \le \ell(b)\Phi(s)$ for every $s \in [0,\ell(b)]$. In fact, if $\frac{1}{2}a \le s \le b$, then

$$\Phi(s) \ge \frac{\Phi(a)}{k\Phi(\frac{1}{2}a)}\,\Phi(s) = \frac{1}{k}\,\frac{\Phi(a)}{\Phi(\frac{1}{2}a)}\,\frac{\Phi(s)}{\Phi(\frac{1}{2}a)}\,\Phi(2s) \ge \frac{1}{k}\,\frac{\Phi(a)}{\Phi(2b)}\,\Phi(2s)\,.$$

Now, if $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$ and $\eta \in \mathrm{BV}^{\Phi}(\Delta,E)$, there exists a $b > 0$ such that $|\xi(X\cap Y)| \le b$ and $|\eta(X\cap Y)| \le b$, for every $X \in \mathcal{Q}$ and every set Y belonging to any partition from Δ. Consequently,

$$v_{\Phi}(\xi+\eta,\Delta;X) \le \ell(b)(v_{\Phi}(\xi,\Delta;X) + v_{\Phi}(\eta,\Delta;X))\,,$$

for every $X \in \mathcal{Q}$. If, further, c is a number, let m be the least positive integer such that $|c| \le 2^m$. Then

$$v_{\Phi}(c\xi,\Delta;X) \le (\ell(2^{m-1}b))^m v_{\Phi}(\xi,\Delta;X)\,,$$

for every $X \in \mathcal{Q}$.

For every $\xi \in \mathrm{BV}^1(\Delta,E)$, let

$$V_1(\xi,\Delta) = \sup\{v_1(\xi,\Delta;X) : X \in \mathcal{Q}\}.$$

Then the functional $\xi \mapsto V_1(\xi,\Delta)$, $\xi \in \mathrm{BV}^1(\Delta,E)$, is a norm making the space $\mathrm{BV}^1(\Delta,E)$ complete.

If the Young function, Φ, satisfies condition (Δ_2) for small values of the argument (see Section 1G), then a norm still can be introduced in the space $\mathrm{BV}^{\Phi}(\Delta,E)$. It can be naturally done in at least two ways. Thus let

$$V_{\Phi}(\xi,\Delta) = \inf\{k > 0 : v_{\Phi}(k^{-1}\xi,\Delta;X) \le 1\,,\ X \in \mathcal{Q}\}\,,$$

for every $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$. Secondly, given a set function $\xi \in \mathrm{BV}^{\Phi}(\Delta,E)$ and a partition $\mathcal{P} \in \Delta$, let

$$V_\Phi^0(\xi,\Delta) = \sup\left\{ \sum_{Y\in\mathcal{P}} \beta(X\cap Y)\,|\mu(X\cap Y)| : \beta \in B_{X\mathcal{P}} , \mathcal{P}\in\Delta , X\in\mathcal{Q}\right\},$$

where $B_{X\mathcal{P}}$ is the set of all functions

$$\beta : \{X\cap Y : Y\in\mathcal{P}\} \to [0,\infty)$$

such that

$$\sum_{Y\in\mathcal{P}} \Psi(\beta(X\cap Y)) \le 1 ,$$

and Ψ is the Young function complementary to Φ. (See Section 1G.)

By analogy with the usual terminology in Orlicz spaces, the functional $\xi \mapsto V_\Phi(\xi,\Delta)$, $\xi \in \mathrm{BV}^\Phi(\Delta,E)$, will be called the Luxemburg norm and the functional $\xi \mapsto V_\Phi^0(\xi,\Delta)$, $\xi \in \mathrm{BV}^\Phi(\Delta,E)$, the Orlicz norm. It turns out that these functionals are indeed norms on the space $\mathrm{BV}^\Phi(\Delta,E)$ and they are equivalent.

PROPOSITION 4.36. *Assume that the Young function Φ satisfies conditions (0), (∞) and (Δ_2) for small values of the argument. Then the functionals $V_\Phi(\cdot,\Delta)$ and $V_\Phi^0(\cdot,\Delta)$ are norms on the space $\mathrm{BV}^\Phi(\Delta,E)$ such that*

$$V_\Phi(\xi,\Delta) \le V_\Phi^0(\xi,\Delta) \le 2V_\Phi(\xi,\Delta) \tag{J.1}$$

for every $\xi \in \mathrm{BV}^\Phi(\Delta,E)$. The space $\mathrm{BV}^\Phi(\Delta,E)$ is complete in each of these norms.

Proof. The inequalities (J.1) follow directly from the definitions of the functionals $V_\Phi(\cdot,\Delta)$ and $V_\Phi^0(\cdot,\Delta)$ and from Proposition 1.15. We omit the proofs that these functionals are indeed norms and of the completeness of the space $\mathrm{BV}^\Phi(\Delta,E)$.

Let us note that, if $1 < p < \infty$ and $\Phi(s) = s^p$, for every $s \in [0,\infty)$, then

$$V_p(\xi,\Delta) = \left[\sup\left\{ \sum_{Y\in\mathcal{P}} |\xi(X\cap Y)|^p : \mathcal{P}\in\Delta , X\in\mathcal{Q}\right\}\right]^{1/p},$$

for every $\xi \in \mathrm{BV}^p(\Delta,E)$.

K. Let $\mathcal{Q}$ be a multiplicative quasiring of sets in a space Ω which is directed upward by inclusion and E a Banach space. Let $\Delta \subset \Pi = \Pi(\mathcal{Q})$ be a set of partitions and let Φ be a Young function satisfying condition (Δ_2) for small values of the argument. (See Section 1G.)

Let us note first that, if the additive set function $\mu : \mathcal{Q} \to E$ has finite Φ-variation with respect to the set of partitions Δ and f is a $\mathcal{Q}$-simple function, then $f\mu \in \mathrm{BV}^{\Phi}(\Delta,E)$. Now, assuming that μ is such a set function, the closure of the vector space $\{f\mu : f \in \mathrm{sim}(\mathcal{Q})\}$ in $\mathrm{BV}^{\Phi}(\Delta,E)$ will be denoted by $\mathrm{BV}^{\Phi}(\Delta,\mu)$. Then $\mathrm{BV}^{\Phi}(\Delta,\mu)$ is a Banach space, being a closed subspace of $\mathrm{BV}^{\Phi}(\Delta,E)$. Again, we write $\mathrm{BV}^{\Phi}(\mathcal{Q},\mu) = \mathrm{BV}^{\Phi}(\Pi,\mu)$.

If ι is a real valued positive σ-additive set function on $\mathcal{Q}$, then

$$V_1(f\iota,\Pi) = \int_{\Omega} |f|\,\mathrm{d}\iota \tag{K.1}$$

for every $f \in \mathrm{sim}(\mathcal{Q})$. Therefore, the elements of the space $\mathrm{BV}^1(\mathcal{Q},\iota)$ are canonically associated with ι-integrable functions, or, more accurately, with the equivalence classes of such functions. In other words, the space $\mathrm{BV}^1(\mathcal{Q},\iota)$ is identified with $L^1(\iota)$.

In this section, those set functions, $\mu : \mathcal{Q} \to E$, are isolated for which an analogous identification of $\mathrm{BV}^{\Phi}(\Delta,\mu)$ with a space of (equivalence classes of) functions on Ω is possible. The definition is immediate.

An additive set function $\mu : \mathcal{Q} \to E$ will be called (Φ,Δ)-closable if it has finite Φ-variation with respect to the set of partitions Δ and the seminorm $\rho = \rho_{\mu,\Phi,\Delta}$ on $\mathrm{sim}(\mathcal{Q})$, defined by

$$\rho(f) = V_{\Phi}(f\mu,\Delta)$$

for every $f \in \mathrm{sim}(\mathcal{Q})$, is integrating. In that case, we write $\mathcal{L}(\mu,\Phi,\Delta) = \mathcal{L}(\rho,\mathrm{sim}(\mathcal{Q}))$ and $\|\cdot\|_{\mu,\Phi,\Delta} = \rho_{\mu,\Phi,\Delta} = \rho = q_{\rho}$. Also, $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}(\mu,\Phi,\Pi)$.

Because $\mathrm{sim}(\mathcal{Q})$ is dense in $\mathcal{L}(\mu,\Phi,\Delta)$, for every $f \in \mathcal{L}(\mu,\Phi,\Delta)$, there is a unique element $\nu_f \in \mathrm{BV}^{\Phi}(\Delta,\mu)$ such that $\nu_f = f\mu$ for $f \in \mathrm{sim}(\mathcal{Q})$ and the map $f \mapsto \nu_f$, from $\mathcal{L}(\mu,\Phi,\Delta)$ onto $\mathrm{BV}^{\Phi}(\Delta,\mu)$, is continuous. We write, of course,

$f\mu = \nu_f$, for every $f \in \mathcal{L}(\mu,\Phi,\Delta)$, and call $f\mu$ the indefinite integral of the function f with respect to μ.

To introduce an interesting class of (Φ,Δ)-closable additive set functions, we adopt the following definition. An additive set function $\mu : \mathcal{Q} \to E$ will be called Φ-scattered if the set function $X \mapsto \Phi(|\mu(X)|)$, $X \in \mathcal{Q}$, is σ-additive.

This notion originates from the case when E is a Hilbert space and for any disjoint sets, X and Y, belonging to $\mathcal{Q}$, the values $\mu(X)$ and $\mu(Y)$ are orthogonal. Such a set function is called orthogonally scattered. It is immediate that, if μ is an orthogonally scattered set function, then the set function $X \mapsto |\mu(X)|^2$, $X \in \mathcal{Q}$, is additive and, if E is a real Hilbert space, then also the converse is true. Since, however, the converse is not necessarily true in a complex Hilbert space and σ-additivity is built in the notion of a 2-scattered set function, which is convenient for the purpose of this example, we keep the notions of an orthogonally scattered and a 2-scattered set function distinct. For a systematic treatment of orthogonally scattered additive set functions, see [49].

PROPOSITION 4.37. *Assume that the Young function Φ satisfies condition (Δ_2). Let $\mu : \mathcal{Q} \to E$ be a Φ-scattered additive set function. Denote $\iota(X) = \Phi(|\mu(X)|)$ for every $X \in \mathcal{Q}$. Assume that the measure generated by the set function ι is σ-finite. Then the set function μ is (Φ,Π)-closable, $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}^{\Phi}(\iota)$ and*

$$\mathrm{V}^0_{\Phi}(f\mu;\Pi) = \|f\|^0_{\iota,\Phi}, \tag{K.2}$$

for every $f \in \mathcal{L}(\mu,\Phi,\mathcal{Q})$.

Proof. First we prove (K.2) for $f \in \mathrm{sim}(\mathcal{Q})$. So, let

$$f = \sum_{j=1}^{n} c_j X_j$$

with an arbitrary $n = 1,2,\ldots,$ numbers c_j and pairwise disjoint sets $X_j \in \mathcal{Q}$, $j = 1,2,\ldots,n$. Let Ψ be the Young function complementary to Φ. Then

$$\|f\|^0_{\iota,\Phi} = \sup\left\{ \int_\Omega fg\,d\iota : g \in \mathrm{sim}(\mathcal{Q}) \,,\ \int_\Omega \Psi(|g|)\,d\iota \le 1 \right\}$$

and

$$V^0_\Phi(f\mu;\Pi) = \sup\left\{ \sum_{Y\in\mathcal{P}} |(f\mu)(Y)|\beta(Y) : \beta \in \mathrm{B}_{\mathcal{P}} \,,\ \mathcal{P} \in \Pi \right\} ,$$

where $\mathrm{B}_{\mathcal{P}}$ is the family of all functions $\beta : \mathcal{P} \to [0,\infty)$ such that

$$\sum_{Y\in\mathcal{P}} \Psi(\beta(Y)) \le 1 .$$

Because $V^0_\Phi(\cdot,\Pi)$ is a norm in the space $\mathrm{BV}^\Phi(\mu,\mathcal{Q})$, it suffices to calculate the supremum over partitions $\mathcal{P} \in \Pi$ such that every set X_j, $j = 1,2,\ldots,n$, is equal to the union of some elements of $\mathcal{P}$. Furthermore, it suffices to take $\beta \in \mathrm{B}_{\mathcal{P}}$ such that $\beta(Y) = 0$, whenever $Y \cap X_j = \emptyset$ for each $j = 1,2,\ldots,n$. Then, given such a β, we put

$$g = \sum_{Y\in\mathcal{P},\,\iota(Y)\ne 0} \Psi^{-1}\left[\frac{\Psi(\beta(Y))}{\iota(Y)}\right] Y .$$

Because, in calculating $\|f\|^0_{\iota,\Phi}$, it suffices to take those functions $g \in \mathrm{sim}(\mathcal{Q})$ which are obtained in this manner, the equality (K.2) is indeed true.

The equality (K.2) is analogous to, or a generalization of, (K.1). It implies that the set function μ is σ-additive, (Φ,Π)-closable and that $\mathcal{L}(\mu,\Phi,\mathcal{Q}) = \mathcal{L}^\Phi(\iota)$.

It seems difficult to prove the (Φ,Δ)-closability of set functions which are not in a sense equivalent to Φ-scattered ones. None-the-less, the norms V_Φ and V^0_Φ could still be helpful. For, if the additive set function $\mu : \mathcal{Q} \to E$ has finite Φ-variation, then the gauge ρ, defined by

$$\rho(X) = V_\Phi(X\mu,\Delta) ,$$

for every $X \in \mathcal{Q}$, is usually very sub-additive (see Section 2J) and so, in many cases, Proposition 2.25 applies. Then this gauge can be used instead of the one studied in Section C.

5. VECTOR VALUED FUNCTIONS AND PRODUCTS

The title practically gives away the content of this chapter. We present first a Bochner-type integration theory, that is, one based on absolute summability, for Banach space valued functions. Then we consider direct products of integrating gauges along with the corresponding Fubini- and Tonelli-type theorems. These two themes are related in the formulation of the mentioned theorems; the notion of a measurable function is avoided by stating them in terms of Bochner integrability.

A. Let ρ be a gauge on a nontrivial family, $\mathcal{K}$, of scalar valued functions on a space Ω. (See Section 2A.)

Let E be a Banach space. To avoid some obvious trivialities, we assume that E contains a non-zero vector. The convention of writing interchangeably $ca = ac$, for every $c \in E$ and a scalar a, will be used throughout the chapter.

A function $f : \Omega \to E$ will be called Bochner integrable with respect to ρ, or, briefly, ρ-integrable, if there exist vectors $c_j \in E$ and functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, such that

$$\text{(A.1)} \qquad \sum_{j=1}^{\infty} |c_j| \rho(f_j) < \infty$$

and

$$\text{(A.2)} \qquad f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

$$\text{(A.3)} \qquad \sum_{j=1}^{\infty} |c_j| \, |f_j(\omega)| < \infty .$$

The family of all E-valued functions on Ω, Bochner integrable with respect to ρ, is denoted by $\mathcal{L}(\rho,\mathcal{K},E)$. If the space E happens to be one-dimensional, that is, just the space of scalars, then, consistently with the notation introduced in Chapter 2, we write $\mathcal{L}(\rho,\mathcal{K}) = \mathcal{L}(\rho,\mathcal{K},E)$.

For any function $f \in \mathcal{L}(\rho,\mathcal{K},E)$, let

$$q_\rho(f) = \inf \sum_{j=1}^{\infty} |c_j| \rho(f_j) ,$$

where the infimum is taken over all choices of the vectors $c_j \in E$ and the functions $f_j \in \mathcal{K}$, $j = 1,2,\ldots$, satisfying condition (A.1), such that the equality (A.2) holds for every $\omega \in \Omega$ for which the inequality (A.3) does.

Clearly, $\mathcal{L}(\rho,\mathcal{K},E)$ is a vector space and $\operatorname{sim}(\mathcal{K},E)$ is a vector subspace of it. (See Section 1D.) Also, it is not difficult to see that q_ρ is a seminorm on $\mathcal{L}(\rho,\mathcal{K},E)$. Consequently, we can speak of q_ρ-Cauchy and q_ρ-convergent sequences of functions from $\mathcal{L}(\rho,\mathcal{K},E)$.

The ρ-equivalence class of a function $f \in \mathcal{L}(\rho,\mathcal{K},E)$ consisting of all functions $g \in \mathcal{L}(\rho,\mathcal{K})$ such that $q_\rho(f-g) = 0$, is denoted by $[f]_\rho$. The set $\{[f]_\rho : f \in \mathcal{L}(\rho,\mathcal{K},E)\}$ of all ρ-equivalence classes of functions from $\mathcal{L}(\rho,\mathcal{K},E)$ is denoted by $L(\rho,\mathcal{K},E)$. Then $L(\rho,\mathcal{K},E)$ is a normed space with respect to the linear operations induced by those of $\mathcal{L}(\rho,\mathcal{K},E)$ and the norm induced by the seminorm q_ρ . This norm is still denoted by the same symbols q_ρ .

A function $f : \Omega \to E$ is said to be ρ-null if $f \in \mathcal{L}(\rho,\mathcal{K},E)$ and $q_\rho(f) = 0$. As to the null sets, their definition remains of course the same as in Section 2B. Namely, a set $Z \subset \Omega$ is ρ-null if its characteristic function is a ρ-null element of $\mathcal{L}(\rho,\mathcal{K})$.

The introduced definitions do not differ in form from those concerning scalar valued functions given in Chapter 2. Because the space E is non-trivial, the treatment of scalar valued integrable functions presented in Sections 2A-2D is applicable practically without a change to E-valued functions. This fact was first noted by J. Mikusiński who exploited it, in [50], for his definition of Bochner integrable functions (in the usual sense). It may be useful to note explicitly that Proposition 2.2 remains valid if by a function is meant an E-valued function and if $\mathcal{L}(\rho,\mathcal{K})$ is replaced by $\mathcal{L}(\rho,\mathcal{K},E)$. It implies that a set $Z \subset \Omega$ is ρ-null if and only if there exist functions $f_j \in \mathcal{L}(\rho,\mathcal{K},E)$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} q_\rho(f_j) < \infty \tag{A.4}$$

and

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty .$$

for every $\omega \in Z$.

The following theorem, which is analogous to Theorem 2.3, is singled out because of its central importance. Its proof is of course omitted.

THEOREM 5.1. *Let the functions* $f_j \in \mathcal{L}(\rho,\mathcal{K},E)$, $j = 1,2,\ldots,$ *satisfy condition* (A.4). *Then*

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty$$

for ρ-*almost every* $\omega \in \Omega$. *Furthermore, if* $f : \Omega \to E$ *is a function such that*

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

for ρ-*almost every* $\omega \in \Omega$, *then* $f \in \mathcal{L}(\rho,\mathcal{K},E)$ *and*

$$\lim_{n\to\infty} q_\rho\left[f - \sum_{j=1}^{n} f_j\right] = 0 .$$

Among the implications of this theorem is that $L(\rho,\mathcal{K},E)$ is a Banach space.

