UNITARY SPINOR METHODS IN GENERAL RELATIVITY

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ABSTRACT

A survey is given of the structure and applications of spinor fields in three-dimensional (pseudo-) Riemannian manifolds. A systematic treatment, independent of the metric signature, is possible since there exists a fairly general structure, to be associated with unitary spinors, which encompasses all but the reality properties. The discussion begins with the algebraic and analytic properties of unitary spinors, the Ricci identities and curvature spinor, followed by the spinor adjungation as space reflection, and the SU(2) and SU(1,1) spin coefficients with some applications. The rapidly increasing range of applications includes space-times with Killing symmetries, the initial-value formulation, positivity theorems on gravitational energy and topologically massive gauge theories.

1. A LITTLE HISTORY

One of the earliest successes of spinor techniques in general relativity is Witten's [1] version of the Petrov classification which was later perfected by Penrose [2] and complemented by the spin coefficient techniques of Newman and Penrose [3]. In the following years, the $SL(2,C)$ spinors have gradually been accepted as a useful mathematical working tool in relativity, even though they never ranked to the straightforward physical utility which spinors in particle physics enjoy. The prevailing view as to why spinors sail so well in curved space-time is that they are seen closely related to null and causal structures. Other insights of more physical nature have only recently been provided by the supersymmetric theories [4].

The applications of the spinors of unitary and pseudo-unitary subgroups of the Lorentz group escaped the attention of relativists until as late as 1970. Then, in an attempt to establish a spinor approach to stationary space-times, the present author has worked out an SU(2) spinor formalism [5]. These techniques have been used for obtaining exact solutions of the gravitational field equations [6]. A corresponding formalism for the non-compact group SU(1,1) has also been developed [7]. It has been shown possible to formulate the stationary axisymmetric vacuum gravitational equations such that the complete description is carried by a set of SU(1,1) spin coefficients, and with no curvature quantities appearing.

In 1980, Sommers [8] introduced the notion of 'space spinors' in connection with SU(2) spinors relative to a space-like foliation of the space-time. This approach has been taken up by Sen [9] for quantizing the spin-3/2 massless field in a curved background.

Recently, the pace of applications has increased, due partly to the SU(2) spinorial nature of many supergravity models [4]. But SU(2) spinors enter Witten's expression for the gravitational energy [10], as well as the theory of topologically massive gauge fields [11]. A new spate of works followed Ashtekar's discovery of new canonical variables for the quantization of the gravitational degrees of freedom [12]. His configuration space is coordinatized essentially by the soldering forms of SU(2) spinors. Unitary spinors have been introduced on manifolds of arbitrary non-null congruences [13]. The ensuing description of space-time contains both the spinors in hypersurfaces and in the manifolds of Killing trajectories as special cases.

It has already been noted by Barut [14] that SU(2) and SU(1,1) spinors can be treated in a unified formalism in which one does not specify the signature of the 3-metric. The signature-independent properties, characterizing what will be called *unitary spinors*, embrace all spinorial relations but those involving adjunction. In the present note we shall show that there exists a remarkably elaborate structure shared by both genres of spinor fields in (pseudo-) Riemannian 3-spaces. Adjoining of spinors is relegated to Sec. 5. Applications in general relativity will be discussed at the end of each section.

2. UNITARY SPINORS

The algebra of unitary spinors encompasses those properties of the SU(2) and SU(1,1) spinors which do not depend on the signature of the associated metric.

2.0 DEFINITION [15] *Spin space* is a pair (Σ, ϵ) where Σ is a two-dimensional vector space over the field of complex (or real) numbers and ϵ a symplectic structure on Σ.

