THE MOTION OF MEMBRANES IN SPACE–TIME

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ABSTRACT

The equation of a hypersurface interacting with external fields in space-time is established. A class of models is based on the properties of a symmetric divergence-free distribution-valued stress tensor for the interacting system. The elastodynamic properties are encoded into the induced metric and shape tensor of the immersion together with invariants constructed from the Weingarten map. Two particular examples are discussed that are relevant to recent developments in the theories of relativistic extended particles.

Introduction

The simplest matter models that enter into the description of standard cosmological space-times (g,M) are given in terms of a unit time-like future pointing vector field V and scalars ρ and p that define the matter stress tensor

$$\tilde{T} = \left(\rho + \frac{p}{c^2}\right) V \otimes V + \frac{p}{c^2} g \qquad (1)$$

It is well known that for a gas of collisionless dust (p=0) the condition that $\tilde{T}$ be divergence free implies that V must be a geodesic vector field. Thus the integral curves of V extremalise the integral of arc length and are models for the world lines of point particles. Recent activity has focussed attention on models for extended particles; either as basic entities for hypothetical unified models [1] or as alternative matter models. Particular attention has been devoted to 2 and 3 dimensional immersions in a (pseudo–) Riemannian manifold. In their simplest form such immersions have vanishing mean curvature. Their ("superspace") extensions[2] offer plausable avenues towards a unification programme. Other investigations have begun to study particular extrinsic geometrical properties of immersions as well as their gravitational and electromagnetic interactions.

In [3] simple models of extended matter that provide a foliation of space-time were studied in the context of Einstein's equations. It is of interest, however, to derive the equation of motion of a single immersion without necessarily expecting it to be the leaf of any particular foliation.

In [4] we studied a model for a "charged bubble" that generalised an earlier model discussed by Dirac[5] to describe the muon as an extended particle. In this note a general class of distribution–valued stress tensors will be presented and these models will be recovered as special cases.

We are interested here in the equations of motion of hypersurfaces in space-time that have a Lorentzian induced metric, interacting with prescribed external fields.

I. Discontinuities in the stress tensor

Suppose C: $D \rightarrow$ M is a time-like 3–dimensional immersion into space-time, D being some 3–dimensional parameter domain. Denote by h the shape tensor of the image Σ of C and $\Pi_N g$ its induced metric. We shall suppose that locally the hypersurface is given by the equation $\Phi = 0$ and that I and II label the disjoint regions of space-time given by $\Phi > 0$ and $\Phi < 0$ respectively. In terms of Φ the unit normal to a regular hypersurface is given by $\widetilde{N} = d\Phi / | d\Phi |$ where $|\ \ |$ denotes the norm with respect to g.In the following, a tilda over a tensor will denote the tensor associated to it by the space-time metric g. Let Y_Φ be the Heaviside function on M with support on $\Phi > 0$. Introduce further, the distribution δ_Φ by the relation

$$d\,Y_\Phi = \delta_\Phi \;\, d\Phi \tag{2}$$

and suppose that the total (2,0) symmetric stress tensor (including all matter and fields) takes the form

$$\tilde{T} = \tilde{T}_I Y_\Phi + \tilde{T}_{II} Y_{-\Phi} + \tilde{T}_{III} \delta_\Phi \tag{3}$$

where the coefficients of Y and δ_Φ are smooth tensors. $\tilde{T}_{III}$ will describe the stress properties of the immersion. Now for any scalar β on M we have:

$$\nabla \cdot \, (\beta\tilde{T}) = \tilde{T}(d\beta,-) + \beta\nabla\cdot\tilde{T} \tag{4}$$

Thus $\tilde{T}$ is divergence free if $\nabla\cdot\tilde{T}_{II} = 0$ in I, $\nabla\cdot\tilde{T}_{II} = 0$ in II and

$$[\tilde{T}]\,(d\Phi,-) = -\,\nabla\cdot\,\tilde{T}_{III}\,\big|_\Sigma \tag{5}$$

$$\tilde{T}_{III}(d\Phi,-)\,\big|_\Sigma = 0 \tag{6}$$

where $[\tilde{T}] \equiv \tilde{T}_I - \tilde{T}_{II}$ denotes the discontinuity of $\tilde{T}$ across $\Phi = 0$. Since $d\Phi$ is space-like we may interpret (5) as saying that the jump in the local momentum current 1–form $[J_{d\Phi}] = -\,[\widetilde{\tilde{T}(d\Phi,-)}]$ gives the normal force on the hypersurface. (In terms of any time-like observer with unit vector V tangent to his world line, $i_V * J$ is the local force 2–form associated with any momentum current 1–form J, $*$ being the Hodge map assoicated with g).