Also, it may seem that the natural seminorm of the space $\mathcal{L}(\rho,\mathcal{K},E)$ should be denoted more accurately by $q_{\rho,E}$ rather than simply by q_ρ. For if E is a subspace of a Banach space F and $f \in \mathcal{L}(\rho,\mathcal{K},E)$, then also $f \in \mathcal{L}(\rho,\mathcal{K},F)$ and $q_{\rho,F}(f) \le q_{\rho,E}(f)$. However, Theorem 5.1 implies that actually $q_{\rho,F}(f) = q_{\rho,E}(f)$, and so, the simpler notation suffices.

The introduced notions are useful perhaps only if the gauge ρ is integrating. (See Section 2D.) The proof of the following straightforward proposition is omitted.

PROPOSITION 5.2. *The gauge* ρ *is integrating if and only if* $|c|\rho(f) = q_\rho(cf)$, *for every function* $f \in \mathcal{K}$ *and a vector* $c \in E$.

B. We maintain the notation of the previous section.

Let us start with two observations. It seems that we would obtain a larger class of Bochner integrable functions if we called Bochner integrable all functions belonging to $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}),E)$ rather than those belonging to $\mathcal{L}(\rho,\mathcal{K},E)$, that is, if we admitted for f_j, $j = 1,2,\ldots,$ in (A.2), any functions from $\mathcal{L}(\rho,\mathcal{K})$ and not merely from $\mathcal{K}$. However, it is not the case, because Theorem 5.1 clearly implies that $\mathcal{L}(q_\rho,\mathcal{L}(\rho,\mathcal{K}),E) = \mathcal{L}(\rho,\mathcal{K},E)$. This is an extension of the last statement of Proposition 2.7.

More interestingly, we can look at $L(\rho,\mathcal{K},E)$ as the projective tensor product of the spaces E and $L(\rho,\mathcal{K})$. (See Section 1C.) Formally, we have the following

PROPOSITION 5.3. *There is a unique isometric isomorphism of the projective tensor product, $E\hat{\otimes}L(\rho,\mathcal{K})$, of the spaces E and $L(\rho,\mathcal{K})$ onto the space $L(\rho,\mathcal{K},E)$, that maps the tensor product, $c\otimes[f]_\rho$, of any element, c, of E and element, $[f]_\rho$, of $L(\rho,\mathcal{K})$ to the element $[cf]_\rho$ of the space $L(\rho,\mathcal{K},E)$.*

Proof. Every element, z, of the projective tensor product $E\hat{\otimes}L(\rho,\mathcal{K})$ can be written in the form

(B.1) $$z = \sum_{j=1}^{\infty} c_j \otimes [f_j]_\rho ,$$

where the vectors $c_j \in E$ and the functions $f_j \in \mathcal{L}(\rho,\mathcal{K})$, $j = 1,2,\ldots,$ satisfy the condition

(B.2) $$\sum_{j=1}^{\infty} |c_j| q_\rho(f_j) < \infty .$$

Moreover, the norm of z in the space $E\hat{\otimes}L(\rho,\mathcal{K})$ is equal to the infimum of the numbers (B.2) subject to the equality (B.1). By Theorem 5.1, any function, f, on Ω, such that (A.2) holds for every $\omega \in \Omega$ for which (A.3) does, belongs to $\mathcal{L}(\rho,\mathcal{K},E)$ and its seminorm, $q_\rho(f)$, is equal to the norm of z. Therefore, if we let correspond to z the element, $[f]_\rho$, of the space $L(\rho,\mathcal{K},E)$ determined by any such function f, we obtain an unambiguously defined map of $E\hat{\otimes}L(\rho,\mathcal{K})$ into $L(\rho,\mathcal{K},E)$. Clearly, this map is a linear isometry. Because, however, every element of the space $L(\rho,\mathcal{K},E)$ is the image of an element of $E\hat{\otimes}L(\rho,\mathcal{K})$, this map is an isometric isomorphism of the spaces $E\hat{\otimes}L(\rho,\mathcal{K})$ and $L(\rho,\mathcal{K},E)$ which, for any $c \in E$ and $f \in \mathcal{L}(\rho,\mathcal{K})$, maps $c\otimes[f]_\rho$ to $[cf]_\rho$.

The spaces $L(\rho,\mathcal{K})$ and $L(\rho,\mathcal{K},E)$ cannot be replaced, in this proposition, by $\mathcal{L}(\rho,\mathcal{K})$ and $\mathcal{L}(\rho,\mathcal{K},E)$, respectively, even if the notion of a tensor product were extended to seminormed spaces. For, any E-valued function that vanishes ρ-almost everywhere belongs to $\mathcal{L}(\rho,\mathcal{K},E)$. Consequently, the space $\mathcal{L}(\rho,\mathcal{K},E)$ may contain functions that cannot be canonically specified by a sequence of vectors from E and a sequence of functions from $\mathcal{L}(\rho,\mathcal{K})$.

Now, let F and G be Banach spaces and let $b : E\times F \to G$ be a continuous bilinear map. (See Section 1C.) Let $\mu : \mathcal{K} \to F$ be an additive map. Assume that the gauge ρ integrates for the map μ. (See Section 3A.)

PROPOSITION 5.4. *There exists a unique continuous linear map,* $\mu_{\rho,b} : \mathcal{L}(\rho,\mathcal{K},E) \to G$, *such that*

(B.3) $$\mu_{\rho,b}(cf) = b(c,\mu(f)),$$

for any vector $c \in E$ *and a function* $f \in \mathcal{K}$.

Proof. By the basic property of projective tensor products, there exists a unique continuous linear map, $\ell : E\hat{\otimes}L(\rho,\mathcal{K}) \to G$, such that $\ell(c\otimes[f]_\rho) = b(c,\mu(f))$, for every $c \in E$ and $f \in \mathcal{L}(\rho,\mathcal{K})$. Because the vector space spanned by $\{[f]_\rho : f \in \mathcal{K}\}$ is dense in $L(\rho,\mathcal{K})$, ℓ is the unique continuous linear map from $E\hat{\otimes}L(\rho,\mathcal{K})$ to G, such that $\ell(c\otimes[f]_\rho) = b(c,\mu(f))$, for every $c \in E$ and $f \in \mathcal{K}$. Now, for every $f \in \mathcal{L}(\rho,\mathcal{K},E)$, let $\mu_{\rho,b}(f) = \ell(z)$, where z is the element of the space $E\hat{\otimes}L(\rho,\mathcal{K})$ such that the element $[f]_\rho$ of $L(\rho,\mathcal{K},E)$ is the image of z under the isomorphism of Proposition 5.3. By the definition of the space $L(\rho,\mathcal{K},E)$ and Proposition 5.3, this defines a unique continuous linear map, $\mu_{\rho,b} : \mathcal{L}(\rho,\mathcal{K},E) \to G$, such that (B.3) holds for every $c \in E$ and every $f \in \mathcal{K}$.

Under the assumptions of this proposition, we write

$$\int_\Omega b(f,\mathrm{d}_\rho\mu) = \int_\Omega b(f(\omega),\mu(\mathrm{d}_\rho\omega)) = \mu_{\rho,b}(f) ,$$

for every function $f \in \mathcal{L}(\rho,\mathcal{K},E)$. Of course, if a different notation is used for the bilinear map b , then it is also used in the symbol for integral. So, for example, if we write $b(x,y) = xy$, for any $x \in E$ and $y \in F$, using simple juxtaposition, then we also write

$$\int_\Omega f\mathrm{d}_\rho\mu = \mu_{\rho,b}(f) ,$$

for every $f \in \mathcal{L}(\rho,\mathcal{K},E)$. Or, if the function f is F-valued and the map μ is E-valued, we denote the integral by

$$\int_\Omega b(\mathrm{d}_\rho\mu,f) = \int_\Omega b(\mu(\mathrm{d}_\rho\omega),f(\omega)) .$$

C. Let the space Ω be equal to the Cartesian product of the spaces Ξ and Υ . That is, $\Omega = \Xi\times\Upsilon$.

If g is a function on Ξ and h a function on Υ , then $f = g\otimes h$ will stand for the function on Ω such that $f(\omega) = g(\xi)h(v)$, for every $\omega = (\xi,v)$ with $\xi \in \Xi$ and $v \in \Upsilon$.

Let $\mathcal{G}$ be a nontrivial family of functions on the space Ξ and $\mathcal{H}$ a nontrivial family of functions on the space Υ . Let $\mathcal{K} = \{g\otimes h : g \in \mathcal{G} ,\ h \in \mathcal{H}\}$.

Let σ be a gauge on $\mathcal{G}$ and τ a gauge on $\mathcal{H}$. By $\rho = \sigma\otimes\tau$ is denoted the gauge on $\mathcal{K}$ such that

$$\rho(f) = \sigma(g)\tau(h) ,$$

for any function $f = g\otimes h$ with $g \in \mathcal{G}$ and $h \in \mathcal{H}$. The gauge ρ is called the direct product of the gauges σ and τ .

PROPOSITION 5.5. *If the gauges σ and τ are both integrating, then their direct product, $\rho = \sigma\otimes\tau$, too is integrating.*

Proof. Let $f = g\otimes h$, $g \in \mathcal{G}$, $h \in \mathcal{H}$. Let c_j be numbers and $f_j \in \mathcal{K}$ functions, $f_j = g_j\otimes h_j$, $g_j \in \mathcal{G}$, $h_j \in \mathcal{H}$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j|\rho(f_j) < \infty \tag{C.1}$$

and

$$f(\omega) = \sum_{j=1}^{\infty} c_j f_j(\omega) ,$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j|\,|f_j(\omega)| < \infty .$$

Let ℓ be a continuous linear functional of norm not greater than one on the space $L(\sigma,\mathcal{G})$ such that $|\ell([g])_\sigma| = q_\sigma(g) = \sigma(g)$, and m a continuous linear functional of norm not greater than one on the space $L(\tau,\mathcal{H})$ such that $|m(h)| = q_\tau([h]_\tau) = \tau(h)$.

Let $u = \ell(g)h$ and $u_j = c_j\ell(g_j)h_j$, for every $j = 1,2,\ldots$. Then $q_\tau(u_j) = |c_j|\,|\ell(g_j)|\,\tau(h_j) \le |c_j|\,\sigma(g_j)\tau(h_j) = |c_j|\rho(f_j)$, for every $j = 1,2,\ldots$, and, by (C.1),

$$\sum_{j=1}^{\infty} q_\tau(u_j) < \infty . \tag{C.2}$$

Now, for every $v \in \Upsilon$, such that

$$\sum_{j=1}^{\infty} |c_j|\,\sigma(g_j)\,|h_j(v)| < \infty , \tag{C.3}$$

let Ξ_v be the set of all points $\xi \in \Xi$ such that

$$\sum_{j=1}^{\infty} |c_j|\,|g_j(\xi)|\,|h_j(v)| < \infty .$$

By (5.10) and Proposition 2.2, the set $\Xi\backslash\Xi_v$ is σ-null, and, by Theorem 2.3,

$$\lim_{n\to\infty} q_\sigma\left[h(v)g - \sum_{j=1}^{n} c_j h_j(v) g_j\right] = 0 .$$

Hence, by the continuity of the functional ℓ,

$$\sum_{j=1}^{\infty} u_j(v) = u(v)$$

for every $v \in \Upsilon$ such that (C.3) holds. However, by (C.2) and Theorem 2.3, (C.3) holds for τ-almost every $v \in \Upsilon$. Therefore, by (C.2) and Theorem 2.3,

$$\lim_{n\to\infty} q_\tau\left[u - \sum_{j=1}^{n} u_j\right] = 0 .$$

So, by the continuity of the functional m,

$$\ell(g)m(h) = m(u) = \sum_{j=1}^{\infty} m(u_j) = \sum_{j=1}^{\infty} c_j\ell(g_j)m(h_j) .$$

Consequently,

$$\rho(f) = \sigma(g)\tau(h) = |\ell(g)m(h)| \le \sum_{j=1}^{\infty} |c_j|\rho(f_j) ,$$

because $|\ell(g_j)m(h_j)| \le q_\sigma(g_j)q_\tau(h_j) = \sigma(g_j)\tau(h_j) = \rho(f_j)$, for every $j = 1,2,\dots$. Hence, by Proposition 2.7, the gauge ρ is integrating.

In many situations, for example when $\mathcal{G}$ and $\mathcal{H}$ are quasirings of sets and σ and τ non-negative σ-additive set functions or $\mathcal{G}$ and $\mathcal{H}$ are vector spaces and σ and τ seminorms, a simpler direct proof of this proposition, avoiding the duality considerations, can be given. The proof presented here was suggested by Brian Jefferies.

D. Let Ξ, Υ, Ω, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ have the same meaning as in Section C .

PROPOSITION 5.6. *Let σ be an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$ and let $\rho = \sigma\otimes\tau$ be their direct product.*

If $g \in \mathcal{L}(\sigma,\mathcal{G})$ and $h \in \mathcal{L}(\tau,\mathcal{H})$, then the function $f = g\otimes h$ is ρ-integrable and $q_\rho(f) = q_\sigma(g)q_\tau(h)$.

For every function $f \in \mathcal{L}(\rho,\mathcal{K})$, *there exist functions* $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ *and* $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots,$ *such that*

(D.1) $$\sum_{j=1}^{\infty} q_\sigma(g_j)q_\tau(h_j) < \infty$$

and

(D.2) $$f(\xi,v) = \sum_{j=1}^{\infty} g_j(\xi)h_j(v) ,$$

for every $\xi \in \Xi$ *and* $v \in \Upsilon$ *such that*

(D.3) $$\sum_{j=1}^{\infty} |g_j(\xi)h_j(v)| < \infty .$$

Furthermore,

(D.4) $$\lim_{n\to\infty} q_\rho\Big[f - \sum_{j=1}^{n} g_j\otimes h_j\Big] = 0 .$$

Conversely, if f *is a function on* Ω *for which there exists functions* $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ *and* $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots,$ *satisfying condition* (D.1), *such that the equality* (D.2) *holds for every* $\xi \in \Xi$ *and* $v \in \Upsilon$ *for which the inequality* (D.3) *does, then* $f \in \mathcal{L}(\rho,\mathcal{K})$.

Proof. By a straightforward application of Proposition 2.1, if $g \in \mathcal{G}$, $h \in \mathcal{L}(\tau,\mathcal{K})$ and $f = g\otimes h$, then $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = \sigma(g)q_\tau(h)$. By a second application of Proposition 2.1, if $g \in \mathcal{L}(\sigma,\mathcal{G})$, $h \in \mathcal{L}(\tau,\mathcal{K})$, then $f \in \mathcal{L}(\rho,\mathcal{K})$ and $q_\rho(f) = q_\sigma(g)q_\tau(h)$.

If $f \in \mathcal{L}(\rho,\mathcal{K})$, then such functions $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ and $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots,$ as claimed exist trivially because $\mathcal{G} \subset \mathcal{L}(\sigma,\mathcal{G})$, $\mathcal{H} \subset \mathcal{L}(\tau,\mathcal{H})$, $q_\sigma(g) = \sigma(g)$, for $g \in \mathcal{G}$; and $q_\tau(h) = \tau(h)$, for $h \in \mathcal{H}$. The equality (D.4) follows by Proposition 2.1. Conversely, if such functions g_j and h_j do exist, then, as we have just noted, the functions $f_j = g_j\otimes h_j$ belong to $\mathcal{L}(\rho,\mathcal{K})$, for every $j = 1,2,\ldots,$ and, hence, by Proposition 2.1, $f \in \mathcal{L}(\rho,\mathcal{K})$.

COROLLARY 5.7. *There is a canonical isometric isomorphism of the space* $L(\rho,\mathcal{K})$ *onto the projective tensor product of the spaces* $L(\sigma,\mathcal{G})$ *and* $L(\tau,\mathcal{H})$.

Proof. Every element, t, of the projective tensor product of the spaces $L(\sigma,\mathcal{G})$ and $L(\tau,\mathcal{H})$ can be written in the form

$$t = \sum_{j=1}^{\infty} [g_j]_\sigma \otimes [h_j]_\tau ,$$

where the functions $g_j \in \mathcal{L}(\sigma,\mathcal{G})$ and $h_j \in \mathcal{L}(\tau,\mathcal{H})$, $j = 1,2,\ldots,$ satisfy condition (D.1). Moreover, the infimum of the numbers (D.1) over all such representations of t is equal to the projective tensor product norm of the element t.

PROPOSITION 5.8. *Let E, F and G be Banach spaces and let $b : F \times G \to E$ be a continuous bilinear map. Let σ be a gauge integrating for an additive map $\nu : \mathcal{G} \to F$ and τ a gauge integrating for an additive map $\lambda : \mathcal{H} \to G$. Let $\rho = \sigma \otimes \tau$ be their direct product. Let $\mu(f) = b(\nu(g),\lambda(h))$, for every $f = g \otimes h$ such that $g \in \mathcal{G}$ and $h \in \mathcal{H}$.*

Then $\mu : \mathcal{K} \to E$ is an additive map and the gauge ρ integrates for μ.

Proof. By Proposition 3.1, there exist a unique continuous linear map $\nu_\sigma : \mathcal{L}(\sigma,\mathcal{G}) \to F$ that extends ν and a unique continuous linear map $\lambda_\tau : \mathcal{L}(\tau,\mathcal{H}) \to G$ that extends λ. Let $\ell : L(\sigma,\mathcal{G}) \hat{\otimes} L(\tau,\mathcal{H}) \to E$ be the continuous linear map such that $\ell([g]_\sigma \otimes [h]_\tau) = b(\nu_\sigma(f),\lambda_\tau(h))$, for every $g \in \mathcal{L}(\sigma,\mathcal{G})$ and $h \in \mathcal{L}(\tau,\mathcal{H})$. Now, given a function $f \in \mathcal{L}(\rho,\mathcal{K})$, let t be the element of the tensor product $L(\sigma,\mathcal{G}) \hat{\otimes} L(\tau,\mathcal{H})$ that corresponds to the element $[f]_\rho$ of the space $L(\rho,\mathcal{K})$ under the isomorphism of Corollary 5.7 and let $\mu_\rho(f) = \ell(t)$. This defines a continuous linear map $\mu_\rho : \mathcal{L}(\rho,\mathcal{K}) \to E$ such that $\mu_\rho(f) = \mu(f)$, whenever $f = g \otimes h$ with $g \in \mathcal{G}$ and $h \in \mathcal{H}$. So, the map $\mu : \mathcal{K} \to E$ is indeed additive and the gauge ρ integrates for it.