Note that ϵ provides an ismorphism

$$\epsilon : \Sigma \to \Sigma^* \tag{2.1}$$

between Σ and the dual space Σ^*. We shall use an index notation such that $\xi^A \in \Sigma$ is an element of the vector space Σ and $\xi_A \in \Sigma^*$ an element of the dual space. Capital Roman spinor indices $A, B, ...$ may take the values 0 and 1. Then (2.1) can be written $\xi^A \to \xi_B = \xi^A \epsilon_{AB}$ where $(\epsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Note that $\epsilon_{AB} = -\epsilon_{BA}$. The inverse map employs the spinor ϵ^{AB} satisfying

$$\epsilon^{AB} \epsilon_{CB} = \delta^A_C \tag{2.2}$$

where δ^A_C is the identity map on Σ.

2.1 DEFINITION *A spinor* $\eta^{ABC...}_{KL...}$ of p upstairs and q downstairs indices is a (p, q) tensor over Σ.

2.2 PROPOSITION The linear map $L : \Sigma \to \Sigma$ preserving the spinor ϵ_{AB} has the determinant 1. The group of the maps L is $SL(2, C)$.

The proof is all too easy.

2.3 DEFINITION The spinor

$$g_{ABCD} = \epsilon_{A(C}\epsilon_{D)B}, \tag{2.3}$$

called the *metric*, yields a bilinear map $V \otimes V \to C$ where V is the vector space

(2.4) $$V = \{v^A{}_B \,|\, v^A_A = 0\}.$$

The inclusion of V in $\Sigma \otimes \Sigma^*$ defines the map $\sigma : \Sigma \otimes \Sigma^* \to V$ rendering to each $v^A_B \in \Sigma \otimes \Sigma^*$ its trace-free part. This map is sometimes called a *soldering form* and has the index structure $\sigma^i{}_{AB}$ where

(2.5) $$\sigma^i{}_{AB} = \sigma^i{}_{BA}.$$

Lower case Roman indices refer concisely to the vector space V.

2.4 DEFINITION *Unitary spin space* is the triplet $(\Sigma, \epsilon, \sigma)$ where $\sigma : \Sigma \otimes \Sigma^* \to V$ is the map satisfying

(2.6) $$\sigma^i{}_C{}^A \sigma^j{}_B{}^C + \sigma^j{}_C{}^A \sigma^i{}_B{}^C = g^{ij} \delta^A_B$$

and $g^{ij} = \sigma^i{}_{AB} \sigma^j{}_{CD} g^{ABCD}$.

The vector space V is oriented because $Trv_1[v_2, v_3]$ defines canonically a three-form $\epsilon(v_1, v_2, v_3)$ on V [16]. In the index notation we have $\epsilon_{ijk} = \varepsilon_{ijk}\sqrt{g}$ where ε_{ijk} is the skew numerical Levi-Civita symbol and $g = det[g_{ik}]$. (Thus ϵ_{ijk} is imaginary when the signature of the metric is negative.)

2.5 COROLLARY [5] The soldering forms σ^{iA}_B satisfy the Lie product rule

(2.7) $$\sigma_{iAC}\sigma^C_{jB} - \sigma_{jAC}\sigma^C_{iB} = \sqrt{2} i \epsilon_{ijk} \sigma^k{}_{AB}.$$

Here the right-hand side changes sign under reversal of the orientation of V.

2.6 PROPOSITION Let $F_{ik} = F_{[ik]}$ be a bivector. Then the spinor $F_{ABCD} = F_{ik}\sigma^i{}_{AB}\sigma^k{}_{CD}$ has the decomposition

(2.8) $$F_{ABCD} = \frac{1}{2}(\epsilon_{BD}\phi_{AC} + \epsilon_{AC}\phi_{BD}).$$

Proof. In three dimensions, the tensor F_{ik} can be written equivalently

(2.9) $$F_i = \sqrt{1/2}\,\epsilon_{ijk}F^{jk}.$$

The (axial) vector F_i has the spinor components

$$\phi_{AB} = \sigma^i_{AB} F_i .$$

Using the identity $\sigma_i^{AB}\sigma^k_{AB} = -\delta_i^k$, we have $F_i = -\sigma_i^{AB}\phi_{AB}$. Substitution of (2.9) in ϕ_{AB} yields

