Kerr-Schild Spacetimes Revisited

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Abstract

The main thrust of this paper is on real vacuum (single) Kerr-Schild (VKS) metrics. The conditions are given for real VKS metrics to have twistfree expanding principal null directions and to have Weyl tensors of Petrov type D. It follows that there are no twistfree type II metrics - i.e. no Robinson-Trautman ones. VKS metrics with twist either belong to the Kerr family or else are of type II. Non-expanding VKS metrics are of type N. The condition for a VKS metric to be of type D leads naturally to a complex translation which has been used to obtain the Kerr metric from the Schwarzschild metric.

1 Introduction

This paper is concerned with metrics which are solutions of Einstein's vacuum equations

$$R_{ab} = 0 \tag{1}$$

and which obey the Kerr-Schild *ansatz*. That is, they can be written as

$$g = g_o + V l \otimes l \tag{2}$$

where g_o is a flat space metric, V is a function of all coordinates and l is a real one-form which is null with respect to g_o and hence g (see [9] and [3]). Vacuum Kerr-Schild metrics of this type are here called *VKS* metrics.

Many properties of VKS metrics with an expanding principal null direction are discussed in Chapter 28 of the book on exact solutions by Kramer *et al* [5]. The basic formulation in that chapter closely follows the formulation by Debney *et al* [2], whose paper can be referred to for earlier references.

This paper highlights and enlarges results which are given by McIntosh and Hickman [7]. Further details of a number of ideas and statements, some proofs, a discussion about real VKS metrics from a complex viewpoint and their relationship with double-Kerr-Schild metrics can also be found in [7].

2 Basic Equations - Expanding VKS Metrics

In coordinates $(\xi, \eta, \zeta, \tilde{\zeta})$, where ξ and η are real and ζ and $\tilde{\zeta}$ are complex conjugates, a tetrad for VKS metrics with an expanding principal null direction can be written as

$$\begin{aligned} \theta^1 &= d\xi + V\theta^2 \\ \theta^2 &= d(\eta + Y\tilde{\zeta}) + \tilde{Y}d(\zeta + Y\xi) - (\tilde{\zeta} + \tilde{Y}\xi)dY \\ \theta^3 &= d(\tilde{\zeta} + \tilde{Y}\xi) - \xi d\tilde{Y} \\ \theta^4 &= d(\zeta + Y\xi) - \xi dY \end{aligned} \tag{3}$$

Here V is a real function of all coordinates and Y is a complex function which satisfies

$$DY = 0, \ \delta Y = 0 \tag{4}$$

where the dual base vectors are

$$\begin{aligned} D = \partial_\xi - Y\partial_\zeta - \tilde{Y}\delta, \ \Delta = \partial_\eta - VD \\ \delta = \partial_{\tilde{\zeta}} - Y\partial_\eta, \ \tilde{\delta} = \partial_\zeta - \tilde{Y}\partial_\eta \end{aligned} \tag{5}$$

The metric

$$g = 2[\theta^1 \otimes \theta^2 - \theta^3 \otimes \theta^4] \tag{6}$$

formed from (3) is of the form (2) where the background flat space metric is

$$g_o = 2[d\xi \otimes d\eta - d\zeta \otimes d\tilde{\zeta}] \tag{7}$$

Equation (4) is equivalent to the following equivalent statements:
(a) exactly two of the one-forms

$$dY, \ d(\eta + Y\tilde{\zeta}), \ d(\zeta + Y\xi) \tag{8}$$

are linearly independent,
(b) Y satisfies the equations

$$Y_{\tilde{\zeta}} = YY_\eta, \ Y_\xi = YY_\zeta \tag{9}$$

(which imply $\Box Y = 0$, i.e. Y satisfies the wave equation in the flat background space - see [4]),
(c) there exists an analytic function F such that

$$F(Y, \eta + Y\tilde{\zeta}, \zeta + Y\xi) = 0 \tag{10}$$

(such that also $DF = \delta F = 0$),
(d) θ^2 and θ^4 span an integrable co-distribution (i.e. D and δ are surface forming vectors).

For the tetrad (3), the statement (d) is equivalent to the Newman-Penrose (NP) spin coefficients κ and σ being zero. This condition, which holds (for some tetrad) for all algebraically degenerate vacuum metrics, is the basis of the ability to look at such metrics from a complex viewpoint, as shown by Plebañski and others; see, e.g. [8] and the references therein. In the series of papers to which [8] belongs, Y is called a *left-leaf constant*, being constant on the complex, null, totally geodesic leaves of the *left foliation* which arises from this integrable co-distribution.

The vacuum requirements are that (i) V has the form

$$V = (m/2P^3)(\rho + \tilde{\rho}) \tag{11}$$

where

$$P = a + Y\tilde{c} + \tilde{Y}c + Y\tilde{Y}b, \tag{12}$$

and

$$\rho = \tilde{\delta}Y \tag{13}$$

(so that $DP = 0$) and that (ii) F takes the form

$$F \equiv \Phi(Y) + (\tilde{c}Y + a)(\zeta + Y\xi) - (bY + c)(\eta + Y\tilde{\zeta}) = 0 \tag{14}$$

where Φ is an arbitrary complex function of the complex variable Y. The other NP spin coefficients for this tetrad are listed in [7].