EXAMPLE 5.9. Let E, F and G be Banach spaces, $b : F \times G \to E$ a continuous bilinear map. Let $\mathcal{Q}$ and $\mathcal{R}$ be σ-algebras of sets in the spaces Ξ and Υ, respectively. Let $\nu : \mathcal{Q} \to F$ and $\lambda : \mathcal{R} \to G$ be σ-additive set functions. Let $\mu(X \times Y) = b(\nu(X),\lambda(Y))$, for every $X \in \mathcal{Q}$ and $Y \in \mathcal{R}$. It is known that μ is not necessarily a σ-additive set function on the semialgebra $\mathcal{P} = \{X \times Y : X \in \mathcal{Q}, Y \in \mathcal{R}\}$; not even if $F = G$ is a Hilbert space, E is the space of scalars and b is the inner product in F. Cf. [13] and [58]. However, if

$$\sigma(g) = \sup\{v(y'\circ\nu,|g|) : y' \in F', \|y'\| \le 1\} ,$$

for every $g \in \mathrm{sim}(\mathcal{Q})$, and

$$\tau(h) = \sup\{v(z'\circ\lambda,|h|) : z' \in G', \|z'\| \le 1\} ,$$

for every $h \in \mathrm{sim}(\mathcal{R})$, then σ is a seminorm on $\mathrm{sim}(\mathcal{Q})$ integrating for ν and τ a seminorm on $\mathrm{sim}(\mathcal{R})$ integrating for λ. (See Section 3F, formula (F.2).) Therefore, $\rho = \sigma\otimes\tau$ is a gauge on the family of functions $\mathcal{K} = \{g\otimes h : g \in \mathrm{sim}(\mathcal{Q}), \ h \in \mathrm{sim}(\mathcal{R})\}$ which integrates for μ.

E. Let Ξ, Υ, Ω, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ have the same meaning as in Section C.

Given a function f on Ω and a point $\xi \in \Xi$, by $f(\xi,\cdot)$ is denoted the function $v \mapsto f(\xi,v)$, $v \in \Upsilon$. The meaning of $f(\cdot,v)$, for a given $v \in \Upsilon$, is analogous.

PROPOSITION 5.10. *Let σ be an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$. Let $\rho = \sigma\otimes\tau$. Let $f \in \mathcal{L}(\rho,\mathcal{K})$.*

Then, for σ-almost every $\xi \in \Xi$, the function $f(\xi,\cdot)$ is τ-integrable. Furthermore, if φ is an $L(\tau,\mathcal{H})$-valued function on Ξ such that $\varphi(\xi) = [f(\xi,\cdot)]_\tau$, for σ-almost every $\xi \in \Xi$, then the function φ is Bochner integrable with respect to σ and $q_\sigma(\varphi) = q_\rho(f)$.

Similarly, for τ-almost every $v \in \Upsilon$, the function $f(\cdot,v)$ is σ-integrable. Furthermore, if ψ is an $L(\sigma,\mathcal{G})$-valued function on Υ such that $\psi(v) = [f(\cdot,v)]_\sigma$, for τ-almost every $v \in \Upsilon$, then the function ψ is Bochner integrable with respect to τ and $q_\tau(\psi) = q_\rho(f)$.

Proof. Let c_j be numbers and $g_j \in \mathcal{G}$ and $h_j \in \mathcal{H}$ functions, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j|\,\sigma(g_j)\tau(h_j) < \infty \tag{E.1}$$

and

(E.2)
$$f(\xi,v) = \sum_{j=1}^{\infty} c_j g_j(\xi) h_j(v)$$

for every $\xi \in \Xi$ and $v \in \Upsilon$ for which

(E.3)
$$\sum_{j=1}^{\infty} |c_j| |g_j(\xi)| |h_j(v)| < \infty .$$

By Proposition 2.2,

$$\sum_{j=1}^{\infty} |c_j| |g_j(\xi)| \tau(h_j) < \infty$$

for σ-almost every $\xi \in \Xi$. Furthermore, if $\xi \in \Xi$ is a point such that (E.3) holds, then $f(\xi,\cdot) \in \mathcal{L}(\tau,\mathcal{H})$, because (E.1) holds for every $v \in \Upsilon$ for which (E.2) does.

Now, let $\varphi_j(\xi) = c_j g_j(\xi)[h_j]_\tau$, for every $\xi \in \Xi$. Then $\varphi_j \in \mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{H}))$ and $q_\sigma(\varphi_j) = |c_j| q_\sigma(g_j) q_\tau(h_j) = |c_j| \sigma(g_j)\tau(h_j)$, for every $j = 1,2,\dots$. Let φ_0 be an $L(\tau,\mathcal{H})$-valued function on Ξ such that

$$\varphi_0(\xi) = \sum_{j=1}^{\infty} \varphi_j(\xi)$$

for every $\xi \in \Xi$ for which

$$\sum_{j=1}^{\infty} q_\tau(\varphi_j(\xi)) = \sum_{j=1}^{\infty} |c_j| |g_j(\xi)| \tau(h_j) < \infty .$$

Then, by Theorem 5.1, $\varphi_0 \in \mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{H}))$. So, if $\varphi(\xi) = \varphi_0(\xi)$ for σ-almost every $\xi \in \Xi$, then φ too belongs to $\mathcal{L}(\alpha,\mathcal{G},L(\tau,\mathcal{H}))$. The equality $q_\sigma(\varphi) = q_\rho(f)$ follows from the definition of the seminorm q_σ on $\mathcal{L}(\sigma,\mathcal{G},L(\tau,\mathcal{H}))$ and that of the seminorm q_ρ on $\mathcal{L}(\rho,\mathcal{K})$.

This proposition already contains all the ingredients necessary to state the following theorem of Fubini type.

THEOREM 5.11. *Let E, F and G be Banach spaces and $b : F \times G \to E$ a continuous bilinear map. Let σ be a gauge on $\mathcal{G}$ integrating for an additive map $\nu : \mathcal{G} \to F$ and τ a gauge on $\mathcal{H}$ integrating for an additive map $\lambda : \mathcal{H} \to G$. Let*

$\rho = \sigma \otimes \tau$. *Let* $\mu(f) = b(\nu(g),\lambda(h))$, *for every function* $f = g \otimes h$ *such that* $g \in \mathcal{G}$ *and* $h \in \mathcal{H}$.

If $f \in \mathcal{L}(\rho,\mathcal{K})$, *then*

$$\text{(E.4)} \qquad \int_\Omega f(\omega)\mu(\mathrm{d}_\rho\omega) = \int_\Xi b\left[\nu(\mathrm{d}_\sigma\xi), \int_\Upsilon f(\xi,v)\lambda(\mathrm{d}_\tau v)\right] = \\ = \int_\Upsilon b\left[\int_\Xi f(\xi,v)\nu(\mathrm{d}_\gamma\xi), \lambda(\mathrm{d}_\tau v)\right].$$

Proof. The existence of the integrals follows from Proposition 5.10. The equalities (E.4) are obviously true if $f = g \otimes h$ with $g \in \mathcal{L}(\sigma,\mathcal{G})$ and $h \in \mathcal{L}(\tau,\mathcal{H})$. Furthermore, all terms are linear in f. Propositions 5.4 and 5.10 imply that all terms of (E.4) depend continuously on f. Because the algebraic tensor product $L(\sigma,\mathcal{G}) \otimes L(\tau,\mathcal{H})$ is isomorphic to a dense subspace of $L(\rho,\mathcal{K})$, the equalities (E.4) are valid for every $f \in \mathcal{L}(\rho,\mathcal{K})$.

F. We still maintain the notation of the previous section and assume that σ is an integrating gauge on $\mathcal{G}$ and τ an integrating gauge on $\mathcal{H}$. So, $\rho = \sigma \otimes \tau$ is an integrating gauge on $\mathcal{K}$.

We prove a converse to Proposition 5.10, which is a Tonelli-type theorem only under some additional assumptions.

ASSUMPTION 5.12. *Let* $Z \subset \Omega$ *and let* X *be the set of all points* $\xi \in \Xi$ *such that the set* $\{v \in \Upsilon : (\xi,v) \in Z\}$ *is not* τ*-null. If the set* X *is* σ*-null, then the set* Z *is* ρ*-null.*

The following proposition gives a convenient sufficient condition for Assumption 5.12 to be satisfied.

PROPOSITION 5.13. *If there exists a function* $h \in \mathcal{L}(\tau,\mathcal{H})$ *such that* $h(v) \neq 0$, *for every* $v \in \Upsilon$, *then Assumption* 5.12 *is satisfied.*

Proof. Let $Z \subset \Omega$ be a set and let X be the set of all points $\xi \in \Xi$ such that the set $\{v \in \Upsilon : (\xi,v) \in Z\}$ is not τ-null. If the set X is σ-null, then, by Proposition 2.2, there exist numbers c_j and functions $g_j \in \mathcal{G}$, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} |c_j|\,\sigma(g_j) < \infty$$

but

$$\sum_{j=1}^{\infty} |c_j|\,|g_j(\xi)| = \infty$$

for every $\xi \in \Xi$. Let $h \in \mathcal{L}(\tau,\mathcal{X})$ be a function such that $h(v) \neq 0$, for every $v \in \Upsilon$, and let $f_j(\omega) = c_j g_j(\xi)h(v)$, for every $\omega = (\xi,v)$, $\xi \in \Xi$, $v \in \Upsilon$, so that $q_\rho(f_j) = |c_j|\,\sigma(g_j)q_\tau(h)$, for every $j = 1,2,\ldots$. Then

$$\sum_{j=1}^{\infty} q_\rho(f_j) < \infty ,$$

but

$$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every $\omega \in Z$. By Proposition 2.2, the set Z is ρ-null.

PROPOSITION 5.14. *Let Assumption* 5.12 *be satisfied. Let f be a function on Ω such that, for σ-almost every $\xi \in \Xi$, the function $f(\xi,\cdot)$ is τ-integrable and, if φ is an $L(\tau,\mathcal{X})$-valued function on Ξ such that $\varphi(\xi) = [f(\xi,\cdot)]_\tau$, for σ-almost every $\xi \in \Xi$, then the function φ is Bochner integrable with respect to σ.*

Then $f \in \mathcal{L}(\rho,\mathcal{K})$.

Proof. Let $g_j \in \mathcal{G}$ and $h_j \in \mathcal{L}(\tau,\mathcal{X})$, $j = 1,2,\ldots$, be functions such that

$$\sum_{j=1}^{\infty} q_\rho(g_j \otimes h_j) = \sum_{j=1}^{\infty} \sigma(g_j)q_\tau(h_j) < \infty$$

and

$$\varphi(\xi) = \sum_{j=1}^{\infty} g_j(\xi)[h_j]_\tau ,$$

in the sense of convergence in the space $L(\tau,\mathcal{X})$, for every $\xi \in \Xi$ for which

$$\text{(F.1)} \qquad \sum_{j=1}^{\infty} |g_j(\xi)|\,q_\tau(h_j) < \infty .$$

By a modification of the function φ and/or f on a set of points $\xi \in \Xi$ which is negligible with respect to σ, we can achieve that $\varphi(\xi) = [f(\xi,\cdot)]_\tau$, for every point $\xi \in \Xi$ for which the inequality (F.1) holds. Then, given such a ξ, the equality

$$f(\xi,v) = \sum_{j=1}^{\infty} g_j(\xi)h_j(v) \tag{F.2}$$

holds for τ-almost every $(\xi,v) \in \Omega$. Therefore, by Theorem 2.3, $f \in \mathcal{L}(\rho,\mathcal{K})$.

If Assumption 5.12 is not satisfied, then the conclusion of this proposition does not necessarily hold.

EXAMPLE 5.15. Let $\Xi = (0,1]$, $\mathcal{Q} = \{(s,t] : 0 \le s \le t \le 1\}$ and let ι be the Lebesgue measure on $\mathcal{Q}$. Let $\Upsilon = (0,1]$, let $\mathcal{R}$ be the family of all finite subsets of Υ and, for every $Y \in \mathcal{R}$, let $\kappa(Y)$ be the number of elements in Y.

Let f be the characteristic function of the set $\{1\}\times\Upsilon$. Then, for ι-almost every $\xi \in \Xi$, $f(\xi,\cdot)$ is the zero-function on Υ, but the function f does not belong to $\mathcal{L}(\iota\otimes\kappa,\mathcal{K})$, where $\mathcal{K} = \{X\times Y : X \in \mathcal{Q}, Y \in \mathcal{R}\}$.

Obviously, the roles of the spaces Ξ and Υ, and of the structures they carry, are not symmetric in Assumption 5.12, Proposition 5.13 and Proposition 5.14. Although, it is quite clear how to formulate analogous assumptions and propositions with these roles interchanged, for the record we formulate the analogies of Assumption 5.12 and Proposition 5.14.

ASSUMPTION 5.16. *Let $Z \subset \Omega$ and let Y be the set of all points $v \in \Upsilon$ such that the set $\{\xi \in \Xi : (\xi,v) \in Z\}$ is not σ-null. If the set Y is τ-null, then the set Z is ρ-null.*

PROPOSITION 5.17. *Let Assumption 5.16 be satisfied. Let f be a function on Ω such that, for τ-almost every $v \in \Upsilon$, the function $f(\cdot,v)$ is σ-integrable and, if ψ is an $L(\sigma,\mathcal{G})$-valued function on Υ such that $\psi(v) = [f(\cdot,v)]_\sigma$, for τ-almost every $v \in \Upsilon$, then the function ψ is Bochner integrable with respect to τ.*

Then $f \in \mathcal{L}(\rho,\mathcal{K})$.

6. SCALAR OPERATORS

The most important applications of integration with respect to Banach space valued measures undoubtedly arise in the theory of spectral operators. To describe its central notion, let E be a complex Banach space, $\mathrm{BL}(E)$ the algebra of all bounded linear operators on E and I the identity operator. A spectral measure is an additive and multiplicative map $P : \mathcal{Q} \to \mathrm{BL}(E)$, whose domain, $\mathcal{Q}$, is an algebra of sets in a space Ω, such that $P(\Omega) = I$. An operator $T \in \mathrm{BL}(E)$ is said to be of scalar type if there exists a σ-additive (in the strong operator topology) spectral measure, P, whose domain is a σ-algebra and a P-integrable function f such that

$$(*) \qquad T = \int_\Omega f \mathrm{d}P .$$

This notion, due to N. Dunford, extends to arbitrary Banach space the idea of an operator with diagonalizable matrix on a finite-dimensional space. It proved to be very fruitful as shows the exposition in Part III of the monograph [14]. Many powerful techniques in which scalar operators play a role are based on the requirements that $\mathcal{Q}$ be a σ-algebra and that P be σ-additive. But precisely these requirements are responsible for excluding many operators of prime interest from the class of scalar-type operators.

In this chapter, we present a suggestion for extending this class, [35]. It is based on the fact that the integral (*) exists if and only if there exist $\mathcal{Q}$-simple functions f_j, $j = 1,2,\ldots$, such that

$$\sum_{j=1}^{\infty} \left\| \int_\Omega f_j \mathrm{d}P \right\| < \infty$$

and the equality

$$f(\omega) = \sum_{j=1}^{\infty} f_j(\omega)$$

holds for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |f_j(\omega)| < \infty .$$

In that case,

$$\int_{\Omega} f\mathrm{d}P = \sum_{j=1}^{\infty} \int_{\Omega} f_j \mathrm{d}P .$$

So, the integral with respect to P can be characterized purely in terms of the operator-norm convergence. Moreover, to use this characterization as a definition of the integral with respect to P, it is not necessary to assume that the set function P be bounded, let alone σ-additive, nor that $\mathcal{Q}$ be a σ-algebra. It suffices to assume that the seminorm

$$f \mapsto \| \int_{\Omega} f\mathrm{d}P \| ,$$

on $\mathcal{Q}$-simple functions, be integrating. (See Section 2D.)

Thus, as scalar operators in a wider sense, we propose operators which can be expressed in the form (*) assuming that P is a spectral measure such that the mentioned seminorm is indeed integrating. Such operators can also be characterized intrinsically, that is, without the reference to any particular definition of integral. Namely, an operator $T \in \mathrm{BL}(E)$ turns out to be scalar in this sense if and only if there exists a (not necessarily bounded) Boolean algebra of projections belonging to $\mathrm{BL}(E)$ such that the Banach algebra of operators it generates is semisimple and contains T. However, in contrast with the classical theory, the Gelfand representations of such a Banach algebra is not necessarily the algebra of all continuous functions on its structure space but only a dense subalgebra.

A. Let E be a complex Banach space. Let $\mathrm{BL}(E)$ be the algebra of all bounded linear operators on E. Then $\mathrm{BL}(E)$ is a Banach algebra with respect to the operator (uniform) norm, defined by $\|T\| = \sup\{ |Tx| : |x| \le 1 , x \in E\}$, for every $T \in \mathrm{BL}(E)$. The identity operator is denoted by I.

Let $\mathcal{Q}$ be a quasialgebra of sets in the space Ω. (See Section 1D.) A map $P : \mathcal{Q} \to \mathrm{BL}(E)$ is said to be multiplicative if $P(fg) = P(f)P(g)$ for every $f \in \mathrm{sim}(\mathcal{Q})$ and $g \in \mathrm{sim}(\mathcal{Q})$. For an additive (see Section 1E) map, P, to be multiplicative it suffices that $P(X \cap Y) = P(X)P(Y)$ for every $X \in \mathcal{Q}$ and $Y \in \mathcal{Q}$.

An additive and multiplicative map $P : \mathcal{Q} \to \mathrm{BL}(E)$ such that $P(\Omega) = I$ will be called a $\mathrm{BL}(E)$-valued spectral set function on $\mathcal{Q}$. If $\mathcal{Q}$ happens to be an algebra of sets, then a spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ is called a spectral measure; see [14], Definition XV.2.1.

The generality of the theory presented in this chapter is not substantially increased by the admission of arbitrary spectral set functions instead of spectral measures only. This admission is dictated mainly by convenience in considering the families of sets which classically occur in integration and spectral theories but are merely quasialgebras and not algebras. It also allows for the possibility of distinguishing certain nuances in the presented theory. However, with the exception of a single remark in the last section, this possibility will not be pursued here.

A spectral set fnction $P : \mathcal{Q} \to \mathrm{BL}(E)$ is said to be σ-additive if, for every $x \in E$, the E-valued set function $X \mapsto P(X)x$, $X \in \mathcal{Q}$, is σ-additive. (See Section 1F.) That is to say, σ-additivity of spectral set functions is understood in the strong operator topology of $\mathrm{BL}(E)$.

In virtue of the Stone representation theorem, a set $W \subset \mathrm{BL}(E)$ is a Boolean algebra of projection operators if and only if there exist an algebra of sets, $\mathcal{R}$, in a space Ω and a spectral measure, $P : \mathcal{R} \to \mathrm{BL}(E)$, such that $W = \{P(X) : X \in \mathcal{R}\}$. Accordingly, a set of operators $\mathrm{W} \subset \mathrm{BL}(E)$ is called a Boolean quasialgebra of projection operators if it is the range of a $\mathrm{BL}(E)$-valued spectral set function, that is, if there exist a quasialgebra of sets, $\mathcal{Q}$, in a space Ω and a spectral set function, $P : \mathcal{Q} \to \mathrm{BL}(E)$, such that $W = \{P(X) : X \in \mathcal{Q}\}$.