$$\phi_{AB} = \sqrt{1/2}\sigma^i_{AB}\epsilon_{ijk}F^{jk} .$$

This contains the terms on the right hand side of Eq. (2.7), hence

$$\phi_{AB} = -iF_{AC}{}^{C}{}_{B} . \tag{2.10}$$

Now adding the terms $\frac{1}{2}(F_{ADCB} - F_{ADCB})$ to F_{ABCD}, it can be written identically as $F_{ABCD} = F_{A[B|C|D]} + F_{[A|D|C]B}$. For an arbitrary spinor η with a pair of skew indices we have $\eta_{AB} - \eta_{BA} = \epsilon_{AB}\eta_R{}^R$. Thus $F_{ABCD} = \frac{1}{2}(\epsilon_{BD}F_{ARC}{}^R + \epsilon_{AC}F_{RD}{}^R{}_B)$. Inserting here (2.10), the decomposition (2.8) follows.

2.7 DIGRESSION TO SL(2,C) The algebraic properties of unitary spinors can be derived [5] from the $SL(2,C)$ spinor algebra in the presence of a preferred non-null four-vector a. The soldering forms σ of $SL(2,C)$ satisfy the defining relation [3]

$$\sigma_{\mu AC'}\sigma_{\nu B}{}^{C'} + \sigma_{\nu BC'}\sigma_{\mu A}{}^{C'} = \hat{g}_{\mu\nu}\epsilon_{AB} \tag{2.11}$$

where Greek indices $\mu, \nu, \ldots$ range through the values 0, 1, 2 and 3 and primed spinor indices refer to the complex conjugate representation. A hat will signify that the entity refers to the 4-space-time whenever such an emphasis is necessary. Choose coordinates adapted to the non-null vector a such that the metric $\hat{g}_{\mu\nu}$ is independent of $x^0 = t$. Then, given a solution σ at $t = t_0$ of (2.11), this σ will continue to be a solution for all possible values of t. We may choose the soldering form σ to be independent of t.

Let us denote the norm of the vector a by

$$f = a^\mu a_\mu = \hat{g}_{00} . \tag{2.12}$$

Then the 4-metric has the decomposition

$$(\hat{g}_{\mu\nu}) = \begin{pmatrix} -f^{-1}g_{ij} + f\omega_i\omega_j & f\omega_i \\ f\omega_j & f \end{pmatrix} \tag{2.13}$$

with the inverse

$$(\hat{g}^{\mu\nu}) = \begin{pmatrix} -fg^{ij} & f\omega^i \\ f\omega^j & f^{-1} - f\omega^2 \end{pmatrix}$$

where g is the metric in the orthogonal 3-complement.

We now decompose Eq. (2.11). From the mixed components with $(\mu, \nu) = (i, 0)$ we get:

$$\text{(2.14)} \qquad \sigma^i_{AC'}\sigma_{0B}{}^{C'} + \sigma^i_{AC'}\sigma_{0B}{}^{C'} = 0.$$

The spinor $\sigma_{0AC'} = a_{AC'}$ is invariantly defined. We now introduce the soldering forms in the 3-space by defining

$$\text{(2.15)} \qquad \sigma^i_{AB} = -\frac{\sqrt{2}}{f}\sigma^i_{AC'}a_B{}^{C'}.$$

The multiplying factor here serves later convenience. Eq. (2.11) then expresses the symmetry (2.5) of the soldering forms in their spinor indices.

The components of Eq. (2.11) with $(\mu, \nu) = (i, j)$ yield the anticommutation property (2.6) of the soldering forms of unitary spinors.

The product relation of soldering forms in the 4-space-time has the form [3]

$$\text{(2.16)} \qquad \sigma_{\mu AB'}\sigma^{\mu}{}_{CD'} = \epsilon_{AC}\epsilon_{B'D'}.$$

By use of the 3+1 decomposition of scalar products

$$\text{(2.17)} \qquad \hat{u}^\mu \hat{v}^\nu = f^{-1}(u_o v_o - u^i v^k g_{ik})$$

we obtain the metric (2.3) for the 3-space:

$$\text{(2.18)} \qquad \sigma^i{}_{AB}\sigma_{iCD} = \epsilon_{A(C}\epsilon_{D)B}$$

3. SPINOR DERIVATIVES

In this section we consider covariant derivatives of unitary spinor fields on (pseudo-) Riemannian 3-manifolds.