It follows that, for the tetrad (3), the tetrad components of the Weyl tensor are

$$\begin{aligned} \Psi_0 &= \Psi_1 = 0 \\ \Psi_2 &= m\tilde{\rho}^3/P^3 \\ \Psi_3 &= (3m/2P^3)(\tilde{\rho}\delta\tilde{\rho} + 2\tilde{\rho}^3\tau/\rho) \\ \Psi_4 &= (m/2P^3)[\delta\delta\tilde{\rho} + 9\tau\delta\tilde{\rho}(\tilde{\rho}/\rho) + 12\tau^2\tilde{\rho}^3/\rho^2] \end{aligned} \tag{15}$$

All expanding VKS metrics admit the Killing vector field

$$X = a\partial_\eta + b\partial_\xi + c\partial_\zeta + \tilde{c}\partial_{\tilde{\zeta}} \tag{16}$$

which is simultaneously a Killing vector field of the flat background metric g_o. Without loss of generality, the constants a, b and c can be chosen so that $P = (1 + kY\tilde{Y})/\sqrt{2}$ where k, with respect to g_o, is (i) 1 when X is timelike, (ii) 0 when X is null and (iii) -1 when X is spacelike.

3 Key Results for Expanding VKS

Two results follow easily from the equations outlined in the last section. It is obvious from the equation (15) that non-flat expanding VKS metrics can only have Weyl tensors of Petrov types *II* or *D*. It also can be shown reasonably easily that the assumption that the repeated principal vector field l is twistfree, such that $\rho = \bar{\rho}$, implies $\delta\rho = 0$. In this case it is obvious from (15) that $3\Psi_2\Psi_4 = 2(\Psi_3)^2$, i.e. the metric has a Weyl tensor of type D. Thus it follows that:

Theorem 1 *If an expanding vacuum Kerr-Schild metric has a Weyl tensor with a repeated principal null direction which is*
(i) twistfree, then the Weyl tensor is of type D,
(ii) twisting, then the Weyl tensor is of type D or II.

It is surprising that, before being mentioned in [7], this result does not appear to be in the literature. For example, it is not mentioned in [5]. What else is surprising is that the condition for this type of metric to be of type D also does not appear to be in the literature, before being mentioned in [7]. This is particularly true when the condition is so simple and helps in understanding the complex 'translation' which has been used to obtain the Kerr metric from the Schwarschild metric.

It can be shown that

$$\Phi_{,YYY}(Y) = 0 \Leftrightarrow \text{metric is of type D} \tag{17}$$

In this case

$$\begin{aligned} \Phi(Y) &= \alpha Y^2 + \beta Y + \gamma \\ &= Y^2[b\tilde{\zeta} - \tilde{c}\xi] + Y[b\eta + c\tilde{\zeta} - \tilde{c}\zeta - a\xi] + [c\eta - a\zeta] \end{aligned} \tag{18}$$

where $\alpha, \beta, \gamma \in C$. Translations of the four coordinates, which leave the flat background metric g_o invariant, may be used to set

$$\alpha = \beta + \tilde{\beta} = \gamma = 0 \tag{19}$$

In this case

$$\Phi(Y) = \Im(\beta)Y \tag{20}$$

so that the most general type D VKS metric arises in this case. Metrics of this class are members of the Kerr family of metrics (and include members of the Schwarzschild family when $\Im(\beta) = 0$).

These results can be summarised as follows:

Theorem 2 *Expanding VKS metrics, given by (3), (6), (7) and (14) and where* Φ *is an arbitrary function of* Y *as defined in (14), belong to one of three families; coordinates for these families can be chosen such that*
(i) $\Phi(Y) = 0 \Leftrightarrow \rho = \tilde{\rho} \Rightarrow$ *Member of Schwarzschild family (Petrov type D)*
(ii) $\Phi(Y) =$*(imaginary constant)*$.Y \Rightarrow$ *Member of Kerr family (Petrov type D)*
(iii) $\Phi_{,YYY} \neq 0 \Rightarrow$ *Twisting type II metric.*

The complex translations of coordinates required in the type D case to get $\Phi(Y)$ into the form (20) obviously keep the background metric (7) real and keep Φ and $\tilde{\Phi}$ complex conjugates. They are listed in full in [7]. These translations can be extended to include a complex translation of $b\eta + a\xi$ which may be used to set $\Im(\beta) = 0$ such that $\Phi(Y) = 0$. $b\eta + a\xi$ is a spacelike coordinate when the Killing vector field X is timelike. However this translation does not simultaneously set $\tilde{\Phi}(\tilde{Y}) = 0$ and is not a proper coordinate transformation. It is the basis of the "complex trick", as discussed in [1] and in Chapter 30 of [5] (and in the references quoted there), which has been used to derive the Kerr metric from the Schwarzschild metric. This translation, together with the real translations used to set $\Phi(Y)$ in the form (20), can be obtained from exponentiation of a Killing vector field of the form (16).

4 Other Comments

It can be shown that VKS metrics, in which the repeated principal null congruence is not expanding, arise in the two distinct classes of metrics where the Weyl tensor is of type N, namely those in which this congruence is rotating and not rotating - see [6]. Indeed the converse holds: all metrics in these two classes are VKS ones. The theorem in [5] concerning this area (Theorem 28.6) is not at all clear as the rotating (non-expanding) case appears to have been overlooked. Thus VKS metrics arise in the cases with different Weyl tensor types and with different rotation and expansion properties as shown in Table 1.

	I	II	D	III	N
$\rho = 0, \tau = 0$					All
$\rho = 0, \tau \neq 0$					All
$\rho = \tilde{\rho} \neq 0$			Some		
$\rho \neq \tilde{\rho}$		Some	Some		

Table 1: A list of cases where vacuum Kerr-Schild metrics exist for various Petrov types of the Weyl tensor. τ and ρ measure the *rotation* and *expansion + i(twist)* of a repeated principal null congruence of algebraically degenerate metrics.

A number of further properties of VKS metrics can be found in [7] and references discussed there. In particular, the relationship between real VKS and complex Double-Kerr-Schild metrics is dealt with in [7], together with reasons why VKS metrics arise as linear (though exact) solutions of the vacuum equations.

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