If $W \subset \mathrm{BL}(E)$, then by $A(W)$ is denoted the least uniformly closed algebra of operators which contains W. If $W = \{P(X) : X \in \mathcal{Q}\}$ is the range of a spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$, we write $A(W) = A(P)$. Clearly, $A(P)$ is then the closure of the family of operators $\{P(f) : f \in \mathrm{sim}(\mathcal{Q})\}$ in the space $\mathrm{BL}(E)$.

Recall that, if A is a commutative Banach algebra with unit, then the structure space, Δ, of A is the set of all homomorphisms of A onto the field of complex numbers. For an element T of A, by $\hat{T}$ is denoted the Gelfand transform of T; it is the function on Δ defined by $\hat{T}(h) = h(T)$, for every $h \in \Delta$. It is well-known (see e.g. [46], 23B) that $\sup\{|\hat{T}(h)| : h \in \Delta\} \leq \|T\|$ and that the coarsest topology on Δ which makes all the functions $\hat{T}$, $T \in A$, continuous turns Δ into a compact Hausdorff space. Hence the Gelfand transform is a norm-decreasing homomorphism of the algebra A into the algebra, $C(\Delta)$, of all complex continuous functions on Δ. If the Gelfand transform is injective, then the algebra A is called semisimple.

Recall that an operator $T \in \mathrm{BL}(E)$ is called nonsingular if it is invertible in $\mathrm{BL}(E)$, that is, if there exists an operator $S \in \mathrm{BL}(E)$ such that $ST = TS = I$. Then of course $S = T^{-1}$ is the inverse of each of T. A full algebra of operators is uniformly closed algebra of operators which contains the inverse of each of its nonsingular elements; see [14], Definition XVII.1.1.

LEMMA 6.1. *Let Q be a quasialgebra of sets in a space Ω and let $P : Q \to \mathrm{BL}(E)$ be a spectral set function.*

(i) *If $f \in \mathrm{sim}(Q)$, then the operator $P(f)$ is nonsingular if and only if the function f can be represented in the form*

$$f = \sum_{j=1}^{n} c_j X_j , \tag{A.1}$$

where the n is a natural number, the c_j are non-zero complex numbers and the X_j are pair-wise disjoint sets from Q, $j = 1,2,\ldots,n,$ such that

$$\sum_{j=1}^{n} P(X_j) = I .$$

In that case, $(P(f))^{-1} = P(g)$, where

$$g = \sum_{j=1}^{n} c_j^{-1} X_j .$$

(ii) *Let $f \in \mathrm{sim}(Q)$ be a function expressed in the form (A.1) where $X_j \in Q$ are pair-wise disjoint sets such that $P(X_j) \neq 0$, for every $j = 1,2,\ldots,n,$ and let*

$c = \sup\{|c_j| : j = 1,2,\ldots,n\}$ *and*

$$d = \sup\{\|\sum_{j\in J} P(X_j)\| : J \subset \{1,2,\ldots,n\}\} .$$

Then $c \leq \|P(f)\| \leq 4cd$.

(iii) $A(P)$ *is a full algebra of operators.*

Proof. Let $n \geq 1$ be an integer. Let $X_j \in Q$ be pair-wise disjoint sets, such that $P(X_j) \neq 0$, for every $j = 1,2,\ldots,n$ and the sum of the operators $P(X_j)$, $j = 1,2,\ldots,n$, is equal to I . Then the family of operators

$$\sum_{j=1}^{n} c_j P(X_j) ,$$

with arbitrary complex $c_j = 1,2,\ldots,n$, is a closed algebra of operators generated by the Boolean algebra of projections

$$\sum_{j\in J} P(X_j) ,$$

where J varies over all subsets of $\{1,2,\ldots,n\}$. Then (i) holds by Lemma XVII.2.1 and (ii) by Lemma XVII.2.2 in [14].

To show that $A(P)$ is a full algebra of operators let T be a non-singular element of $A(P)$. Let $f_n \in \mathrm{sim}(Q)$, $n = 1,2,\ldots,$ be functions such that $\|T-P(f_n)\| \to 0$, as $n \to \infty$. Then for all sufficiently large n , the operator $P(f_n)$ is nonsingular and $\|T^{-1} - (P(f_n))^{-1}\| \to 0$. But, by (i), for each such n , there exists a function $g_n \in \mathrm{sim}(Q)$ such that $(P(f_n))^{-1} = P(g_n)$. Therefore, $T^{-1} \in A(P)$.

B. With a spectral set function $P : Q \to \mathrm{BL}(E)$, we shall associate the seminorm ρ_P on $\mathrm{sim}(Q)$ defined by

(B.1) $$\rho_P(f) = \|P(f)\|$$

for every $f \in \mathrm{sim}(Q)$.

PROPOSITION 6.2. *A set* $Y \subset \Omega$ *is* ρ_P*-null if and only if there exist sets* $X_j \in \mathcal{Q}$ *such that* $P(X_j) = 0$, *for every* $j = 1,2,\ldots,$ *and*

(B.2) $$Y \subset \bigcup_{j=1}^{\infty} X_j .$$

Proof. Let $X_j \in \mathcal{Q}$ be sets such that $P(X_j) = 0$ for every $j = 1,2,\ldots$ and (B.2) holds. Let us repeat each set countably many times, arrange the resulting family of sets into a single sequence and call their characteristic functions f_j, $j = 1,2,\ldots$. Then $f_j \in \operatorname{sim}(\mathcal{Q})$, for every $j = 1,2,\ldots,$

(B.3) $$\sum_{j=1}^{\infty} \rho_P(f_j) < \infty$$

and

(B.4) $$\sum_{j=1}^{\infty} |f_j(\omega)| = \infty$$

for every $\omega \in Y$. So, by Proposition 2.2, the set Y is ρ_P-null.

Conversely, assume that $f_j \in \operatorname{sim}(\mathcal{Q})$, $j = 1,2,\ldots,$ are functions, satisfying (B.3), such that (B.4) holds for every $\omega \in Y$. Let

$$f_j = \sum_{j=1}^{n_j} c_{jk} X_{jk} ,$$

with some integer $n_j \geq 1$, numbers c_{jk} and pair-wise disjoint sets $X_{jk} \in \mathcal{Q}$, $k = 1,2,\ldots,n_j$, for every $j = 1,2,\ldots$. By Lemma 6.1, $\|P(f_j)\| \geq |c_{jk}|$, whenever $P(X_{jk}) \neq 0$. Therefore if we modify each function f_j by omitting those sets X_{jk} , together with the corresponding numbers c_{jk} , for which $P(X_{jk}) \neq 0$, then (B.4) will remain satisfied for every $\omega \in Y$. But then, Y is covered by the remaining sets X_{jk} , $k = 1,2,\ldots,n_j$, $j = 1,2,\ldots$.

In view of this proposition, ρ_P-null sets will be called simply P-null.

For a function f on Ω , let

$$\|f\|_\infty = \inf\{\sup\{|f(\omega)| : \omega \in \Omega \setminus Y\} : Y \in \mathcal{N}\} ,$$

where $\mathcal{N}$ is the family of all P-null sets. Then $0 \le \|f\|_\infty \le \infty$. The function f is said to be P-essentially bounded if $\|f\|_\infty < \infty$. In that case, the infimum is actually a minimum because any subset of the union of countably many P-null sets is P-null. That is to say, for any P-essentially bounded function f, there exists a P-null set, Y, such that

$$\|f\|_\infty = \sup\{|f(\omega)| : \omega \in \Omega \setminus Y\} .$$

Following the custom, we shall call P-null any function f on Ω such that $\|f\|_\infty = 0$. The P-equivalence class of a function f will be denoted by $[f]$, or by $[f]_P$ if the spectral set function P needs to be indicated. To be sure, $[f]$ is the set of all functions g on Ω such that $\|f-g\|_\infty = 0$.

Let $\mathcal{L}^\infty(P)$ be the family of all functions f on Ω such that, for every $\epsilon > 0$, there exists a function $g \in \operatorname{sim}(\mathcal{Q})$ for which $\|f-g\|_\infty < \epsilon$. Then $\mathcal{L}^\infty(P)$ is an algebra under the point-wise operations.

Let $L^\infty(P) = \{[f] : f \in \mathcal{L}^\infty(P)\}$. Then $L^\infty(P)$ is a Banach algebra with respect to the operations induced by the operations in the algebra $\mathcal{L}^\infty(P)$ and the norm, $\|\cdot\|_\infty$, induced by the seminorm $f \mapsto \|f\|_\infty$, $f \in \mathcal{L}^\infty(P)$.

The Banach algebra $L^\infty(P)$ is semisimple (see e.g. [46], Theorem 24C). Actually, if Δ is the structure space of $L^\infty(P)$, then the Gelfand transform is an isometric isomorphism of $L^\infty(P)$ onto the whole of $C(\Delta)$. Moreover, for any function $f \in \mathcal{L}^\infty(P)$, the equality

$$\{[f]^\wedge(h) : h \in \Delta\} = \bigcap_{Y \in \mathcal{N}} \overline{\{f(\omega) : \omega \in \Omega \setminus Y\}} \tag{B.5}$$

holds, where $\mathcal{N}$ is the family of all P-null sets and the bar indicates the closure in the complex plane. The set (B.5) is called the P-essential range of the function f.

C. A spectral set function $P : \mathcal{Q} \to BL(E)$ will be called closable if the associated seminorm, ρ_P, defined by (B.1) on $\operatorname{sim}(\mathcal{Q})$ is integrating. Obviously, in that case, ρ_P integrates for P. Because ρ_P is determined by P, we shall write

$\mathcal{L}(P) = \mathcal{L}(\rho_P, \mathrm{sim}(Q))$, $L(P) = L(\rho_P, \mathrm{sim}(Q))$, $q_{\rho_P} = \rho_P$ and

$$P(f) = \int_\Omega f(\omega)P(\mathrm{d}\omega) = \int_\Omega f\mathrm{d}P = \int_\Omega f\mathrm{d}_{\rho_P} P ,$$

for every $f \in \mathcal{L}(P)$, omitting the subscript.

PROPOSITION 6.3. *Let $P : Q \to \mathrm{BL}(E)$ be a closable spectral set function.*

The equality $\|f\|_\infty = 0$ holds for a function f on Ω if and only if $f \in \mathcal{L}(P)$ and $(f) = 0$. Furthermore, $\mathcal{L}(P) \subset \mathcal{L}^\infty(P)$ and $\|f\|_\infty \le \|P(f)\|$, for every function $f \in \mathcal{L}(P)$.

If $f \in \mathcal{L}(P)$ and $g \in \mathcal{L}(P)$ then $fg \in \mathcal{L}(P)$ and $P(fg) = P(f)P(g)$. So, $\mathcal{L}(P)$ is an algebra of functions.

The range of the integration map $P : \mathcal{L}(P) \to \mathrm{BL}(E)$ is equal to $A(P)$. The Banach algebra $A(P)$ is semisimple. The integration map $P : L(P) \to A(P)$ is an isomorphism of the algebra $L(P)$ onto the algebra $A(P)$.

If $f \in \mathcal{L}(P)$, then the spectrum of the operator $T = P(f)$ is equal to the P-essential range of the function f .

Proof. If f is a function on Ω such that $\|f\|_\infty = 0$, then by the definitions of the P-null sets, P-null functions and integral, $f \in \mathcal{L}(P)$ and $P(f) = 0$.

Let $f \in \mathcal{L}(P)$. Let $f_j \in \mathrm{sim}(Q)$, $j = 1,2,\ldots$, be functions, satisfying condition (B.3), such that

(C.1) $$f(\omega) = \sum_{j=1}^\infty f_j(\omega)$$

for every $\omega \in \Omega$ for which

(C.2) $$\sum_{j=1}^\infty |f_j(\omega)| < \infty .$$

Then, by Lemma 6.1,

(C.30) $$\sum_{j=1}^\infty \|f_j\|_\infty < \infty .$$

By the completeness of the space $L^\infty(P)$, there exists a function $g \in \mathcal{L}^\infty(P)$ such that

$$[g] = \sum_{j=1}^{\infty} [f_j] \tag{C.4}$$

in $L^\infty(P)$. Since, by Proposition 6.2, the set of points $\omega \in \Omega$, for which the equality (C.1) does not hold, is P-null, we have $\|f-g\|_\infty = 0$, and so, $f \in \mathcal{L}^\infty(P)$. Moreover, by Lemma 6.1,

$$\| \sum_{j=1}^{n} f_j\|_\infty \leq \|P\Big[\sum_{j=1}^{n} f_j\Big]\|,$$

for every $n = 1,2,\ldots$. Therefore, by Proposition 2.1 and the continuity of norms, $\|f\|_\infty \leq \|P(f)\|$.

If, moreover, $g \in \operatorname{sim}(\mathcal{Q})$, then, by Lemma 6.1, $P(f_j g) = P(f_j)P(g)$, for every $j = 1,2,\ldots$, and, by (B.3),

$$\sum_{j=1}^{\infty} \|P(f_j g) \leq \sum_{j=1}^{\infty} \|P(f_j)\| \; \|P(g)\| < \infty. \tag{C.5}$$

Hence, $fg \in \mathcal{L}(P)$ and $\mathrm{P}(fg) = P(f)P(g)$. But then, we can write (C.5) for any function $g \in \mathcal{L}(P)$. Consequently, by Proposition 2.1, $fg \in \mathcal{L}(P)$ and $P(fg) = \mathrm{P}(f)P(g)$ for any $f \in \mathcal{L}(P)$ and $g \in \mathcal{L}(P)$.

It is clear, from the definition of the integral, that for any $f \in \mathcal{L}(P)$, the operator $P(f)$ belongs to $A(P)$, the closure of the set $\{P(h) : h \in \operatorname{sim}(\mathcal{Q})\}$. Hence, to show that $\{P(h) : h \in \mathcal{L}(P)\} = A(P)$, it suffices to show that the set $\{P(h) : h \in \mathcal{L}(P)\}$ is closed in $\mathrm{BL}(E)$. So, let the operator T be in the closure of this set. Let $h_j \in \mathcal{L}(P)$ be functions such that $\|T-P(h_j)\| < 2^{-j}$ for every $j = 1,2,\ldots$. Let $f_1 = h_1$ and $f_j = h_j - h_{j-1}$, for every $j = 2,3,\ldots$. Then the condition (B.3) is satisfied, and, so by Proposition 2.1, if f is a function such that (C.1) holds for every $\omega \in \Omega$ for which (C.2) does, then $f \in \mathcal{L}(P)$ and $T = P(f)$.

It is now obvious that the integration map $P : L(P) \to A(P)$ is an isomorphism of the algebras $L(P)$ and $A(P)$. Because the algebra $L(P)$ is semisimple, being a dense subalgebra of $L^\infty(P)$, the algebra $A(P)$ too is semisimple.

By Lemma 6.1, the algebra $A(P)$ is full. Therefore, the spectrum of an operator T belonging to $A(P)$ coincides with its spectrum as an element of this algebra. Because of the isomorphism of $A(P)$ and $L(P)$, this spectrum coincides with the spectrum of the element, $[f]$, of the algebra $L(P)$ such that $T = P(f)$, which is equal to the essential range of the function f.

D. If $P : \mathcal{Q} \to \mathrm{BL}(E)$ is a closable spectral set function, then by Proposition 6.3, $L(P) \subset L^{\infty}(P)$. Clearly, if P is not bounded on the algebra generated by $\mathcal{Q}$, then the integration map is not continuous in the norm of the space $L^{\infty}(P)$ and its domain, $L(P)$, is not equal to the whole of $L^{\infty}(P)$. This domain is of course dense in $L^{\infty}(P)$ and the following proposition implies that the integration map is closed.

PROPOSITION 6.4. *A spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ is closable if and only if there exists an injective map $\Phi : A(P) \to L^{\infty}(P)$ such that $\|\Phi(T)\|_{\infty} \le \|T\|$, for every $T \in A(P)$, and $\Phi(P(f)) = [f]$, for every $f \in \mathrm{sim}(\mathcal{Q})$.*

If the spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ is indeed closable then such a map Φ is unique, its range if equal to $L(P)$ and the map Φ is equal to the inverse of the integration map.

Proof. If such a map $\Phi : A(P) \to L^{\infty}(P)$ exists, then it is unique and linear because $\{P(f) : f \in \mathrm{sim}(\mathcal{Q})\}$ is a dense subspace of $A(P)$. Let then $f_j \in \mathrm{sim}(\mathcal{Q})$, $j = 1,2,\ldots,$ be functions satisfying condition (B.3) and let

$$\sum_{j=1}^{\infty} f_j(\omega) = 0$$

for every $\omega \in \Omega$ for which (C.2) holds. Let $T \in \mathrm{BL}(E)$ be the operator such that

$$\lim_{n\to\infty} \|T - \sum_{j=1}^{n} P(f_j)\| = 0 .$$

Then of course $T \in A(P)$. Because the map Φ is norm-decreasing, condition (B.3) implies that (C.3) holds and, if $[g] = \Phi(T)$, then (C.4) does. Now, by Proposition 6.2, the set of the points $\omega \in \Omega$ for which (B.4) holds is P-null, and so, $[g] = 0$.

Consequently, $T = 0$ because the map Φ is injective. That is

$$\sum_{j=1}^{\infty} P(f_j) = 0 ,$$

and, by Proposition 2.8, the set function P is closable.

If the set function P is closable, then by Proposition 6.2, such a map $\Phi : A(P) \to L^{\infty}(P)$ exists: it is the inverse of the integration map.

Let us now mention a sufficient condition for a spectral set function to be closable. But first a definition:

A spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ is said to be stable if $P(Y) = 0$ for every P-null set Y which belongs to $\mathcal{Q}$.

PROPOSITION 6.5. *If $\mathcal{Q}$ is an algebra of sets and $P : \mathcal{Q} \to \mathrm{BL}(E)$ a bounded and stable spectral set function, then P is closable.*

Proof. Let $[\mathrm{sim}(\mathcal{Q})] = \{[f] : f \in \mathrm{sim}(\mathcal{Q})\}$. Because P is stable, there is a map $\tilde{P} : [\mathrm{sim}(\mathcal{Q})] \to \mathrm{BL}(E)$, unambiguously defined by $\tilde{P}([f]) = P(f)$, for every $f \in \mathrm{sim}(\mathcal{Q})$. Because P is bounded and $\mathcal{Q}$ is an algebra, by Lemma 1, the map $\tilde{P}$ is bounded. Then $\tilde{P}$ has a unique continuous extension onto the whole of $L^{\infty}(P)$. By Lemma 1, $\tilde{P}$ and its extension are norm-increasing. Therefore, $\tilde{P}$ so extended has an inverse, Φ, which is norm-decreasing. Because both maps, $\tilde{P}$ and Φ, are bounded, the domain of Φ is closed and, hence, equal to $A(P)$. So, by Proposition 6.4, the set function P is closable.

COROLLARY 6.6. *Let $P : \mathcal{Q} \to \mathrm{BL}(E)$ be a spectral set function such that, for every $x \in E$ and $x' \in E'$, the set function $X \mapsto x'P(X)x$, $X \in \mathcal{Q}$, generates a σ-additive measure of finite variation. The the set function P is closable.*

Proof. The assumption implies that the additive extension of P onto the algebra of sets generated by $\mathcal{Q}$ is bounded and stable.