3.1 DEFINITION A *unitary* (p,q)*- spinor field* is a local section of a bundle of unitary (p,q)-spinors over a (pseudo-) Riemannian 3-manifold $(\mathcal{M},g)$.

The covariant derivative of a spinor with in reference to the metric $\hat{g}$ is defined in the contemporary literature such that the soldering forms and the fundamental spinor ϵ_{AB} are covariantly constant.

3.2 DEFINITION The *covariant derivative* ∇ of a (p,q)-spinor field is a map into a $(p+1,q+1)$-spinor field with the usual linearity and Leibnitz properties of derivatives. The map ∇ coincides with the gradient map of scalars when $(p,q)=(0,0)$. and has the further properties

(3.1)
$$\nabla_k \sigma^i{}_A{}^B = 0, \qquad \nabla_i \epsilon_{AB} = 0.$$

3.3 DEFINITION The spinor affine connection $\Gamma^{\;C}_{\mu A}$ in the covariant derivative

(3.2)
$$\nabla_i \xi_A = \xi_{A,i} - \Gamma^{\;C}_{iA}\,\xi_C$$

can be expressed in the form [5]

(3.3)
$$\Gamma^{\;B}_{iA} = -\frac{1}{2}\sigma_j{}^{BC}(\sigma^j_{\;AC,i} + \sigma^k_{\;AC}\Gamma^j_{\;ik}).$$

3.4 PROPOSITION The *spinor Ricci identity* has the form

(3.4)
$$\nabla_{P(B}\nabla^P_{C)}\xi_A = \phi_{ABCD}\,\xi^D + 2\epsilon_{A(B}\,\xi_{C)}\Lambda$$

Proof. Eq. (3.4) can be derived from the Ricci identity

(3.5)
$$(\nabla_k\nabla_j - \nabla_j\nabla_k)v_i = R_{rijk}v^r$$

by choosing v_i to be the null vector $v_i = \sigma_i^{AB}\,\xi_A\,\xi_B$. Convecting with $\sigma^i_{\ BN}\,\sigma^j_{\ DQ}\,\sigma^k_{\ CP}$ and using (2.15) we get

$$(\nabla_{CP}\nabla_{DQ} - \nabla_{DQ}\nabla_{CP})(\xi_B\,\xi_N) = -R_{AMBNCPDQ}\,\xi^A\,\xi^M \tag{3.6}$$

where $\nabla_{AB} = \sigma^i_{\ AB}\,\nabla_i$ and the spinor structure of the Riemann tensor R_{ijkl} is given by $R_{AMBNCPDQ} = \sigma^i_{\ AM}\,\sigma^j_{\ BN}\,\sigma^k_{\ CP}\,\sigma^l_{\ DQ}\,R_{ijkl}$. The curvature tensor can be decomposed into irreducible parts by use of the scheme (2.8) for pairs of skew indices:

$$\begin{aligned} R_{AMBNCPDQ} = \\ & - \epsilon_{AB}\,\epsilon_{CD}\,\phi_{MNPQ} - \epsilon_{AB}\,\epsilon_{PQ}\,\phi_{MNCD} - \epsilon_{MN}\,\epsilon_{CD}\,\phi_{ABPQ} - \epsilon_{MN}\,\epsilon_{PQ}\,\phi_{ABCD} \\ & - \Lambda[\epsilon_{MN}\,\epsilon_{PQ}\,(\epsilon_{AC}\,\epsilon_{BD} + \epsilon_{AD}\,\epsilon_{BC}) + \epsilon_{MN}\,\epsilon_{CD}\,(\epsilon_{AP}\,\epsilon_{BQ} + \epsilon_{AQ}\,\epsilon_{BP}) \\ & + \epsilon_{AB}\,\epsilon_{PQ}\,(\epsilon_{MC}\,\epsilon_{ND} + \epsilon_{MD}\,\epsilon_{AC}) + \epsilon_{AB}\,\epsilon_{CD}\,(\epsilon_{MP}\,\epsilon_{NQ} + \epsilon_{MQ}\,\epsilon_{NP})] \end{aligned} \tag{3.7}$$