Now for any (in general mixed) tensor S we may define $\Pi_N S$ by

$$(\Pi_N S)(X_1, X_2, \dots, \alpha_1, \alpha_2, \dots) = S(\Pi_N X_1, \Pi_N X_2, \dots \tilde{\Pi}_N \alpha_1, \tilde{\Pi}_N \alpha_2 \dots) \quad (7)$$

where $\Pi_N = 1 - \tilde{N}(N)\, N \otimes \tilde{N}$ and $\tilde{\Pi}_N = 1 - \tilde{N}(N)\, \tilde{N} \otimes N$ are (1,1) projection tensors. $\Pi_N S$ will be said to be orthogonal to N, or N–orthogonal. Hence (6) says that the hypersurface stress tensor must be orthogonal to its normal field.

It is straightforward to project $\nabla \cdot \tilde{T}_{III}$ into its normal and tangental parts with respect to Π_N:

$$\nabla \cdot \tilde{T}_{III} = \Pi_N(\nabla \cdot \tilde{T}_{III}) + \tilde{N}(\nabla \cdot \tilde{T}_{III})N \quad (8)$$

where as usual

$$(\nabla \cdot \tilde{T}_{III}) = \sum_{a=0}^{3} (\nabla_{X_a} \tilde{T}_{III})(e^a, -) \quad (9)$$

$\{X_a\}$, $\{e^a\}$ being arbitrary dual bases with X_0 time-like. Since $\Pi_N \tilde{T}_{III} = \tilde{T}_{III}$

$$\tilde{N}(\nabla \cdot \tilde{T}_{III}) = (\nabla_{X_a} \tilde{T}_{III})(e^a, \tilde{N}) = -T_{III}(e^a, \nabla_{X_a} \tilde{N}). \quad (10)$$

But

$$\Pi_N(\nabla_{\Pi_N X} N) = -A_N(\Pi_N X) \quad (11)$$

where A_N is the Weingarten map.. In terms of the shape tensor with h = HN :

$$g(h(X,Y), N) = g(A_N X, Y) = H(X,Y) \quad (12)$$

Hence

$$\tilde{N}(\nabla \cdot \tilde{T}_{III}) = \tilde{T}_{III}(e^a, \widetilde{A_N X_a}) \equiv Tr(A_N T^{\sim}) \quad (13)$$

where $T^{\sim}$ is the associated 1–1 tensor.

Thus if we write

$$[J_{d\Phi}] = \tilde{\Pi}_N [J_{d\Phi}] + N([J_{d\Phi}])\tilde{N} \quad (14)$$

equation (5) gives as normal and tangental parts

$$[J_{d\Phi}](N) = Tr(A_N T^{\sim}_{III})\,|_{\Sigma} \quad (15)$$

and

$$\Pi_N [\tilde{J}_{d\Phi}] = \Pi_N(\nabla \cdot \tilde{T}_{III})\,|_{\Sigma} \quad (16)$$

where $\tilde{T}_{III} = \Pi_N \tilde{T}_{III}$.

II. Geometrical Elastodynamics

We assign to the "material" hypersurface $\Phi = 0$ (with space-like $d\Phi$) the $d\Phi$–orthogonal symmetric tensor

$$T_{III} = |d\Phi| \{ L_1(\Pi_N g) + L_2(\Pi_N H) \} \quad . \tag{17}$$

The scalars L_1 and L_2 may be chosen to depend on a set of geometric invariants κ_1, κ_2, κ_3, $\Delta\kappa_1$, $\Delta\kappa_2$, $\Delta\kappa_3$, ... Δ here denotes the Laplacian (with respect to $\Pi_N g$). The invariants κ_i may be taken as the elementary symmetric functions of the Weingarten map eigenvalues. Thus if $\{X_i\}$ i = 0,2,3 is a local g–orthonormal basis of vector fields tangent to Σ such that

$$A_N X_i = \lambda_i X_i \qquad i = 0,2,3$$

then we define

$$\kappa_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$\kappa_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$\kappa_3 = \lambda_1\lambda_2\lambda_3 \; .$$

We shall consider two examples that have featured in recent investigations.