Corollary 6.6 implies, in particular, that a σ-additive spectral measure whose domain is a σ-algebra of sets is closable.

E. Let us call a Boolean quasialgebra of projections $W \subset \mathrm{BL}(E)$ semisimple if the Banach algebra, $A(W)$, it generates is semisimple.

PROPOSITION 6.7. *A Boolean quasialgebra of projection operators, $W \subset \mathrm{BL}(E)$, is semisimple if and only if there exists a quasialgebra of sets, $\mathcal{Q}$, in a space Ω, and a closable spectral set function, $P : \mathcal{Q} \to \mathrm{BL}(E)$, such that $A(W) = A(P)$.*

Proof. Let W be semisimple. Let Ω be the structure space of the Banach algebra $A(W)$. Let us denote by Φ the Gelfand transform and put $\mathcal{Q} = \{\Phi(S) : S \in W\}$. Because we identify sets with their characteristic functions, $\mathcal{Q}$ is a quasialgebra of sets in the space Ω. Let $P(\Phi(S)) = S$, for every $S \in W$. This defines a spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ such that the empty set is the only P-null set. Therefore, $L^\infty(P) = C(\Omega)$ and the Gelfand transform is clearly a norm-decreasing injective map from $A(P) = A(W)$ into $L^\infty(P)$ such that $\Phi(P(f)) = [f]$ for every $f \in \mathrm{sim}(\mathcal{Q})$. So, by Proposition 6.4, the spectral set function P is closable.

Conversely, if a closable spectral set function P such that $A(W) = A(P)$ exists, then, by Proposition 6.3, the Banach algebra $A(W)$ is semisimple.

COROLLARY 6.8. *Any bounded Boolean algebra of projections is semisimple.*

Proof. By the Stone representation theorem, for any Boolean algebra of operators, W, there exists an algebra of sets, $\mathcal{Q}$, and a spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ such that $\emptyset$ is the only P-null set and $\{P(X) : X \in \mathcal{Q}\} = W$. By Proposition 4.5, the set function P is closable.

Let us call an operator $T \in \mathrm{BL}(E)$ scalar in the wider sense if there exists a semisimple Boolean quasialgebra of operators $W \subset \mathrm{BL}(E)$ such that $T \in A(W)$. By Proposition 4.7, and Proposition 4.3, an operator T is scalar in the wider sense if and only if there exist a quasialgebra of sets, $\mathcal{Q}$, in a space Ω, a closable spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ and a P-integrable function f such that $T = P(f)$.

An operator is said to be scalar in the sense of N. Dunford if there exist a σ-algebra of sets, $\mathcal{Q}$, in a space Ω, a σ-additive spectral measure $P : \mathcal{Q} \to \mathrm{BL}(E)$ and a function $f \in \mathcal{L}(P)$ such that $T = P(f)$. We may also call such operators σ-scalar. By Corollary 6.6, operators which are scalar in the sense of Dunford are scalar in the wider sense. Moreover, these operators can be characterized in terms introduced here.

By a Boolean σ-algebra of projection operators is understood a Boolean algebra of projection operators which contains the strong limit of every monotonic sequence of its elements.

PROPOSITION 6.9. *An operator $T \in \mathrm{BL}(E)$ is scalar in the sense of Dunford if and only if there exists a Boolean σ-algebra of projection operators, $W \subset \mathrm{BL}(E)$, such that $T \in A(W)$ and every element of W commutes with every operator from $\mathrm{BL}(E)$ which commutes with T.*

Proof. If the operator $T \in \mathrm{BL}(E)$ is scalar in the sense of Dunford, then there exist a σ-algebra of sets, $\mathcal{Q}$, in a space Ω, a σ-additive spectral measure $P : \mathcal{Q} \to \mathrm{BL}(E)$ and a function $f \in \mathcal{L}(P)$ such that $T = P(f)$. Let $\mathcal{Q}_f$ be the minimal σ-algebra of sets such that $\mathcal{Q}_f \subset \mathcal{Q}$ and, if P_f is the restriction of P to $\mathcal{Q}_f$, then $f \in \mathcal{L}(P_f)$. The range, $\mathrm{W} = \{P(X) : X \in \mathcal{Q}_f\}$, of the spectral measure P_f is then a Boolean σ-algebra of projections such that $T \in A(W)$ and every element of W commutes with every operator commuting with T.

Conversely, let $W \subset \mathrm{BL}(E)$ be a Boolean σ-algebra of projections such that $T \in A(W)$. By the Stone representation theorem there exist a compact space Ω, an algebra $\mathcal{R}$ consisting of its compact and open subsets and a spectral set function $P : \mathcal{R} \to \mathrm{BL}(E)$ such that $W = \{P(X) : X \in \mathcal{R}\}$. Let $\mathcal{Q}$ be the σ-algebra of sets generated by $\mathcal{R}$. Because P is in fact σ-additive and W is a σ-algebra of operators, the set function P has a strongly σ-additive extension onto $\mathcal{Q}$, still denoted by P, whose range remains equal to W; see, for example, [30]. Then $P : \mathcal{Q} \to \mathrm{BL}(E)$ is a spectral measure such that, by Proposition 4.3, $T = P(f)$, for some function $f \in \mathcal{L}(P)$.

Operators which are scalar in the wider sense but not scalar in the sense of Dunford abound. A way of producing a wealth of such operators is indicated by the following

EXAMPLE 6.10. Let $\Omega = (0,1]$, $\mathcal{Q} = \{(s,t] : 0 \le s \le t \le 1\}$. Let $p > 1$ and $\rho(X) = (\iota(X))^{1/p}$, for every $X \in \mathcal{Q}$, where ι is the one-dimensional Lebesgue measure. By Proposition 2.13 and Proposition 2.26, ρ is an integrating gauge on $\mathcal{Q}$. Let $E = L(\rho,\mathcal{Q})$.

For every $X \in \mathcal{Q}$, let $P(X)$ be the operator of point-wise multiplication by the characteristic function of the set X. That is, $P(X)[u]_\rho = [Xu]_\rho$, for every $u \in \mathcal{L}(\rho,\mathcal{Q})$. Because $\mathcal{L}(\rho,\mathcal{Q}) \neq \mathcal{L}^p(\iota)$ (see Example 4.16(ii) in Section 4C) the so-defined spectral set function $P : \mathcal{Q} \to \mathrm{BL}(E)$ is surely not σ-additive; indeed, its additive extension on the algebra of sets generated by $\mathcal{Q}$ is not bounded. Nevertheless, P is closable. Moreover, if $n \ge 1$ is an integer and a set X is equal to the union of n pair-wise disjoint sets, X_k, $k = 1,2,\dots,n$, belonging to $\mathcal{Q}$, then $\|P(X)\| \le n^{(p-1)/p}$. In fact, let u be a function belonging to $\mathcal{L}(\rho,\mathcal{Q})$. Let c_j be numbers and $Y_j \in \mathcal{Q}$ sets, $j = 1,2,\dots,$ such that

$$\sum_{j=1}^{\infty} |c_j| \rho(Y_j) < \infty$$

and

$$u(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j|\, Y_j(\omega) < \infty .$$

Then

$$q_\rho(Y_j \cap X) \le \sum_{k=1}^{n} \rho(Y_j \cap X_k) \le n^{(p-1)/p} \rho(Y_j) ,$$

for every $j = 1,2,\ldots,$ and

$$(Xu)(\omega) = \sum_{j=1}^{\infty} c_j Y_j(\omega) X(\omega)$$

for every $\omega \in \Omega$ for which

$$\sum_{j=1}^{\infty} |c_j| (Y_j \cap X)(\omega) < \infty .$$

Therefore, $Xu \in \mathcal{L}(\rho,Q)$ and $q_\rho(Xu) \le n^{(p-1)/p} q_\rho(u)$.

Now, let Z be the function on $\mathbb{R}$ which is periodic with period 1 and its restriction to Ω is equal to the characteristic function of the interval $(\frac{1}{2},1]$. For every $j = 1,2,\ldots,$ let X_j be the function $\omega \mapsto Z(2^{j-1}\omega)$, $\omega \in \Omega$. Hence, $X_j \in \mathrm{sim}(Q)$ and $\|P(X_j)\| \le 2^{(j-1)(p-1)/p}$, for every $j = 1,2,\ldots$. Also, if $f(\omega) = \omega$, then

$$f(\omega) = \sum_{j=1}^{\infty} 2^{-j} X_j(\omega) ,$$

for every $\omega \in \Omega$. Therefore, $f \in \mathcal{L}(P)$ and

$$\|P(f)\| \le \sum_{j=1}^{\infty} 2^{-j} 2^{(j-1)(p-1)/p} = \frac{1}{2}\,\frac{2^{1/p}}{2^{1/p} - p} .$$

F. This and the next sections are devoted to an example, or, rather, a class of examples, which is sufficiently rich to display all the features of the presented theory.

Let G be a locally compact Abelian group and Γ its dual group. The value of a character $\xi \in \Gamma$ on an element $x \in G$ is denoted by $\langle x,\xi \rangle$.

Let $1 < p < \infty$ and let $E = L^p(G)$, with respect to a fixed Haar measure on the group G .

Let $\mathcal{M}^p(\Gamma)$ be the family of all individual functions on Γ which determine multiplier operators on E . That is, $f \in \mathcal{M}^p(\Gamma)$ if and only if there exists an operator $T_f \in \mathrm{BL}(E)$ such that $(T_f\varphi)^\wedge = f\hat{\varphi}$, for every $\varphi \in L^2 \cap L^p(G)$. Here, of course, $\hat{\varphi}$ denotes the Fourier-Plancherel transform of an element φ of $L^2(G)$.

Functions belonging to $\mathcal{M}^p(\Gamma)$ are essentially bounded. In fact, $\|f\|_\infty \le \|T_f\|$, for every $f \in \mathcal{M}^p(\Gamma)$, where $\|f\|_\infty$ is the essential supremum norm of f with respect to the Haar measure. The operator T_f depends only on the equivalence class of the function f. That is, if $f \in \mathcal{M}^p(\Gamma)$ and if g is a function on Γ such that $g(\xi) = f(\xi)$ for almost every $\xi \in \Gamma$, relative to the Haar measure, then $g \in \mathcal{M}^p(\Gamma)$ and $T_g = T_f$.

It is well-known that an operator $T \in \mathrm{BL}(E)$ commutes with all translations of G if and only if there exists a function $f \in \mathcal{M}^p(\Gamma)$ such that $T = T_f$. So, $\{T_f : f \in \mathcal{M}^p(\Gamma)\}$ is a commutative algebra of operators, containing the identity operator, which is closed in $\mathrm{BL}(E)$. Clearly, $\mathcal{M}^p(\Gamma)$ is an algebra of functions and the map $f \mapsto T_f$, $f \in \mathcal{M}^p(\Gamma)$, is multiplicative and linear.

Let $\mathcal{R}^p(\Gamma)$ be the family of all sets $X \subset \Gamma$ such that $X \in \mathcal{M}^p(\Gamma)$. Let $P^p_\Gamma(X) = T_X$, for every $X \in \mathcal{R}^p(\Gamma)$.

PROPOSITION 6.11. *The family* $\mathcal{R}^p(\Gamma)$ *is an algebra of sets in* Γ *and* $P^p_\Gamma : \mathcal{R}^p(\Gamma) \to B(L^p(G))$ *is a closable spectral set function.*

Proof. It follows from the mentioned properties of the map $f \mapsto T_f$, $f \in \mathcal{M}^p(\Gamma)$, that $\mathcal{R}^p(\Gamma)$ is an algebra of sets and the set function $P = P^p_\Gamma$ is spectral. Furthermore, a set $Y \subset \Gamma$ is P-null if and only if it is null with respect to the Haar measure on Γ. Consequently the Haar measure equivalence classes of functions on Γ are the same as the P-equivalence classes and so are their ∞-norms. Therefore, $L^\infty(P)$ is a Banach subspace of $L^\infty(\Gamma)$. Now, $A(P)$ is a closed subalgebra of the Banach algebra $\{T_f : f \in \mathcal{M}^p(\Gamma)\}$. For every $T \in A(P)$, let $\Phi(T) = [f]$, where $f \in \mathcal{M}^p(\Gamma)$ is a function such that $T = T_f$. Then Φ is an unambiguously defined norm-decreasing map from $A(P)$ into $L^\infty(P)$ such that $\Phi(P(f)) = [f]$, for every $f \in \mathrm{sim}(\mathcal{R}^p(\Gamma))$. Therefore, by Proposition 6.4, the set function P is closable.

The usefulness of this proposition depends of course on how rich is the algebra of sets $\mathcal{R}^p(\Gamma)$. A result of T.A. Gillespie implies that it is rich enough to permit complete spectral analysis of translation operators. Let us introduce the necessary relevant notation.

Let $\mathbb{T}$ be the circle group, $\{z \in \mathbb{C} : |z| = 1\}$, with its usual topology of a subset of the complex plane. Connected subsets of $\mathbb{T}$ will be called arcs. For an element x of the group G and an arc $Z \subset \mathbb{T}$, let

$$X_{Z,x} = \{\xi \in \Gamma : \langle x,\xi\rangle \in Z\} .$$

Let $\mathcal{K}_1(\Gamma)$ be the family of all sets $X_{Z,x}$ corresponding to arcs $Z \subset \mathbb{T}$ and elements of $x \in G$. The classes of sets $\mathcal{K}_n(\Gamma)$, $n = 2,3,\ldots,$ are then defined recursively by requiring that $\mathcal{K}_n(\Gamma)$ consist of all sets $X \cap Y$ such that $X \in \mathcal{K}_{n-1}(\Gamma)$ and $Y \in \mathcal{K}_1(\Gamma)$.

LEMMA 6.12. *The inclusion* $\mathcal{K}_n(\Gamma) \subset \mathcal{R}^p(\Gamma)$ *is valid for every* $p \in (1,\infty)$ *and every* $n = 1,2,\ldots$. *Moreover, for every* $p \in (1,\infty)$, *there exists a constant* $C_p \geq 1$ *such that* $\|P^p_\Gamma(X)\| \leq C^n_p$, *for every* $X \in \mathcal{K}_n(\Gamma)$, *every* $n = 1,2,\ldots,$ *and every locally compact Abelian group* Γ.

Proof. For $n = 1$, this is a simple re-formulation of Lemma 6 of [18]. (See also Lemma 20.15 in [12].) By induction, the result follows for every $n = 2,3,\ldots$.

Let $\mathcal{J}_1$ be the family of all subsets of $\mathbb{R}$ which contains all members of $\mathcal{K}_1(\mathbb{R})$ and all intervals in $\mathbb{R}$ and no other sets. The families $\mathcal{J}_n$, $n = 2,3,\ldots,$ are then defined recursively by requiring that $\mathcal{J}_n$ consist of all sets $X \cap Y$ such that $X \in \mathcal{J}_{n-1}$ and $Y \in \mathcal{J}_1$.

If we combine Lemma 6.12 with a classical theorem of M. Riesz (interpreted to the effect that intervals belong to $\mathcal{M}^p(\mathbb{R})$ and determine a bounded family of multiplier operators; see e.g. [8], Theorem 6.3.3) we obtain the following

COROLLARY 6.13. *The inclusion* $\mathcal{J}_n \subset \mathcal{R}^p(\mathbb{R})$ *is valid for every* $p \in (1,\infty)$ *and every* $n = 1,2,\ldots$. *Moreover, for every* $p \in (1,\infty)$, *there exists a constant* $D_p \geq 1$ *such that* $\|P^p_{\mathbb{R}}(X)\| \leq D^n_p$, *for every* $X \in \mathcal{J}_n$ *and* $n = 1,2,\ldots$.

G. The (total) variation of a function f of bounded variation on $\mathbb{R}$ or on $\mathbb{T}$ will be denoted by $\mathrm{var}(f)$. Recall that every function, f, of bounded variation has a decomposition, $f = f_1 + f_2 + f_3$, such that the function f_1 is absolutely continuous, f_2 is continuous and singular (its derivative vanishes almost everywhere) and f_3 is a jump-function. If the function f vanishes at a point (or at $-\infty$) then there is only one such decomposition with all the three components, f_1, f_2 and f_3, vanishing at that point. If the continuous singular component, f_2, is identically equal to zero, then the function f is called non-singular.

LEMMA 6.14. *Let α, β and b be real numbers such that $\alpha \le \beta$. Let u be the function on $\mathbb{R}$ such that $u(t) = 0$ for $t < \alpha$, $u(t) = b(t-\alpha)$ for $\alpha \le t \le \beta$, and $u(t) = b(\beta-\alpha)$ for $t \ge \beta$. Then there exist numbers c_j and sets $X_j \in \mathcal{J}_2$, $j = 0,1,2,\ldots$, such that*

$$\sum_{j=0}^{\infty} |c_j| \, \|P_{\mathbb{R}}^p(X_j)\| \le 2D_p^2 \mathrm{var}(u)$$

and

$$\sum_{j=0}^{\infty} c_j X_j(t) = u(t)$$

for every $t \in \mathbb{R}$.

Proof. Because $\mathrm{var}(u) = |b|(\beta-\alpha)$, by Corollary 6.13, the statement holds with $c_j = 2^{-j} b(\beta-\alpha)$, $j = 0,1,2,\ldots$, $X_0 = [\beta,\infty)$ and

$$X_j = \left\{ t \in \mathbb{R} : \exp\left[\frac{2^j \pi(t-\alpha)}{\beta - \alpha} \mathrm{i}\right] \in \{\exp s\mathrm{i} : \pi \le s < 2\pi\} \right\} \cap [\alpha,\beta) ,$$

$j = 1,2,\ldots$.

PROPOSITION 6.15. *Let f be a real non-singular function of bounded variation on $\mathbb{R}$ such that $f(-\infty) = 0$. Then $f \in \mathcal{L}(P_{\mathbb{R}}^p)$ and*

$$P_{\mathbb{R}}^p(f) \le 3D_p^2 \mathrm{var}(f) , \tag{G.1}$$

for every $p \in (1,\infty)$.

Proof. Let $f = f_1 + f_3$ for a function g, integrable on $\mathbb{R}$, such that

$$f_1(t) = \int_{-\infty}^{t} g(s)\mathrm{d}s ,$$

$t \in \mathbb{R}$, and a jump-function f_3 vanishing at $-\infty$. Then $\operatorname{var}(f) = \operatorname{var}(f_1) + \operatorname{var}(f_3)$. Moreover, there exist numbers c_j and intervals X_j, $j = 1,2,3,\ldots$, such that

$$f_3(t) = \sum_{j=1}^{\infty} c_j X_j(t) ,$$

for every $t \in \mathbb{R}$, and

$$\operatorname{var}(f_3) = \sum_{j=1}^{\infty} |c_j| .$$

There also exist numbers b_j and bounded intervals Y_j, $j = 1,2,\ldots$, such that, if

$$u_j(t) = \int_{-\infty}^{t} b_j Y_j(s) ds ,$$

for every $t \in \mathbb{R}$ and $j = 1,2,\ldots$, then

$$\sum_{j=1}^{\infty} \operatorname{var}(u_j) = \sum_{j=1}^{\infty} |u_j(\infty)| < \frac{3}{2} \int_{-\infty}^{\infty} |g(s)| ds = \frac{3}{2} \operatorname{var}(f_1)$$

and

$$f_1(t) = \sum_{j=1}^{\infty} u_j(t)$$

for every $t \in \mathbb{R}$. Hence, by Lemma 6.14, and Proposition 2.1, $f \in \mathcal{L}(P_{\mathbb{R}}^p)$ and the inequality (G.1) holds.