where $\phi_{ABCD} = \frac{1}{2}(R_{ik} - \frac{1}{3}g_{ik}R)\sigma^i_{\ AB}\,\sigma^k_{\ CD}$, $R_{ik} = R_{irsk}\,g^{rs}$ is the Ricci tensor and $\Lambda = R/24$. Transvecting Eq. (3.6) with $\epsilon^{PQ}\,\eta^B\,\eta^N$ where η^N is an arbitary non-vanishing spinor field, and removing the overall factor $2(\eta^B\,\xi_B)\eta^N$, we obtain the spinor version (3.4) of the Ricci identity.

2.4 DIGRESSION TO SL(2,C) One can decompose the connection $\hat{\Gamma}$ in terms of Γ. Introduce the complex *gravitational vector* [5]

$$G_i = \frac{1}{2}\Big(\frac{f_{,i}}{f} + i\varepsilon_{ikl}\sqrt{|g|}\omega^{k;l} f\Big). \tag{3.8}$$

This can be expressed in spinor terms as

$$G_{AB} = \sigma^i_{\ AB}\,G_i = \hat{\nabla}^{B'}_{(A}\,a_{B)B'}. \tag{3.9}$$

Straightforward computation yields

$$\hat{\Gamma}_{0A}^{\ \ C} = -\frac{1}{\sqrt{2}} f G^B_A$$

$$\hat{\Gamma}_{iA}^{\ \ C} = \Gamma_{iA}^{\ \ C} - \frac{1}{\sqrt{2}}\omega_i f G^B_A - i\frac{1}{\sqrt{2}}\epsilon_{ikl}\,G^k\,\sigma^l{}_A{}^B\,\sqrt{|g|}. \tag{3.10}$$

4. ADJUNCTION

The spinor $a_{AC'}$ establishes, in a natural way, a map between the primed and unprimed spin spaces. The 'unpriming' of spinor indices proceeds by contraction of any primed spinor index with $\sqrt{\frac{2}{|f|}}a^{AB'}$. As an example, the unprimed version $\hat{v}_{AB} = \sqrt{\frac{2}{|f|}}a_B{}^{B'}\hat{v}_{AB'}$ of the vector $\hat{v}$ has the irreducible parts

$$\hat{v}_{(AB)} = v^i\sigma_{iAB}, \qquad \hat{v}_{[AB]} = \epsilon_{AB}v_o.$$

Thus the 3+1 decomposition of tensors becomes a symmetry operation over spinor indices.

4.0 DEFINITION The *adjoint spinor* $\xi^{\dagger A}$ is defined by

(4.1)
$$\xi^{\dagger A} \stackrel{\text{def}}{=} \sqrt{\frac{2}{|f|}}a^{AB'}\bar{\xi}_{B'}.$$

4.1 PROPOSITION The double adjoint has the property

(4.2)
$$(\xi^\dagger)^{\dagger A} = \begin{cases} -\xi^A & \text{if } f > 0 \ \textit{(time-like a)}, \\ \xi^A & \text{if } f < 0 \ \textit{(space-like a)}. \end{cases}$$

Proof. Contraction of Eq. (3.7) with $a^\mu a^\nu$ yields the product rule for the $a_{AB'}$ spinors:

(4.3)
$$a_{AC'}a_B{}^{C'} = \frac{f}{2}\delta_A^B.$$

4.3 DEFINITION The *norm* of the spinor ξ is defined by

(4.4)
$$\|\xi\|^2 \stackrel{\text{def}}{=} \xi^{\dagger A}\xi_A.$$

4.4 PROPOSITION The spinor norm (4.4) is real.

Proof. The complex conjugation of scalar products proceeds as follows,

(4.5)
$$\begin{aligned}\overline{(\xi_A\eta^A)} &= \bar{\xi}_{A'}\bar{\eta}^{A'} = \frac{2}{f}\bar{\xi}_{A'}a_C^{A'}a^{CB'}\bar{\eta}_{B'} \\ &= \pm\,\xi_A^\dagger\eta^{\dagger A}\end{aligned}$$

where the upper and lower sign holds for $f > 0$ and $f < 0$, respectively. In particular,

(4.6) $$\overline{\xi^{\dagger}_A \eta^A} = \pm \xi^{\dagger\dagger}_A \eta^{\dagger A} = \xi^A \eta^{\dagger}_A .$$

4.4 PROPOSITION The soldering forms have the Hermiticity property

(4.7) $$\sigma^{i\dagger}_{AB} = \mp \sigma^{i}_{AB} .$$

Proof. Eq. (4.7) follows from the Hermiticity of the SL(2,C) soldering forms and from (2.12).

Although the timelike case carries here a negative sign, this is the kind of behavior a Hermitian product $\sigma_{AB} = \xi_{(A}\xi^{\dagger}_{B)}$ exhibits.

4.5 PROPOSITION The covariant derivation commutes with adjunction.

The **Proof** is straightforward and will be omitted here.

When the four-vector a is space-like, 3-spinors can be chosen real, but the adjunction will retain a direct geometrical interpretation, to be discussed in the next section.

5. TRIAD FORMALISM

Consider a spinor $\eta \in \Sigma$ of positive norm. Such a spinor always exists and can be scaled to

(5.1) $$\eta_A \tilde{\eta}^A = 1.$$

(In this section we shall adopt the tilde notation of Ref. [11] for adjoints.) Then

(5.2) $$l^i \stackrel{\text{def}}{=} \sqrt{2}\eta_A \sigma^{i}{}_{B}{}^{A} \tilde{\eta}^B$$

is a real and self-adjoint unit vector: $\tilde{l}^i = l^i$ and

(5.3) $$m^i \stackrel{\text{def}}{=} \eta_A \sigma^{i}{}_{B}{}^{A} \eta^B$$

a null vector (real in the space-like case) with $\tilde{m}^i$ the adjoint null vector. Adjunction may be expressed in geometric terms as a swop of the basis spinors η and $\tilde{\eta}$, or as an exchange of the vectors m and $\tilde{m}$ while leaving the vector l intact.

The triad of vectors

$$(z_{\mathrm{p}}{}^i) = \{l^i, m^i, \tilde{m}^i\}, \qquad \mathrm{p} = \mathrm{o}, +, -, \tag{5.4}$$

forms a normalized basis in the vector space V. The 3-metric acquires the form

$$(g_{\mathrm{p\,q}}) = (z_{\mathrm{p}\,i} z_{\mathrm{q}}{}^i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.5}$$

5.1 DEFINITION The *Ricci rotation coefficients* are the invariants

$$\gamma_{\mathrm{p\,q\,r}} = \nabla_k z_{\mathrm{p}\,i} z_{\mathrm{q}}{}^i z_{\mathrm{r}}{}^k. \tag{5.6}$$

The Ricci rotation coefficients will be given the individual notation [11]

mn / p	$-$o	+o	+$-$
o	$\tilde{\kappa}$	κ	ϵ
+	$\tilde{\rho}$	σ	$-\tilde{\tau}$
$-$	$\tilde{\sigma}$	ρ	τ

Table 1. *The rotation coefficients.*

5.2 DEFINITION The *triad derivatives* acting on scalars are defined

$$\partial_{\mathrm{p}} = z_{\mathrm{p}}{}^i \partial_i.$$

We also use the detailed notation

(5.7) $$D = \partial_o \qquad \delta = \partial_+ \qquad \tilde{\delta} = \partial_- .$$

