THE REDUCTION PROBLEM FOR EINSTEIN'S EQUATIONS

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I. MOTIVATION

It has long been known that the vacuum (or electro-vacuum) Einstein equations for spacetimes admitting a single (spacelike) Killing field can be *locally* reexpressed as a $2 + 1$ dimensional system of Einstein-harmonic-map equations [1, 2]. In this formulation the unknown fields are (in the vacuum case) a Lorentzian metric on the (locally defined) space of orbits of the Killing field and the norm and twist potential of this Killing field which define a harmonic map from the space of orbits to hyperbolic two-space. A natural problem is to study this formulation *globally* on suitably chosen manifolds and to apply standard methods of elliptic analysis to reduce the system to its "minimal form". In physics terminology one would like to solve an elliptic set of equations for all the dependent or "non-propagating" variables and be left with an unconstrained hyperbolic system for only the independent "physical degrees of freedom" of the gravitational field.

One mathematical motivation for this reduction program is that it aims to take maximal advantage of the elliptic aspects of Einstein's equations, for which the needed global analysis techniques are already at hand, and to reduce the complementary (and presumably more difficult) hyperbolic aspects of the system as much as possible. The ultimate aim of this project is to study the long time existence properties of Einstein's equations and the hope is that the reduction program should help to make such an analysis more tractable.

Another more "practical" motivation for this program is that it suggests a numerical method for approximately solving the Einstein equations which avoids the notorious problem of the "drifting of the constraints". The latter arises when, in contrast to the method mentioned above, one attempts to impose the elliptic equations only on an initial Cauchy hypersurface and then subsequently evolves most or all of the variables by a suitably defined hyperbolic system. The difficulty is that the elliptic Einstein equations, which should be preserved from one hypersurface to the next by virtue of the Bianchi identities, tend, because of numerical errors, to drift away from being satisfied as the evolution continues. The "reductions" approach sidesteps this difficulty by solving the elliptic "constraint" equations on every time slice and evolving only the unconstrained variables hyperbolically.

II. GLOBAL FORMULATION OF THE REDUCTION PROBLEM

Let us assume that the spacetimes to be considered have compact Cauchy hypersurfaces and that each admits an isometric action of $U(1)$ generated by a nowhere vanishing, everywhere spacelike Killing field X with closed integral curves. Let us further assume that the integral curves of X fiber the spacetime in the (locally trivial) fashion of a circle bundle over $K \times \mathbb{R}$ where K is a compact, orientable two-manifold.

If ${}^{(4)}g$ is a Lorentzian metric which satisfies the vacuum Einstein equations on the chosen four-manifold and is invariant under the flow of X, then ${}^{(4)}g$ induces a Lorentzian metric ${}^{(3)}b$ on the three dimensional base manifold $K \times \mathbb{R}$ (i.e., on the space of orbits of X). In addition, upon making use of Einstein's equations, one finds that ${}^{(4)}g$ and X induce two scalar fields upon the base manifold $K \times \mathbb{R}$, namely the norm and twist potential of the Killing field X.

One way of seeing how the twist potential arises is to consider a foliation of spacetime by a preferred family of Cauchy hypersurfaces chosen tangent to the vector field X. Taking the projection of the momentum constraint equation along X and reexpressing the result as an equation on the base manifold $K \times \mathbb{R}$, one finds that it takes the form of a "Gauss law constraint" on each constant time hypersurface (diffeomorphic to $\mathbb{K}$), i.e., it requires the vanishing of the divergence of a certain one-form induced on each such hypersurface. Solving this equation by means of the Hodge decomposition on $\mathbb{K}$ yields the twist potential (the dual of a two-form) as free data. If $\mathbb{K} \approx S^2$ then the twist potential yields the general solution of the Gauss law constraint--otherwise one must include the harmonic terms from the Hodge decomposition to get the general solution. For this as well as some other reasons to be mentioned below, the choice $\mathbb{K} \approx S^2$ is the simplest one to consider and (aside from the trivial bundle $T^3 \times \mathbb{R} \rightarrow T^2 \times \mathbb{R}$) is the only one to be considered so far in analytical detail [3, 4].