This proposition points at the richness of the space $\mathcal{L}(P_{\mathbb{R}}^p)$. To be sure, this space also contains functions of bounded variation which do not vanish at $-\infty$ and many functions of unbounded variation. In fact, it also contains many functions of unbounded r-variation, for any $r > 1$, because already the characteristic functions of many sets from $\mathcal{J}_2$ are such. (In this context, see [24].) As $\mathcal{L}(P_{\mathbb{R}}^p) \subset \mathcal{M}^p(\mathbb{R})$, we have a large class of multiplier operators which are scalar in the wider sense.

LEMMA 6.16. *Let r, α, β and b be real numbers such that $r \leq \alpha < \beta \leq r + 2\pi$. Let u be the function on $\mathbb{T}$ such that $u(\exp ti) = 0$ for $r \leq t < \alpha$, $u(\exp ti) = b(t-\alpha)$ for $\alpha \leq t < \beta$, and $u(\exp ti) = b(\beta-\alpha)$ for $\beta \leq t < r + 2\pi$. Then there exist numbers c_j and sets $X_j \in \mathcal{K}_2(\mathbb{T})$, $j = 0,1,2,\ldots$, such that*

$$\sum_{j=0}^{\infty} |c_j|\ \|P_{\mathbb{T}}^p(X_j)\| \le C_p^2 \mathrm{var}(u)$$

and

$$\sum_{j=1}^{\infty} c_j X_j(z) = u(z)$$

for every $z \in \mathbb{T}$.

Proof. Let m be the largest integer such that $m(\beta-\alpha) \le 2\pi$. Let $\gamma = \alpha + 2\pi m^{-1}$. Note that $\mathrm{var}(u) = 2|b|(\beta-\alpha)$, $r \le \alpha < \beta \le \gamma \le r+2\pi$ and $m(\gamma-\alpha) = 2\pi$. Hence, by Lemma 6.12, it suffices to take $c_0 = b(\beta-\alpha)$, $X_0 = \{\exp ti : \beta \le t < r+2\pi\}$, $c_j = 2^{1-j}\pi b m^{-1}$ and

$$X_j = \{\exp(\gamma-t)\mathrm{i} : \exp(2^{j-1}mt\mathrm{i}) \in \{\exp s\mathrm{i} : 0 < s \le \pi\}\} \cap \{\exp t\mathrm{i} : \alpha \le t < \beta\}$$

for $j = 1,2,\dots$.

PROPOSITION 6.17. *Let* $r \in \mathbb{R}$ *and let* f *be a real non-singular function of bounded variation on* $\mathbb{T}$ *such that* $f(\exp r\mathrm{i}) = 0$. *Then* $f \in \mathcal{L}(P_{\mathbb{T}}^p)$ *and*

$$P_{\mathbb{T}}^p(f) \le 2C_p^2 \mathrm{var}(f)$$

for every $p \in (1,\infty)$.

Proof. It is analogous to that of Proposition 6.15 only Lemma 6.16 is used instead of Lemma 6.14.

COROLLARY 6.18. *Let* $x \in G$, *let* u *be a non-singular function of bounded variation on* $\mathbb{T}$ *and let* $f(\xi) = u(\langle x,\xi\rangle)$, *for every* $\xi \in \Gamma$. *Then* $f \in \mathcal{L}(P_{\Gamma}^p)$ *for every* $p \in (1,\infty)$.

Proof. A power of a character of a group is a character and all characters of $\mathbb{T}$ are powers of a single one, namely the identity function on $\mathbb{T}$. Interpreting G as the group of characters of Γ we see immediately that, for every $Y \in \mathcal{K}_n(\mathbb{T})$, the set $X = \{\xi \in \Gamma : \langle x,\xi\rangle \in Y\}$ belongs to $\mathcal{K}_n(\Gamma)$, $n = 1,2,\dots$. So, Lemma 6.12 and Proposition 6.17 imply the result.

Now, each element, x, of the group G is interpreted as a function on Γ - the character it generates - that is, the function $\xi \mapsto \langle x,\xi\rangle$, $\xi \in \Gamma$. Then $x \in \mathcal{M}^p(\Gamma)$ and T_x is the operator of translation by x. By Corollary 4.17, $x \in \mathcal{L}(P_\Gamma^p)$ and

(G.2) $$T_x = \int_\Gamma \langle x,\xi\rangle \, P_\Gamma^p(\mathrm{d}\xi) ,$$

for every $x \in G$. For $p = 2$, this is an instance of Stone's theorem (see e.g. [46], 36E).

Some observations about the Stone formula (G.2) could be of interest because they could possibly have somewhat wider implications. Its proof shows that, $x \in \mathcal{L}(\rho_P, \mathcal{K}_2(\Gamma))$, for every $x \in G$, where $P = P_\Gamma^p$ with any $p \in (1,\infty)$. That is to say, for every $x \in G$, there exist numbers c_j and sets $X_j \in \mathcal{K}_2(\Gamma)$, $j = 1,2,\ldots$, which depend of course on x but not on p, such that

$$\sum_{j=1}^{\infty} |c_j| \, \|P_\Gamma^p(X_j)\| < \infty ,$$

the equality

$$\langle x,\xi\rangle = \sum_{j=1}^{\infty} c_j X_j(\xi)$$

holds for every $\xi \in \Gamma$ and

$$T_x = \sum_{j=1}^{\infty} c_j \, P_\Gamma^p(X_j) ,$$

for every $p \in (1,\infty)$. Hence for each $p \in (1,\infty)$, the translation operator, T_x, is expressed as the sum of the same multiples of the projections $P_\Gamma^p(X_j)$, $j = 1,2,\ldots$. These projections too are 'the same' for each p, only the space, $E = L^p(G)$, in which they operate varies with p.

Also the fact that the sets X_j, $j = 1,2,\ldots$, belong to the class $\mathcal{K}_2(\Gamma)$ may possibly be worth noting. The algebra $\mathcal{R}^p(\Gamma)$ contains of course also sets of much greater complexity than those belonging to $\mathcal{K}_2(\Gamma)$. It seems that it would contribute considerably to our understanding of multiplier operators to know what kind of sets, besides those belonging to the classes $\mathcal{K}_n(\Gamma)$, $n = 1,2,\ldots$, are in the algebra $\mathcal{R}^p(\Gamma)$. The classes $\mathcal{J}_n$, $n = 1,2,\ldots$, give us some indication in the case $\Gamma = \mathbb{R}$.

7. SUPERPOSITION OF EVOLUTIONS

The main point of this chapter is to present a vector, or operator, version of the Feynman-Kac formula representing certain perturbations of a given evolution. While for some evolutions, such as the diffusion semigroup, the formula can be stated in terms of classical absolutely convergent integrals, for others, notably the Schrödinger group, the usage of a more general conceptual machinery is inevitable. Needless to say, the notions introduced in earlier chapters will be used here.

A. Let E be a Banach space. The algebra of all bounded linear operators on E is denoted by $\mathrm{BL}(E)$

The basic ingredient of the abstract Feynman-Kac formula, to be stated in the next section, is the $\mathrm{BL}(E)$-valued additive set function determined by an evolution in the space E and a $\mathrm{BL}(E)$-valued spectral measure. In this section, the conventions pertaining to these notions are introduced.

Let Λ be a locally compact Hausdorff space. Although other spaces may be, and indeed are, of considerable interest, in the examples considered in this chapter, Λ will be equal to $\mathbb{R}^d$, for some small or unspecified positive integer d. Let $\mathcal{B} = \mathcal{B}(\Lambda)$ be the σ-algebra of Baire sets in Λ. The $\mathcal{B}$-measurable functions on Λ will be called the Baire functions. (See Section 1D.)

Let $P : \mathcal{B} \to \mathrm{BL}(E)$ be a σ-additive spectral measure. (See Section 6A.) By Corollary 6.6, the spectral measure P is closable. (See Section 6C.) If $\varphi \in E$, by $P\varphi$ is denoted the E-valued set function on $\mathcal{B}$ such that $(P\varphi((B) = P(B)\varphi$, for every $B \in \mathcal{B}$. By the assumption, $P\varphi$ is σ-additive, for every $\varphi \in E$. The integrability with respect to $P\varphi$ is understood in the sense of Proposition 3.13. That is, a function on Λ is called $(P\varphi)$-integrable if it satisfies, *mutatis mutandis*, any of the equivalent conditions (i), (ii) or (iii) of Proposition 3.13.

Given a Baire function, W, on Λ, by

$$P(W) = \int_\Lambda W \mathrm{d}P = \int_\Lambda W(x) P(\mathrm{d}x)$$

will be denoted the operator whose domain consists of the elements, φ, of the space E such that the function W is $(P\varphi)$-integrable and whose value, $P(W)\varphi$, at any such element is given by the formula

$$P(W)\varphi = \int_{\Lambda} W(x)P(\mathrm{d}x)\varphi .$$

The operator $P(W)$ is bounded if and only if the function W is P-essentially bounded, that is, there exists a Baire set, B_0, such that $P(B_0) = 0$ and W is bounded on the complement of B_0. (See Section 6B.) So, $P(W) \in \mathrm{BL}(E)$ if and only if $W \in \mathcal{L}(P)$. (See Section 6C.)

For any real numbers t' and t'' such that $0 \leq t' \leq t''$, let $S(t'',t') \in \mathrm{BL}(E)$ be an operator such that

(i) $S(t,t) = I$, the identity opereator, for every $t \geq 0$;

(ii) $S(t''',t') = S(t''',t'')S(t'',t')$, for any t', t'' and t''' such that $0 \leq t' \leq t'' \leq t'''$; and

(iii) the map $S : \{(t'',t') : 0 \leq t' \leq t''\} \to \mathrm{BL}(E)$ is continuous in the strong operator topology of $\mathrm{BL}(E)$.

Such a map $S : \{(t'',t') : 0 \leq t' \leq t''\} \to \mathrm{BL}(E)$, with properties (i), (ii) and (iii), is called an evolution, or a propagator, in the space E. If $S(t'',t') = S(t''-t',0)$, for any $0 \leq t' \leq t''$, then we speak of a continuous semigroup, or a dynamical propagator, and write without ambiguity $S(t) = S(t,0)$, for every $t \geq 0$. Needless to say, the numbers $t, t', t'',\ldots$ entering into arguments of an evolution are intuitively interpreted as instants of time.

Let $t \geq 0$. For every $s \in [0,t]$, let Υ_s be a set of maps $v : [0,s] \to \Lambda$ to be called paths. We assume that $\{v(s) : v \in \Upsilon_t\} = \Lambda$, for every $s \in [0,t]$. To formulate another assumption, for any $s \in [0,t]$, let $\mathrm{pr}_{t,s}$ be the natural projection of Υ_t onto Υ_s. That is, the value, $\mathrm{pr}_{t,s}(v)$, of the map $\mathrm{pr}_{t,s}$ at an element, v, of Υ_t is equal to the restriction, $v|_{[0,s]}$, of v to the interval $[0,s]$. We shall assume that $\{\mathrm{pr}_{t,s}(v) : v \in \Upsilon_t\} = \Upsilon_s$, for every $s \in [0,t]$.

Of main interest are the cases in which $\Upsilon_t = \Lambda^{[0,t]}$, or Υ_t consists of all continuous paths $v : [0,t] \to \Lambda$, or ones which are right-continuous at each point of the interval $[0,t)$ and have a left limit at each point of the interval $(0,t]$, etc.

Let $t \geq 0$. Given an integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\ldots,k$, let

$$\text{(A.1)} \qquad Y = \{v \in \Upsilon_t : v(t_j) \in B_j,\ j = 1,2,\ldots,k\} .$$

Whenever it is necessary to indicate the parameters on which the set Y depends we write $Y = Y(t_1,\ldots,t_k; B_1,\ldots,B_k)$.

The family of sets (A.1) formed for all choices of $k = 1,2,\ldots,$ sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\ldots,k$, is denoted by $\mathcal{R}_t$. It is classical and comparatively easy to show, that $\mathcal{R}_t$ is a semialgebra of sets in the space Υ_t. (See Section 1D.)

Now we define the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ determined by the evolution S and the spectral measure P. Namely, if $k \geq 1$ is an integer, $B_j \in \mathcal{B}$ sets and t_j numbers, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for $j = 2,3,\ldots,k$, and the set Y is given by (A.1), let

$$\text{(A.2)} \qquad M_t(Y) = S(t,t_k)P(B_k)S(t_k,t_{k-1})P(B_{k-1}) \ldots P(B_2)S(t_2,t_1)P(B_1)S(t_1,0) .$$

PROPOSITION 7.1. *For every set $Y \in \mathcal{R}_t$, the operator $M_t(Y)$ is defined by* (A.2) *unambiguously. The resulting set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ is additive.*

Proof. Let $Y \in \mathcal{R}_t$. If Y is given by (A.1), then $Y = \emptyset$ if and only if $B_j = \emptyset$ for some $j = 1,2,\ldots,k$. So, let $Y \neq \emptyset$. If $Y = Y(t_1,\ldots,t_k; B_1,\ldots,B_k)$, for some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and pair-wise different numbers t_j, $j = 1,2,\ldots,k$, and also $Y = Y(s_1,\ldots,s_\ell; C_1,\ldots,C_\ell)$, for some integer $\ell \geq 1$, sets $C_m \in \mathcal{B}$ and pair-wise different numbers s_m, $m = 1,2,\ldots,\ell$, then $C_m = B_j$ whenever $s_m = t_j$, $B_j = \Lambda$ for every $j = 1,2,\ldots,k$ such that $t_j \neq s_m$ for every $m = 1,2,\ldots,\ell$, and $C_m = \Lambda$ for every $m = 1,2,\ldots,\ell$ such that $s_m \neq t_j$ for every $j = 1,2,\ldots,k$. Therefore, property (ii) of an evolution and the equality $P(\Lambda) = I$ imply that the operator $M_t(Y)$ is defined unambiguously by (A.2).

To prove the additivity of the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$, by Proposition 1.8, it suffices to prove that this set function is 2-additive. However, the 2-additivity follows immediately from the following general set theoretical fact: If $X \in \mathcal{R}_t$, $Y \in \mathcal{R}_t$ and $Z \in \mathcal{R}_t$ are sets such that $Y \cap Z = \emptyset$ and $X = Y \cup Z$, then there exist an integer $k \geq 1$, sets A_j, B_j and C_j, belonging to $\mathcal{B}$, pair-wise different numbers t_j, $j = 1,2,\ldots,k$, and an integer $m \in [1,k]$ such that

$$X = \{v \in \Upsilon_t : v(t_j) \in A_j ,\ j = 1,\ldots,k\} ,\quad Y = \{v \in \Upsilon_t : v(t_j) \in B_j ,\ j = 1,\ldots,k\} ,$$

$$Z = \{v \in \Upsilon_t : v(t_j) \in C_j ,\ j = 1,\ldots,k\} ,$$

$A_j = B_j = C_j$ for every $j \neq m$, $j = 1,2,\ldots,k$, $B_m \cap C_m = \emptyset$ and $A_m = B_m \cup C_m$.

It should be noted that the set function $M_t\varphi$, for some given $\varphi \in E$, is usually of more direct interest than M_t itself. To be sure, $M_t\varphi$ is the E-valued function on $\mathcal{R}_t$ whose value at any set $Y \in \mathcal{R}_t$ is equal to $M_t(Y)\varphi$.

Let ρ be a gauge on some quasialgebra $\mathcal{Q} \subset \mathcal{R}_t$ integrating for the restriction of $M_t\varphi$ to $\mathcal{Q}$. Let $f \in \mathcal{L}(\rho,\mathcal{Q})$. By

$$\int_{\Upsilon_t} f(v) M_t(\mathrm{d}_\rho v)\varphi = \int_{\Upsilon_t} f \mathrm{d}_\rho (M_t\varphi) = (M_t)_\rho(f)\varphi$$

will be denoted the 'integral of the function f with respect to $M_t\varphi$,' that is, the value, $\ell(f)$, of the continuous linear functional, ℓ, on $\mathcal{L}(\rho,\mathcal{Q})$ such that $\ell(X) = M_t(X)\varphi$, for every $X \in \mathcal{Q}$. (See Section 3A.) We should note though that, usually, ρ does not integrate for (the corresponding restriction of) M_t, so that the symbol '$(M_t)_\rho(f)$' is meaningless as are other symbols for the 'integral of f with respect to M_t.'

EXAMPLE 7.2. Let $\varphi \in E$. Let $\mathcal{Q} \subset \mathcal{R}_t$ be a quasialgebra. Let ρ be a gauge on $\mathcal{Q}$ integrating for the set function $M_t\varphi$ restricted to $\mathcal{Q}$. Let $0 \leq t_1 < t_2 < \ldots < t_{n-1} < t_n \leq t$ and $W_1, W_2, \ldots, W_n$ be Baire functions on Λ such that the function f, defined by

$$f(v) = \prod_{j=1}^{n} W_j(v(t_j))$$

for every $v \in \Upsilon_t$, is ρ-integrable. Then

$$\int_{\Upsilon_t} f \mathrm{d}_\rho(M_t\varphi) = S(t,t_n)P(W_n)S(t_n,t_{n-1})P(W_{n-1}) \dots P(W_2)S(t_2,t_1)P(W_1)S(t_1,0)\varphi .$$

EXAMPLE 7.3. Let $\varphi \in E$. Let $\mathcal{Q} \subset \mathcal{R}_t$ be a quasialgebra. Let ρ be a gauge on $\mathcal{Q}$ integrating for the set function $M_t\varphi : \mathcal{Q} \to E$. Let $0 \le s \le t$. Let $\mathcal{Q}_s = \{Z \subset \Upsilon_s : \mathrm{pr}_{t,s}^{-1}(Z) \in \mathcal{Q}\}$ and $\rho_s(Z) = \rho(\mathrm{pr}_{t,s}^{-1}(Z))$, for every $Z \in \mathcal{Q}_s$. Assume that the gauge ρ_s integrates for the set function on $M_s\varphi : \mathcal{Q}_s \to E$. Let g be a ρ_s-integrable function on Υ_s, W a Baire function on Λ and

$$f(v) = W(v(s))g(\mathrm{pr}_{t,s}(v))$$

for every $v \in \Upsilon_t$. If the function f is ρ-integrable, then

$$\int_{\Upsilon_t} f \mathrm{d}_\rho(M_t\varphi) = S(t_1 s)P(W) \int_{\Upsilon_s} g \mathrm{d}_{\rho_s}(M_s\varphi) .$$

B. We maintain the notation of Section A.

Assume that an evolution, S, in the space E and a spectral measure, P, on $\mathcal{B} = \mathcal{B}(\Lambda)$ are given. Let φ be an element of the space E.