The triad derivatives have the commutators

(5.8) $$\begin{aligned} D\delta - \delta D &= (\tilde{\rho} + \epsilon)\delta + \sigma\tilde{\delta} + \kappa D \\ \delta\tilde{\delta} - \tilde{\delta}\delta &= \tilde{\tau}\tilde{\delta} - \tau\delta + (\rho - \tilde{\rho})D. \end{aligned}$$

The Ricci identities have the detailed form

(5.9) $$\begin{aligned} &D\sigma - \delta\kappa = \tilde{\tau}\kappa + \kappa^2 + \sigma(\rho + \tilde{\rho} + 2\epsilon) - 2\phi_{++} \\ &D\rho - \tilde{\delta}\kappa = \tilde{\kappa}\kappa - \kappa\tau + \sigma\tilde{\sigma} + \rho^2 - \phi_{oo} \\ &D\tau - \tilde{\delta}\epsilon = \tilde{\kappa}\rho - (\kappa + \tilde{\tau})\tilde{\sigma} + \epsilon(\tilde{\kappa} - \tau) + \rho\tau - 2\phi_{o-} \\ &\tilde{\delta}\sigma - \delta\rho = 2\sigma\tau + \kappa(\rho - \tilde{\rho}) + 2\phi_{o+} \\ &\delta\tau + \tilde{\delta}\tilde{\tau} = \rho\tilde{\rho} - \sigma\tilde{\sigma} + 2\tau\tilde{\tau} - \epsilon(\rho - \tilde{\rho}) - 2\phi_{+-} + \phi_{oo} \end{aligned}$$

where

(5.10) $$\phi_{\mathrm{pq}} = \frac{1}{2}(R_{\mathrm{pq}} - \frac{1}{3}g_{\mathrm{pq}}R).$$

5.3 DIGRESSION TO SL(2,C) One can decompose the null tetrad of Newman & Penrose [3] in terms of the triad. The null tetrad is fixed by the $\{o_A, \iota_A\}$ basis in the spin space, normalized by $o_A\,\iota^A = 1$. This defines a tetrad

(5.11). $$\begin{aligned} \hat{l}^\mu &= o_A\,\sigma^{\mu A B'}\bar{o}_{B'} \\ \hat{m}^\mu &= o_A\,\sigma^{\mu A B'}\bar{\iota}_{B'} \\ \hat{n}^\mu &= \iota_A\,\sigma^{\mu A B'}\bar{\iota}_{B'} \end{aligned}$$

The decomposition will depend on the relation of the o, ι and the $\eta, \tilde{\eta}$ basis in the spin space. These spinor bases are are locally connected by an $SL(2,C)$ rotation. It is sometimes convenient [5] to choose, for example,

(5.12) $$o_A = \left(\frac{2}{f}\right)^{1/4}\eta_A, \qquad \iota_A = \left(\frac{f}{2}\right)^{1/4}\eta^\dagger_A.$$

We then have

$$
\begin{aligned}
\hat{l}^{\mu} &= \{l^{i}, f^{-1} - (l^{j}\omega_{j})\} \\
\hat{m}^{\mu} &= \sqrt{|f|}\{m^{i}, -(m^{j}\omega_{j})\}\,. \\
n^{\mu} &= \frac{1}{2} f \hat{l}^{\mu} + a^{\mu}
\end{aligned}
\tag{5.13}
$$

The spin coefficients of Newman and Penrose [3] have the decomposition

$$
\begin{aligned}
\hat{\epsilon} &= \frac{1}{4}(2\epsilon + G_{o} - \bar{G}_{o}) & \hat{\kappa} &= f^{-1/2}(\kappa - 2G_{+}) \\
\hat{\rho} &= \rho + G_{o} & \hat{\pi} &= \frac{1}{2} f^{1/2}\tilde{\kappa} \\
\hat{\sigma} &= \sigma & \hat{\tau} &= -\frac{1}{2} f^{1/2}\kappa \\
\hat{\lambda} &= \frac{1}{2} f \tilde{\sigma} & \hat{\alpha} &= \frac{1}{4} f^{1/2}(2\tau + G_{-} - \bar{G}_{-}) \\
\hat{\mu} &= \frac{1}{2} f(\tilde{\rho} + G_{o}) & \hat{\beta} &= -\frac{1}{4} f^{1/2}(2\tilde{\tau} + 3G_{+} + \bar{G}_{+}) \\
\hat{\nu} &= \frac{1}{4} f^{3/2}(-\tilde{\kappa} + 2G_{-}) & \hat{\gamma} &= -\frac{1}{8} f(2\epsilon - 3G_{o} - \bar{G}_{o})
\end{aligned}
\tag{5.14}
$$