Having solved the X-projection of the momentum constraint by means of the Hodge decomposition one can solve the complementary projections of this constraint by a straightforward L^2-orthogonal decomposition of the gravitational momentum variable (essentially the second fundamental form) induced on the Cauchy surfaces of the $2 + 1$ dimensional base manifold $(\mathbb{K} \times \mathbb{R}, {}^{(3)}b)$. To facilitate this step it is desirable to specialize the choice of preferred foliation to one of *constant mean curvature* . One of the summands which ordinarily arises in such an L^2-orthogonal splitting drops out when $\mathbb{K} \approx S^2$ since traceless, divergenceless symmetric two tensors vanish identically on S^2. This is another (even more important) simplifying feature of the choice $\mathbb{K} \approx S^2$ and is traceable to the fact that Riemannian metrics on S^2 are all conformally diffeomorphic to the canonical metric on S^2 .

Finally one must solve the Hamiltonian constraint on each hypersurface of the preferred (constant mean curvature) foliation. For this purpose one can exploit the fact that every Riemannian metric on $\mathbb{K}$ is conformal to one of constant curvature. In particular the first fundamental form ${}^{(2)}g$ induced by ${}^{(3)}b$ on any hypersurface of the preferred foliation must be expressible as ${}^{(2)}g = \exp(2\lambda) \cdot {}^{(2)}h$ where ${}^{(2)}h$ is a metric of constant (whose sign is determined by the genus of $\mathbb{K}$). If $\mathbb{K} \approx S^2$ then ${}^{(2)}h$ can be uniquely chosen to be the canonical metric on S^2. For higher genus $\mathbb{K}$, ${}^{(2)}h$ must be allowed to vary over the (finite dimensional) moduli space of conformally inequivalent metrics on $\mathbb{K}$. Indeed, the Einstein evolution equations induce a certain flow of ${}^{(2)}h$ (or, more properly $[{}^{(2)}h]$, the conformal class of ${}^{(2)}h$)

in moduli space which is driven by (and reacts upon) the remaining dynamical fields in the reduced problem.

In any case the Hamiltonian constraint may be viewed as a non-linear elliptic equation for the conformal factor λ . For $K \approx S^2$ the existence and uniqueness of solutions for λ , with data in suitably chosen Sobolev spaces was proven in Ref.(3). This same paper showed how the coordinate conditions of a constant mean curvature slicing for $(S^2 \times \mathbb{R}, {}^{(3)}b)$ and a canonical chart for ${}^{(2)}h$ could be preserved in time by a suitable choice of the lapse function and shift vector field in the 2 + 1 dimensional metric ${}^{(3)}b$. This choice involved the solution of a certain linear elliptic system for the lapse and shift for which existence and uniqueness (up to a trivial freedom due to the lack of complete rigidity of the coordinate system) could be readily proven.

For circle bundles over $S^2 \times \mathbb{R}$ the reduced field equations could thus be shown to be simply a pure "harmonic map" system from the Lorentzian base manifold $(S^2 \times \mathbb{R}, {}^{(3)}b)$ to hyperbolic two-space where the base metric ${}^{(3)}b$ is completely determined (in terms of the independent, harmonic map fields) by the solution of the sequence of elliptic problems described above and where the norm and twist potentials provide natural coordinate functions for the "harmonic map". Of course this reduced system is not a strict harmonic map system in the sense of the usual mathematics literature since ${}^{(3)}b$ is not given a priori. Nevertheless, once ${}^{(3)}b$ is known then the equations for the norm and twist potentials are just the usual harmonic map equations defined for that particular choice of base metric.