For some constant κ consider

$$T_{III} = \kappa |d\Phi| \Pi_N g$$
$$\equiv \kappa |d\Phi| (g - \tilde{N} \otimes \tilde{N}) \tag{18}$$

It follows immediately from (12) that

$$\tilde{N}(\nabla \cdot \Pi_N g) = H(\Pi_N X^a, \Pi_N X_a) \equiv 3|\eta| \tag{19}$$

where $\eta = \frac{1}{3} \sum_{a=0,1,2} h(\Pi_N X^a, \Pi_N X_a) \quad ; \qquad X^a = g^{ab} X_b$

is the mean curvature normal of the hypersurface. Thus if $\tilde{T}_I$ and $\tilde{T}_{II}$ are zero such an immersion is extremal in the sense that it extremalises the induced "volume" of the immersion. If, however, for the parameter domain *D* we take the topology $S^2 \times R$ (so that region II is the world tube traced out by the interior of a space-like bubble) and couple this membrane to the Maxwell field F by taking $F_{II} = 0$ and $T_I = T_{Maxwell}$, then

$$[J_{d\Phi}] = -\frac{1}{2} * \left(i_{\widetilde{d\Phi}} F^I_{\wedge} * F^I - i_{\widetilde{d\Phi}} * F^I_{\wedge} F^I \right) \tag{20}$$

where F^I is the external field of the charged membrane. In Minkowski space-time with F^I the Coulomb field of a spherical bubble (in its proper frame), (15) gives us the equation of motion:

$$\kappa |\eta| = \frac{1}{2} *^{-1} (F^I_\wedge * F^I) |_\Sigma \tag{21}$$

This is the classical equation first studied by Dirac[5] in connection with his particle model.

Instead of (18) we may include the shape tensor in T_{III}. In particular suppose

$$T_{III} = \kappa |d\Phi| \left\{ (\mathrm{Tr}\, A_N) \Pi_N g - \Pi_N H \right\} \quad . \tag{22}$$

It follows from the Gauss equation that $\forall$ X,Y :

$$T_{III} (A_N X, Y) = \kappa |d\Phi| \left\{ (\mathrm{Tr}\, A_N) \Pi_N g\, (A_N X, Y) - \Pi_N g \left(A_N^2 X, Y\right) \right.$$

$$= \kappa |d\Phi| \left\{ \mathrm{Ric}\, (\Pi_N X, \Pi_N Y) - \mathbf{Ric}\, (\Pi_N X, \Pi_N Y) \right\} \tag{23}$$

where Ric is the Ricci tensor of $(\Pi_N g, \Sigma)$ and **Ric** the Ricci tensor of (g, M). Hence

$$\tilde{N} (\nabla \cdot \tilde{T}_{III}) = \kappa |d\Phi| (R - \mathbf{R}) \tag{24}$$

in terms of Gaussian curvatures. If such a membrane is electromagnetically coupled to the Maxwell field as before we obtain the equation of motion (in Minkowski space-time):

$$\kappa R = \frac{1}{2} *^{-1} (F^I_\wedge * F^I) |_\Sigma \, . \tag{25}$$

It may be shown that this is the variational equation that defines an extremum of the integral[4]

$$A\,[C] = -9\kappa \int_C |\eta| \hat{*} 1 + \frac{1}{2} \int_{M_I} F_\wedge * F \tag{26}$$

where $\hat{*}\, 1$ is the induced "volume" 3–form on Σ. In Euclidean spaces such uncharged (F=0) membranes gives rise to Willmore immersions[6]. Such immersions have an extensive mathematical literature. The charged Lorentzian model above forms the basis of an improved semi-classical estimate of the electron–muon mass–ration [7]. The quantum mechanics of immersions with generalised stresses given classsically by (17) is an important unsolved problem.

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