Let $t \ge 0$. Let ι be the Lebesgue measure on the interval $[0,t]$ and $\mathcal{L}(\iota)$ the family of all (individual) Lebesgue integrable functions on $[0,t]$. We write, of course,

$$\iota(f) = \int_0^t f(r)\mathrm{d}r ,$$

for every $f \in \mathcal{L}(\iota)$.

Let $\mathcal{Q}$ be a semialgebra of sets in the space Υ_t such that $\mathcal{Q} \subset \mathcal{R}_t$ and let ρ be a gauge integrating for the restriction of the set function $M_t\varphi$ to $\mathcal{Q}$. For every $s \in [0,t]$, let $\mathcal{Q}_s = \{Z \subset \Upsilon_s : \mathrm{pr}_{t,s}^{-1}(Z) \in \mathcal{Q}\}$ and let $\rho_s(Z) = \rho(\mathrm{pr}_{t,s}^{-1}(Z))$, for every $Z \in \mathcal{Q}_s$.

Let W be a function on $[0,t] \times \Lambda$ such that the function $r \mapsto W(r,v(r))$, $r \in [0,t]$, is ι-integrable for ρ-almost every $v \in \Upsilon_t$. For every $s \in [0,t]$, let e_s be a function on Υ_s, to be called the Feynman-Kac functional, such that

(B.1) $$e_s(v) = \exp\left[\int_0^s W(r,v(r))\mathrm{d}r \right],$$

for every $v \in \Upsilon_s$ for which the integral at the right exists.

If e_s happens to be ρ_s-integrable, let

(B.2) $$u(s) = \int_{\Upsilon_s} e_s(v)M_s(\mathrm{d}_{\rho_s} v)\varphi .$$

In particular,

$$u(t) = \int_{\Upsilon_t} e_t(v)M_t(\mathrm{d}_{\rho} v)\varphi .$$

In order to present concisely an intuitive interpretation of $u(t)$, let us extend the definition of W onto the whole of $[0,\infty) \times \Lambda$ by letting $W(s,x) = W(t,x)$, for every $s \geq t$ and every $x \in \Lambda$. Assume that, for every t' and t'' such that $0 \leq t' \leq t''$,

(B.3) $$T(t'',t') = P\left[\exp \int_{t'}^{t''} W(s,\cdot)ds\right]$$

is a well-defined operator belonging to $\mathrm{BL}(E)$ and the resulting map $T: \{(t'',t'): 0 \leq t' \leq t''\} \to \mathrm{BL}(E)$ is an evolution in the space E.

Then $u(t)$ can be thought of as the element of the space E into which φ evolves under simultaneous action of S and T during the time-interval $[0,t]$. In fact, if the numbers $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = t$ represent a partition, $\not p$, of the interval $[0,t]$, let us denote

$$u_{\not p}(t) = T(t_n,t_{n-1})S(t_n,t_{n-1})\,T(t_{n-1},t_{n-2})S(t_{n-1},t_{n-2}) \dots T(t_2,t_1)S(t_2,t_1)\,T(t_1,t_0)S(t_1,t_0)\varphi .$$

Furthermore, let

$$W_j = \exp\left[\int_{t_{j-1}}^{t_j} W(r,\cdot)\mathrm{d}r\right] ,$$

so that $T(t_{j-1},t_j) = P(W_j)$, for every $j = 1,2,\ldots,n$, and

$$h_{\not p}(v) = \prod_{j=1}^{n} W_j(v(t_j)) ,$$

for every $v \in \Gamma_t$. Then, by Example 7.2,

$$u_{\not p}(t) = \int_{\Gamma_t} h_{\not p}\mathrm{d}_\rho(\mathrm{M}_t\varphi) .$$

Now, if the partition $\not p$ is sufficiently fine, then we may expect that $u_{\not p}(t)$ will be approximately the outcome of the simultaneous action of the evolutions S and T on φ during the time-interval $[0,t]$. On the other hand, we may also expect that the integral of $h_{\not p}$ 'with respect to $M_t\varphi$' will approximate the integral of e_t .

Turning these heuristics into a solid argument would of course require an appeal to a Trotter-Kato type theorem. However, we shall proceed differently. Namely, assuming that the function e_s is ρ_s-integrable, for every $s \in [0,t]$, we are going to present a sufficient condition for the function $s \mapsto u(s)$, $s \in [0,t]$, to satisfy a Duhamel type integral equation which expresses formally the idea of the superposition of the two evolutions. The condition is stated in terms of $(\iota\otimes\rho)$-integrability. (See Section 5C.)

So, let

$$f(s,v) = W(s,v)\exp\left[\int_0^s W(r,v(r))\mathrm{d}r)\right] ,$$

for every $s \in [0,t]$ and $v \in \Upsilon_t$ for which the integral at the right exists.

THEOREM 7.4. *If, for every $s \in [0,t]$, the function e_s is ρ_s-integrable and the function f is $(\iota\otimes\rho)$-integrable, then*

$$\text{(B.4)} \qquad u(t) = S(t,0)\varphi + \int_0^t S(t,s)P(W(s,\cdot))u(s)\mathrm{d}s .$$

Proof. First note that

$$\int_0^t f(s,v)\mathrm{d}s = \exp\left[\int_0^s W(r,v(r))\mathrm{d}r\right] - 1 = e_t(v) - 1 ,$$

for every $v \in \Upsilon_t$ such that $f(\cdot,v) \in \mathcal{L}(\iota)$. Furthermore, by Example 7.3,

$$\int_{\Upsilon_t} f(s,v)M_t(\mathrm{d}_\rho v)\varphi = S(t,s)P(W(s,\cdot))u(s) ,$$

for every $s \in [0,t]$ such that $f(s,\cdot) \in \mathcal{L}(\rho,\mathcal{Q})$. Therefore, by theorem 5.11,

$$\begin{aligned} u(t) - S(t,0)\varphi &= \int_{\Upsilon_t} (e_t(v) - 1)M_t(d_\rho v)\varphi = \\ &= \int_{\Upsilon_t}\left[\int_0^t f(s,v)\mathrm{d}s\right]M_t(\mathrm{d}_\rho v)\varphi = \int_0^t\left[\int_{\Upsilon_t} f(s,v)M_t(\mathrm{d}_\rho v)\varphi\right]\mathrm{d}s = \\ &= \int_0^t S(t,s)P(W(s,\cdot))u(s)\mathrm{d}s . \end{aligned}$$

It should be noted that it may be possible to define $u(s)$ by (B.2), for every $s \in [0,t]$, and to write equation (B.4) independently of whether (B.3) defines an evolution. Indeed, the initial-value problem

$$\dot{u}(t) = P(W(t,\cdot))u(t) ,\ t > 0 ;\ u(0+) = \varphi ,$$

may have a solution for some $\varphi \in E$ but not for others.

Now, assuming that (B.4) holds for every $t \in (0,t_0)$, where $0 < t_0 \leq \infty$, formal differentiation gives that

$$\text{(B.5)} \qquad \dot{u}(t) = A(t)u(t) + P(W(t,\cdot))u(t) ,\ t \in (0,t_0) ,$$

where

$$A(t)\psi = \lim_{r\to 0} r^{-1}(S(t+r,t)\psi - \psi) ,$$

for every $\psi \in E$ such that this limit exists in the sense of convergence in the space E. Furthermore, (B.4) also implies that $u(0+) = \varphi$.

So, another, perhaps more conventional, interpretation of $u(t)$ is that it is the value at t of a generalized solution of the initial-value problem consisting of equation (B.5) and the condition that $u(0+) = \varphi$.

C. Let Λ be a locally compact Hausdorff space. Let $t \mapsto \Sigma_t$, $t \in \mathbb{R}$, be a continuous group of homeomorphisms of the space Λ. That is, for every $t \in \mathbb{R}$, a homeomorphism $\Sigma_t : \Lambda \to \Lambda$ is given such that

(i) $\Sigma_{s+t} = \Sigma_t \circ \Sigma_s$, for every $s \in \mathbb{R}$ and $t \in \mathbb{R}$; and

(ii) for every $x \in \Lambda$, the orbit $t \mapsto \Sigma_t x$, $t \in \mathbb{R}$, of the element x is a continuous map of $\mathbb{R}$ into Λ.

Let $\mathcal{B}$ be the σ-algebra of Baire sets in Λ. Let κ be a Baire measure on Λ. That is to say, there is a vector lattice, $\mathcal{L}(\kappa)$, of functions on Ω and a positive linear functional, κ, on $\mathcal{L}(\kappa)$ such that its restriction to $\mathcal{B}_\kappa = \mathcal{B} \cap \mathcal{L}(\kappa)$ is σ-additive and $\mathcal{L}(\kappa) = \mathcal{L}(\kappa,\mathcal{B}_\kappa)$ and $B\varphi \in \mathcal{L}(\kappa)$ for every set $B \in \mathcal{B}$ and function $\varphi \in \mathcal{L}(\kappa)$. (See Section 3B.) For the sake of simplicity, we assume also that κ is σ-finite, that is, Λ is equal to the union of a sequence of sets belonging to $\mathcal{B}_\kappa$.

Let $1 \le p < \infty$ and $E = L^p(\kappa)$ with the usual norm. (See Section 3C.) To simplify the exposition, we shall use the standard licence and not distinguish between elements of the space E and the individual functions on Λ determining them.

Let $S(t)\varphi = \varphi \circ \Sigma_t$, for every $t \in \mathbb{R}$ and $\varphi \in E$. Assume that

(i) $S(t)\varphi \in E$, for every $t \in \mathbb{R}$ and $\varphi \in E$;

(ii) for every $t \in \mathbb{R}$, the so defined map $S(t) : E \to E$ is an element of $\mathrm{BL}(E)$; and

(iii) for every $\varphi \in E$ the map $t \mapsto S(t)\varphi$, $t \in \mathbb{R}$, is continuous.

So, $S : \mathbb{R} \to \mathrm{BL}(E)$ is a (continuous) group of operators.

For any set $B \in \mathcal{B}$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B. Then the map $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure. The integral, $P(W)$, of a Baire function W is the operator of pointwise multiplication by the function W. So, we may write simply $P(W) = W$.

For any $t \geq 0$, let Υ_t be the space of all continuous maps $v : [0,t] \to \Lambda$. For every $x \in \Lambda$ and $r \in [0,t]$, let

$$\gamma_x(r) = \Sigma_{-r} x .$$

Then, by the assumptions, $\gamma_x \in \Upsilon_t$, for every $x \in \Lambda$. For any set $Y \subset \Upsilon_t$, let

$$B_Y = \{x \in \Lambda : \gamma_x \in Y\} .$$

If the set $Y \in \mathcal{Q}_t$ is given by (A.1) with some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j , $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\ldots,k$, let

$$\text{(C.1)} \qquad M_t(Y) = S(t-t_k)P(B_k)S(t_k-t_{k-1})P(B_{k-1}) \ldots P(B_2)S(t_2-t_1)P(B_1)S(t_1) .$$

This is of course a version of (A.2) for the case when the evolution S happens to be time-homogeneous, that is, it is a semigroup.

Let $\mathcal{S}_t$ be the σ-algebra of sets generated by $\mathcal{R}_t$.

PROPOSITION 7.5. *If* $Y \in \mathcal{S}_t$, *then* $B_Y \in \mathcal{B}$. *If* $Y \in \mathcal{R}_t$, *then* $M_t(Y) = S(t)P(B_Y)$.

Let $\varphi \in E$. *If* $\mu(Y) = S(t)P(B_Y)\varphi$, *for every* $Y \in \mathcal{S}_t$, *then* μ *is an* E-*valued* σ-*additive set function on* $\mathcal{S}_t$ *such that* $\mu(Y) = M_t(Y)\varphi$, *for every* $Y \in \mathcal{R}_t$.

Proof. Because $B_Y \in \mathcal{B}$ for every $Y \in \mathcal{R}_t$ and $\mathcal{B}$ is a σ-algebra, it follows that $B_Y \in \mathcal{B}$ for every $Y \in \mathcal{S}_t$. The equality $M_t(Y) = S(t)P(B_Y)$ can be checked by a direct inspection for any $Y \in \mathcal{R}_t$. Then the last statement is obvious.

Let $\varphi \in E$ and let us keep the notation of Proposition 7.5. Because the set function μ is σ-additive, Proposition 3.13 is applicable. Let $|\varphi' \circ \mu|$ be the variation of the set function $\varphi' \circ \mu$, for any $\varphi' \in E'$. Let

$$\rho(f) = \sup\left\{ \int_{\Upsilon_t} |f| \, \mathrm{d}|\varphi' \circ \mu| : \varphi' \in E' , \|\varphi'\| \leq 1 \right\} ,$$

for every $f \in \mathrm{sim}(\mathcal{R}_t)$. Then, by Proposition 3.13, the seminorm ρ integrates for (the linear extension of) μ .

EXAMPLE 7.6. Let $0 \leq t_1 < t_2 < \ldots < t_{k-1} < t_k \leq t$, let $W_1, W_2, \ldots, W_k$ be Baire functions on Λ and let

$$f(v) = \prod_{j=1}^{k} W(v(t_j))$$

for every $v \in \Upsilon$. The function f is ρ-integrable if and only if the function

$$\varphi \prod_{j=1}^{k} W_j \circ \Sigma_{-t_j}$$

(the multiplication is point-wise) determines an element of the space E. Moreover,

$$\int_{\Upsilon_t} f \mathrm{d}_\rho \mu = \left[\varphi \prod_{j=1}^{k} W_j \circ \Sigma_{-t_j} \right] \circ \Sigma_t ,$$

whenever the function f is in fact integrable.

PROPOSITION 7.7. *Let W be a function on $[0,t] \times \Lambda$ such that the function $r \mapsto W(r, \Sigma_r x)$, $r \in [0,t]$, is integrable for κ-almost every $x \in \Lambda$. Let*

$$\text{(C.2)} \qquad V_{t,W}(x) = \exp\left[\int_0^t W(r, \Sigma_{-r} x) \mathrm{d}r \right] ,$$

for every $x \in \Lambda$ such that the integral on the right exists. Then the function e_t is ρ-integrable if and only if the function $V_{t,W}\varphi$ determines an element of the space E. If the function e_t is indeed ρ-integrable, then

$$\int_{\Upsilon_t} e_t(v) \mu(\mathrm{d}_\rho v) = (V_{t,W}\varphi) \circ \Sigma_t .$$

Proof. When the integral in (C.2) exists in the sense of Riemann, then the statement follows easily from Example 7.6. So, the statement is valid for all functions that are κ-almost everywhere limits of functions for which the integral in (C.2) exists in the sense of Riemann.

The special case when $\Lambda = \mathbb{R}^d$, for some integer $d \geq 1$, and Σ is the fundamental solution of the dynamical system of differential equations $\dot{x} = a(x)$, where $a:\mathbb{R}^d \to \mathbb{R}^d$ is a mapping with components $a_1,a_2,\ldots,a_d$, is of a particular interest. For any $x^0 \in \mathbb{R}^d$, the function $t \mapsto \Sigma_t x^0$, $t \in \mathbb{R}$, is then the solution of this system passing through x^0 at $t=0$. In this case, the infinitesimal generator of the semigroup S is the differential operator

$$A = \sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} .$$

If the function φ is smooth enough and

$$u(t,x) = V_{t,W}(\Sigma_t x)\varphi(\Sigma_t x) ,$$

for every $t \geq 0$ and $x \in \mathbb{R}^d$, then u is a solution of the problem

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} a_j \frac{\partial}{\partial x_j} + Wu ,\ t > 0 ,\ x \in \mathbb{R}^d ;\ u(0+,x) = \varphi(x) ,\ x \in \mathbb{R} .$$

The case of the Feynman-Kac formula suggested in this section admits many variants. None-the-less the set function M_t it gives rise to can be considered rather 'degenerate'. More complex cases are obtained by introducing another parameter.

For every $y \in [0,1]$, let $t \mapsto \Sigma_t^y$, $t \in \mathbb{R}$, be a group of homeomorphisms of the space Λ. Assume that the map $(x,y,t) \mapsto \Sigma_t^y x$, of the space $\Lambda \times [0,1] \times \mathbb{R}$ into Λ, is continuous.

Let $E = L^p(\kappa \otimes \iota)$, where $\kappa \otimes \iota$ is the tensor product of a σ-finite Baire measure on Λ and the Lebesgue measure on $[0,1]$. For a function φ on $\Lambda \times [0,1]$ and $t \in \mathbb{R}$, let

$$(S(t)\varphi)(x,y) = \varphi(\Sigma_t^y x, y) ,$$

for every $x \in \Lambda$ and $y \in [0,1]$. Assume that, for every $\varphi \in E$ and $t \in \mathbb{R}$, the function $S(t)\varphi$ determines again an element of E, that the resulting map $S(t) : E \to E$ is an operator belonging to $\mathrm{BL}(E)$ and, finally, the so defined map $t \mapsto S(t)$, $t \geq 0$, is a continuous group of operators.

Let $t \geq 0$ and let Υ_t have the same meaning as before. Let

$$\gamma_{x,y}(r) = \Sigma_{-r}^y x ,$$

for every $x \in \Lambda$, $y \in [0,1]$ and $r \in [0,t]$. For a set $Y \subset \Upsilon_t$, let $B_Y^y = \{x : \gamma_{x,y} \in Y\}$.

Let $\varphi \in E$. For every $Y \in \mathcal{S}_t$, let $\mu(Y)$ be the element of the space E such that

$$(\mathrm{S}(-t)\mu(Y))(x,y) = B_E^y(x)\varphi(x,y) ,$$

for κ-almost every $x \in \Lambda$ and every $y \in [0,1]$. It is then a matter of direct calculation that $\mu(Y) = M_t(Y)\varphi$, for every $Y \in \mathcal{Q}_t$.

D. Let $d \geq 1$ be an integer. We shall specialize the situation of Sections A and B by taking the d-dimensional arithmetic Euclidean space, $\mathbb{R}^d$, for Λ and the space of all scalar valued σ-additive set functions on the σ-algebra, $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$, of all Baire sets in $\mathbb{R}^d$ for E . Because every element of E has finite and σ-additive variation, we use the standard conventions about integration with respect to elements of E mentioned in Section 3F. Namely, we note that the variation, $|\varphi|$, of an element, φ , of the space E is a gauge on $\mathcal{B}$ which integrates for φ , denote $\mathcal{L}(\varphi) = \mathcal{L}(|\varphi|)$ and do not show the gauge, $|\varphi|$, in symbols for integral with respect to φ . The norm, $\|\varphi\|$, of an element, φ , of the space E is the total variation of φ , that is, the number $|\varphi|(\mathbb{R}^d)$.