III. OUTLOOK FOR FURTHER DEVELOPMENTS

Several natural generalizations of the results mentioned above immediately suggest themselves. First of all one would like to be able to treat the case of higher genus K . So far this has only been done for the case of the trivial bundle over $T^2 \times \mathbb{R}$ where it was used as the basis for a numerical study [4] . For T^2 or any higher genus K' one has to contend, as we have mentioned, with a non-trivial flow in moduli space which is induced by Einstein's equations. But this in itself does not disturb the picture of a reduced system of the pure "harmonic map" type. What does disturb this picture in the higher genus cases is the presence of harmonic terms in the Hodge decomposition of the Gauss law constraint. These arose in the T^2 case but there it was shown that the harmonic terms were conserved quantities which, if set to zero initially, remained zero throughout the evolution, leaving a pure harmonic map system. We expect that this pattern will hold true for arbitrary genus—the harmonic terms occur but are conserved quantities and so could always be turned off to yield a pure harmonic map system as a special case.

The generalization of the aforementioned results for $S^2 \times \mathbb{R}$ to the electro-vacuum case has almost been completed [5] and represents a straightforward extension of the same methods to the Einstein-Maxwell equations. In this case the harmonic map arising in the reduced field equations has, as target space, a certain four dimensional coset space of the group SU(2,1) in place of hyperbolic two-space. The four scalar fields which represent this harmonic map correspond to the two independent gravitational and two independent electromagnetic degrees of freedom (per spacetime point) which remain after reduction.

One interesting feature which emerged from the analysis of the vacuum fields defined over $S^2 \times \mathbb{R}$ concerns the effect of a "Geroch transformation" on the topology of the spacetime being generated. Here the term Geroch transformation refers to the well-known local invariance of the one-Killing-field Einstein equations under an action of $SL(2,\mathbb{R})$—the isometry group of hyperbolic two-space. Though this action locally preserves the Einstein equations it may disturb the integrability condition which is necessary (and sufficient) for the construction of a *global* vacuum metric on any particularly chosen bundle over the given base. This integrability condition takes the form of requiring that a certain dynamically conserved integral must take a specific integral value determined by the Chern class of the chosen bundle. The $SL(2,\mathbb{R})$ action however does not in general leave this quantity fixed. Indeed, the conserved quantity in question is simply one of the Hamiltonian generators of the group action itself and, since the group is non-abelian, fails to commute with this action.

This fact leads to the interesting possibility that one can take a solution of the field equations associated with one particular bundle, apply a suitably chosen Geroch transformation and obtain a solution associated to an inequivalent bundle. The most remarkable feature of this technique is that whereas one can transform an infinite dimensional family of solutions corresponding to the trivial bundle to solutions on any of a sequence of non-trivial bundles, there is an obstruction to transforming *all* of the solutions defined on the trivial bundle in this way. As a particular application of this technique we have shown that it is possible to transform the entire (infinite dimensional) family of *generalized Taub-NUT* solutions defined on $S^3 \times \mathbb{R}$ (which, by construction, all have *compact Cauchy horizons* at the boundaries of their maximal Cauchy developments) to solutions defined on $S^2 \times S^1 \times \mathbb{R}$ which all have *strong curvature singularities* in place of the Cauchy horizons at the boundaries of their maximal Cauchy developments [6,7].

This "topology changing" group action has its analog in the Einstein-Maxwell problem as well, at least for the case of bundles over $S^2 \times \mathbb{R}$. Does it, however, have an analog for bundles over $K \times \mathbb{R}$ for higher genus K ? If so does this action interact in some non-trivial way with the harmonic terms in the Hodge decomposition ? What is the geometrical and/or physical significance of the obstruction to transforming all solutions from the trivial to the non-trivial bundles ? How in general do the SL(2,R) transformations (or,in the electrovacuum case, the SU(2,1) transfor-mations) modify the physically or geometrically significant properties of the spacetimes to which they are applied ? The answers to these questions must await further research.

ACKNOWLEDGMENT

This research was supported in part by NSF grant 85-03072 to Yale University.

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