and the Weyl curvature spinor is given by

$$
\begin{aligned}
\Psi_{0} &= 2[\delta G_{+} - \sigma G_{o} + \tilde{\tau} G_{+} + (2G_{+} + \bar{G}_{+})G_{+}] \\
\Psi_{1} &= -f^{1/2}[DG_{+} - \kappa G_{o} - \epsilon G_{+} + (2G_{o} + \bar{G}_{o})G_{+}] \\
\Psi_{2} &= \frac{1}{2} f[DG_{o} + \tilde{\kappa} G_{+} + \kappa G_{-} + (G_{+} + \bar{G}_{o})G_{o} - 2G_{+}G_{-}] \\
\Psi_{3} &= \frac{1}{2} f^{3/2}[DG_{-} - \tilde{\kappa} G_{o} + \epsilon G_{-} + (2G_{o} + \bar{G}_{o})G_{-}] \\
\Psi_{4} &= \frac{1}{2} f^{2}[\tilde{\delta} G_{-} - \tilde{\sigma} G_{o} + \tau G_{-} + (2G_{-} + \bar{G}_{-})G_{-}].
\end{aligned}
\tag{5.15}
$$

In a *vacuum space-time*, the Ricci tensor of the 3-space is determined by Einstein's gravitational equations as follows [5]: $R_{\mathrm{p}\,\mathrm{q}} + G_{\mathrm{p}}\bar{G}_{\mathrm{q}} + \bar{G}_{\mathrm{p}}G_{\mathrm{q}} = 0$. The triad components G_{p} of the gravitational vector satisfy the vacuum field equations [5]

$$
\begin{aligned}
DG_{o} + \tilde{\delta} G_{+} + \delta G_{-} &= (\rho + \tilde{\rho})G_{o} - (\kappa - \tilde{\tau})G_{-} - (\tilde{\kappa} - \tau)G_{+} \\
&\quad + (G_{o} - \bar{G}_{o})G_{o} + (G_{+} - \bar{G}_{+})G_{-} + (G_{-} - \bar{G}_{-})G_{+} \\
DG_{-} - \tilde{\delta} G_{o} &= (\rho - \epsilon)G_{-} + \tilde{\sigma} G_{+} + \tilde{\kappa} G_{o} - \bar{G}_{o}G_{-} + \bar{G}_{-}G_{o} \\
DG_{+} - \delta G_{o} &= \sigma G_{-} + (\tilde{\rho} + \epsilon)G_{+} + \kappa G_{o} - \bar{G}_{o}G_{+} + \bar{G}_{+}G_{o} \\
\delta G_{-} - \tilde{\delta} G_{+} &= (\tilde{\rho} - \rho)G_{o} - \tau G_{+} + \tilde{\tau} G_{-} - \bar{G}_{+}G_{-} + \bar{G}_{-}G_{+}\,.
\end{aligned}
\tag{5.16}
$$

The choice of the triad has so far been left arbitrary. A convenient orientation for the triad vector l is along the *eigenrays* of the gravitational field. These eigendirections are [5] the principal null directions of the gravitational spinor given by the canonical decomposition

$$G_{AB} = \chi_{(A}\eta_{B)}. \tag{5.17}$$

The eigenray condition can be written in triad terms as $G_{+} = 0$. An eigenray congruence in the 3-space is geodesic if and only if the corresponding null congruence with tangent four-vector $\hat{l}$ is geodesic. Solutions of the vacuum gravitational equations with geodesic eigenrays have been found in Ref. [6].

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