The Lebesgue measure on $\mathbb{R}^d$ is denoted by λ . Identifying the elements of $L^1(\lambda)$ with their indefinite integrals, we identify the space $L^1(\lambda)$ with a subspace of E consisting of those elements which are λ-absolutely continuous.

Given a set $B \in \mathcal{B}$, let $P(B)$ be the operator of restriction to the set B . That is, $(P(B)\varphi)(X) = \varphi(B \cap X)$, for every set $X \in \mathcal{B}$ and every $\varphi \in E$. So, on the subspace $L^1(\lambda)$ of E , the operator $P(B)$ acts as point-wise multiplication by the characteristic function of the set B . For every $B \in \mathcal{B}$, the operator $P(B)$ is an element of $\mathrm{BL}(E)$ and the map $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure.

Let D be a strictly positive real number and let

$$p_D(t,x) = (4\pi Dt)^{-\frac{1}{2}d}\exp(-|x|^2/4Dt) ,$$

for every $t > 0$ and $x \in \mathbb{R}^d$. ($|x|$ stands for the usual Euclidean norm of an element x of $\mathbb{R}^d$.)

Let $S(0) = I$ and

$$(S(t)\varphi)(B) = \int_B \mathrm{d}x \int_{\mathbb{R}^d} p_D(t,x-y)\varphi(\mathrm{d}y) ,$$

for every $t > 0$, $B \in \mathcal{B}$ and $\varphi \in E$. Then the set function $S(t)\varphi$, that is, $B \mapsto (S(t)\varphi)(B)$, $B \in \mathcal{B}$, is an element of the space E. For every $t > 0$, the operator $S(t)$, that is, $\varphi \mapsto S(t)\varphi$, $\varphi \in E$, is an element of $\mathrm{BL}(E)$. Finally, the resulting map $t \mapsto S(t)$, $t \in [0,\infty]$, is a continuous semigroup, $S : [0,\infty] \to \mathrm{BL}(E)$, of operators.

The semigroup S can be interpreted as a mathematical description of isotropic and homogeneous diffusion in $\mathbb{R}^d$ with the diffusion coefficient D. It is called the Poisson semigroup. Its infinitesimal generator is the (closure of the) operator $D\Delta$, where Δ is the Laplacian in $\mathbb{R}^d$.

Given a $t \geq 0$, let Υ_t be the set of all continuous paths $v : [0,t] \to \Lambda$. Because S is a semigroup, the formula (A.2), defining the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$, takes the form (C.1), for every set $Y \in \mathcal{R}_t$ given by (A.1) with some integer $k \geq 1$, sets $B_j \in \mathcal{B}$ and numbers t_j, $j = 1,2,\ldots,k$, such that $0 \leq t_{j-1} < t_j \leq t$, for each $j = 2,3,\ldots,k$.

Let $\varphi \in E$ be a non-negative measure. Let

$$\rho_\varphi(Y) = \|M_t(Y)\varphi\| = (M_t(Y)\varphi)(\mathbb{R}^d) ,$$

for every $Y \in \mathcal{R}_t$. Then ρ_φ is a non-negative σ-additive set function on $\mathcal{R}_t$ and so, it generates a measure in the space Υ_t. This fact, dating back to N. Wiener, is classical; see, for example, [11], Theorem VIII.2.2. If φ is a probability measure on $\mathbb{R}^d$, then ρ_φ is called the d-dimensional Wiener measure of variance $2D$ per unit of time with initial distribution φ. (See Example 4.33.)

Now, if φ is an arbitrary element of the space E, then $\rho = \rho_{|\varphi|}$ is a gauge on $\mathcal{R}_t$ which integrates for φ.

Let W be a Baire function on $[0,t] \times \mathbb{R}^d$. Mark Kac noted, see [26],Chapter IV, that, if the function $r \mapsto W(r,v(r))$, $r \in [0,t]$, is Riemann integrable in $[0,t]$, for ρ-almost every $v \in \Upsilon_t$, then the Feynman-Kac functional,

$$e_t(v) = \exp\left[\int_0^t W(r,v))\mathrm{d}r\right], \quad v \in \Upsilon_t,$$

is ρ-measurable on Υ_t. Consequently, if e_t is also bounded then it is ρ-integrable. This happens, for example, when $W(r,x) = W(0,x)$, for every $r \in [0,t]$ and $x \in \mathbb{R}^d$, and the function $W(0,\cdot)$ is bounded above and continuous on the complement of a set of capacity zero in $\mathbb{R}^d$, because the set of paths $v \in \Upsilon_t$ that avoid a given set of capacity zero has the Wiener measure equal to zero.

Now, if e_t is indeed ρ-integrable, for every $t > 0$, the element

$$u(t) = \int_{\Upsilon_t} e_t(v) M_t(\mathrm{d}_\rho v)\varphi \tag{D.1}$$

of the space E belongs to $L^1(\lambda)$, for every $\varphi \in E$. Let us abuse the notation and denote by $x \mapsto u(t,x)$, $x \in \mathbb{R}^d$, the density of $u(t)$. In terms of densities, the integral equation (B.4) can be re-written in the form

$$u(t,x) = \int_{\mathbb{R}^d} p_D(t,x-y)\varphi(\mathrm{d}y) + \int_0^t \int_{\mathbb{R}^d} p_D(t-s,x-y)\, W(s,y)\, u(s,y)\mathrm{d}y\mathrm{d}s, \tag{D.2}$$

for $x \in \mathbb{R}^d$ and $t > 0$. This equation represents the initial-value problem

$$\begin{aligned} &\dot{u}(t,x) = D\Delta u(t,x) + W(t,x)u(t,x), \ t > 0, \ x \in \mathbb{R}^d \\ &\lim_{t \to 0+} \int_B u(t,x)\mathrm{d}x = \varphi(B), \ B \in \mathcal{B}. \end{aligned} \tag{D.3}$$

If $d \geq 2$, it is easy to produce functions W such that $u(t)$ is well-defined by (D.1) for every $t \geq 0$, but, for many $\varphi \in E$, the integral equation (D.2) does not have a solution. Then the problem (D.3) does not have a solution either. For example, $W(t,x) = -|x|^{-d}$, $t \geq 0$, $x \in \Lambda$, $x \neq 0$, is such a function. Still, $u(t)$ has a perfectly good physical interpretation. (See Section 0C.)

E. Let $d \geq 1$ be an integer. We take, again, $\Lambda = \mathbb{R}^d$. Let $E = L^2(\lambda)$, where λ is the Lebesgue measure in $\mathbb{R}^d$. Elements of the space E and functions on $\mathbb{R}^d$ representing them will not be distinguished. The norm of an element, φ, of E will be denoted by $\|\varphi\|$.

For any $B \in \mathcal{B} = \mathcal{B}(\mathbb{R}^d)$, let $P(B)$ be the operator of point-wise multiplication by the characteristic function of the set B. That is, $P(B)\varphi = B\varphi$, for every $\varphi \in E$. Then $P : \mathcal{B} \to \mathrm{BL}(E)$ is a σ-additive spectral measure.

Let m be a strictly positive number. Let $S(0) = I$ and, for every $t \in \mathbb{R}$, $t \neq 0$, let $S(t) \in \mathrm{BL}(E)$ be the operator such that

$$(S(t)\varphi(x) = \left[\frac{m}{2\pi t \mathrm{i}}\right]^{\frac{1}{2}d} \int_{\mathbb{R}^d} \varphi(y)\exp\left[\frac{m\mathrm{i}}{2t}\,|x-y|^2\right]\mathrm{d}y\,,$$

for every $x \in \mathbb{R}^d$ and every $\varphi \in L^1 \cap L^2(\lambda)$. The root is determined from the branch that assigns positive real values to positive real numbers. It is well-known, and can easily be shown using the Plancherel theorem, say, that such an operator $S(t)$ exists, for every $t \in \mathbb{R}$, is unique and the resulting map $t \mapsto S(t)$, $t \in \mathbb{R}$, is a unitary group of operators. It is called the Schrödinger group. The infinitesimal generator of the Schrödinger group, $S : \mathbb{R} \to \mathrm{BL}(E)$, is (the closure of) the operator

$$A = \frac{\mathrm{i}}{2m}\Delta\,,$$

where Δ is the Laplacian on $\mathbb{R}^d$.

Let $t \geq 0$. Let Υ_t be the set of all continuous paths $v : [0,t] \to \mathbb{R}^d$. Let the set function $M_t : \mathcal{R}_t \to \mathrm{BL}(E)$ be defined by the formula

$$M_t(Y) = S(t-t_k)P(B_k)S(t_k-t_{k-1})P(B_{k-1}) \dots P(B_2)S(t_2-t_1)P(B_1)S(t_1)\,,$$

for every set

$$Y = \{v \in \Upsilon_t : v(t_j) \in B_j\,,\ j = 1,2,\dots,k\}\,,$$

where $k \geq 1$ is an integer, the sets B_j belong to $\mathcal{B}$ and the numbers t_j, $j = 1,2,\dots,k$, satisfy the conditions $0 \leq t_{j-1} < t_j \leq t$ for every $j = 2,3,\dots,k$. Let φ be an element of the space E.

Our aim is to produce a gauge on $\mathcal{R}_t$ which integrates for the set function $M_t\varphi$. Actually, a suggestion for producing such a gauge is presented in Example 4.33, because the construction exhibited there for $d = 1$ can be easily adapted for arbitrary d. However, we present now another construction.

By a special partition of Υ_t we shall understand any $\mathcal{R}_t$-partition, $\mathcal{P}$, obtained in the following manner. (See Section 1D.) Assume that $k \geq 1$ is an integer, $\not{h}_j$ are $\mathcal{B}$-partitions of $\mathbb{R}^d$ and t_j are numbers, $j = 0,1,\ldots,k$, such that $t_0 = 0$, $t_{j-1} < t_j$, for every $j = 1,2,\ldots,k$, and $t_k = t$. The partition $\mathcal{P}$ then consists of all sets of the form

$$Y = Y(t_0,t_1,\ldots,t_k;B_0,B_1,\ldots,B_k) = \{v \in \Upsilon_t : v(t_j) \in B_j, j = 0,1,\ldots,k\},$$

with arbitrary sets B_j belonging to the partition $\not{h}_j$, for every $j = 0,1,\ldots,k$. We say that the partition $\mathcal{P}$ is determined by the numbers t_j and partitions $\not{h}_j$, $j = 0,1,\ldots,k$. The set of all special partitions of Υ_t will be denoted by Γ.

Our construction uses the fact, proved in the following proposition, that the set function $M_t\varphi$ has finite 2-variation with respect to the set of partitions Γ. (See Section 4A.)

PROPOSITION 7.8. *For every special partition, $\mathcal{P}$, we have $v_2(M_t\varphi,\mathcal{P};\Upsilon_t) = \|\varphi\|^2$. Consequently, $v_2(M_t\varphi,\Gamma;\Upsilon_t) = \|\varphi\|^2$.*

Proof. Let the partition $\mathcal{P}$ be determined by the numbers t_j and the $\mathcal{B}$-partitions $\not{h}_j$, $j = 0,1,\ldots,k$. Because the operators $S(t-t_j)$ and $S(t_j-t_{j-1})$ are unitary and $\not{h}_j$ is a $\mathcal{B}$-partition of $\mathbb{R}^d$, we have

$$\|M_t(Y(t_0,\ldots,t_{j-1};B_0,\ldots,B_{j-1}))\varphi\|^2 = \sum_{B_j \in \not{h}_j} \|M_t(Y(t_0,\ldots,t_{j-1},t_j;B_j,\ldots,B_{j-1},B_j))\varphi\|^2,$$

for any sets $B_\ell \in \not{h}_\ell$, $\ell = 0,1,\ldots,j-1$, and any $j = 1,2,\ldots,k$. Moreover, because $S(t)$ is a unitary operator and $\not{h}_0$ is a $\mathcal{B}$-partition of $\mathbb{R}^d$,

$$\|\varphi\|^2 = \|M_t(\Upsilon_t)\varphi\|^2 = \sum_{B_0 \in \not{h}_0} \|M_t(Y(t_0;B_0))\varphi\|^2.$$

Therefore,

$$\|\varphi\|^2 = \sum_{Y\in\mathcal{P}} \|M_t(Y)\varphi\|^2 = v_2(M_t\varphi,\mathcal{P};\Upsilon_t) .$$

Now, let μ be a non-negative real-valued σ-additive set function on $\mathcal{B}$ such that $\mu(B) = 0$ if and only if $\lambda(B) = 0$ and $\mu(\mathbb{R}^d) = 1$. Let ι be the d-dimensional Wiener measure of variance one, say, per unit of time with initial distribution μ. (See Section D.)

Given a partition $\mathcal{P} \in \Gamma$, let

$$\sigma_{\mathcal{P}}(X) = \sum_{Y\in\mathcal{P}} \frac{\|M_t(Y)\,\varphi\|^2}{\iota(Y)}\,\iota(X\cap Y)$$

for every $X \in \mathcal{R}_t$, putting, by convention, $\iota(X\cap Y)/\iota(Y) = 0$, whenever $\iota(Y) = 0$. By Proposition 2.13, the set function $\sigma_{\mathcal{P}}$ is an integrating gauge on $\mathcal{R}_t$, for every $\mathcal{P} \in \Gamma$. So, if P is a non-empty subset of Γ, by Proposition 2.14, the set function σ, defined by

$$\sigma(X) = \sup\{\sigma_{\mathcal{P}}(X) : \mathcal{P} \in \mathrm{P}\} ,$$

for every $X \in \mathcal{R}_t$, is an integrating gauge on $\mathcal{R}_t$.

By Proposition 7.8, however we choose P, the equality $\sigma(\Upsilon_t) = \|\varphi\|^2$ holds. Moreover, the gauge σ is monotonic. (See Section 2G.) We can choose P so that the inequality $\|M_t(X)\varphi\|^2 \le \sigma(X)$ holds for every $X \in \mathcal{R}_t$. To do that it suffices to take $\mathrm{P} = \Gamma$. However, much more economical choices of the set P are possible. In fact, there are countable subsets of Γ which can be chosen for such a P.

Having made such a choice of P, let $\rho(X) = (\sigma(X))^{\frac{1}{2}}$, for every $X \in \mathcal{R}_t$. Then, by Proposition 2.26, the gauge ρ integrates for the set function $M_t\varphi$.

It may be interesting to note that there does not necessarily exist a non-negative σ-additive set function, σ, on $\mathcal{R}_t$ such that the gauge $\sigma^{\frac{1}{2}}$ integrates for the set function $M_t\varphi$. In fact, we have the following proposition, in which $d = 1$, due to Brian Jefferies, which implies that $v_2(M_t\varphi,\Pi(\mathcal{R}_t);\Upsilon_t) = \infty$, for some $\varphi \in E$.

PROPOSITION 7.9. *Let $\mathcal{Q}$ be the semialgebra in the space $\mathbb{R}\times\mathbb{R}$ consisting of all sets of the form $A\times B$ with $A\in\mathcal{B}$ and $B\in\mathcal{B}$. Let $\varphi(x)=\exp(-\frac{1}{2}(1+i)x^2)$, for every $x\in\mathbb{R}$. Let $\nu(A\times B)=P(B)S(1)P(A)\varphi$, for every $A\in\mathcal{B}$ and $B\in\mathcal{B}$. Then $v_2(\nu,\Pi(\mathcal{Q});\mathbb{R}\times\mathbb{R})=\infty$.*

Proof. For any $A\in\mathcal{B}$ and $B\in\mathcal{B}$, we have

$$\|\nu(A\times B)\|^2=\frac{1}{2\pi}\int_A\int_B\int_A\bar{\varphi}(x_2)\exp(-\tfrac{1}{2}i(x_2-x_1)^2)\exp(\tfrac{1}{2}i(x_1-x_0)^2)\varphi(x_0)\mathrm{d}x_0\mathrm{d}x_1\mathrm{d}x_2$$

$$=\frac{1}{2\pi}\int_B\left|\int_A\exp(i(\tfrac{1}{2}x_0^2-x_0x_1))\varphi(x_0)\mathrm{d}x_0\right|^2\mathrm{d}x_1=\frac{1}{2\pi}\int_B\left|\int_A\exp(-ixy-\tfrac{1}{2}x^2)\mathrm{d}x\right|^2\mathrm{d}y\,.$$

The Cauchy-Schwarz inequality implies that

$$\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\mathrm{d}x\mathrm{d}y\right|^2\le\lambda(B)\int_B\left|\int_A\exp(-ixy-\tfrac{1}{2}x^2)\mathrm{d}x\right|^2\mathrm{d}y\,.$$

Now, for every $n=0,1,2,\ldots$, let $a_n=2\pi n+\pi/3$ and

$$C_n=\{(x,y):x\in\mathbb{R},y\in\mathbb{R},|xy-2\pi n|\le\pi/3\,,\ y\ge a_n\}\,.$$

Then $0\le x\le 1$ and $\cos(xy)\ge\frac{1}{2}$, for every $(x,y)\in C_n$, so

$$\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\mathrm{d}x\mathrm{d}y\right|\ge\tfrac{1}{2}\exp(-\tfrac{1}{2})\lambda(A)\lambda(B)\,,$$

whenever $A\in\mathcal{B}$, $B\in\mathcal{B}$, $A\times B\subset C_n$, $n=0,1,2,\ldots$. Consequently,

$$\frac{1}{\lambda(B)}\left|\int_B\int_A\exp(-ixy-\tfrac{1}{2}x^2)\mathrm{d}x\mathrm{d}y\right|^2\ge\tfrac{1}{4}\exp(-1)(\lambda(A))^2\lambda(B)\,,$$

and, hence,

$$\text{(E.1)}\qquad \|\nu(A\times B)\|^2\ge\frac{\exp(-1)}{8\pi}(\lambda(A))^2\lambda(B)\,,$$

for any sets $A\in\mathcal{B}$ and $B\in\mathcal{B}$ such that $A\times B\subset C_n$, $n=0,1,2,\ldots$.

If B is sufficiently small interval centered around a point $y>a_n$, then there exists an interval, A, of length arbitrarily close to $2\pi/3y$ such that $A\times B\subset C_n$, $n=0,1,2,\ldots$. Moreover, for every $n=0,1,2,\ldots$, the set C_n contains a pair-wise disjoint family, $\mathcal{J}_n$, of such sets, $A \times B$, which can be chosen so that

$$\sum_{A\times B\in\mathcal{J}_n} (\lambda(A))^2\lambda(B) > \int_{a_n}^{\infty} \left[\frac{2\pi}{3y}\right]^2 dy - 2^{-n} .$$

Because the sets C_n, $n = 0,1,2,\ldots,$ are pair-wise disjoint, by (E.1), the 2-variation, $v_2(\nu,\Pi(\mathcal{Q});\mathbb{R}\times\mathbb{R})$, of the set function ν is not less than

$$\frac{\exp(-1)}{8\pi} \sum_{n=0}^{\infty} \left[\int_{a_n}^{\infty} \left[\frac{2\pi}{3y}\right]^2 dy - 2^{-n}\right] = \infty .$$

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