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It was a truly splendid
structure and Yossarian
throbbed with a mighty
sense of accomplishment
each time he gazed at
it and reflected that
none of the work that
had gone into it was his.

Catch 22

Joseph Heller

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PREFACE

The following notes represent a course of lectures delivered at the Australian National University in the second semester of 1982 as part of the mathematics honours programme. Most of the material contained in the notes is standard although a few new refinements and variations are included. The course consisted of twenty six one-hour lectures and this sufficed to present about ninety five per cent of the content of the notes.

I am indebted to Mayda Shahinian of the University of New South Wales for typing a first draft of part of these notes and to Helen Daish of the Australian National University for typing the final draft.

1.1. Introduction.

Continuous one-parameter semigroups of bounded operators occur in many branches of mathematics, both pure and applied. The calculus of functions of one real variable can be formulated in terms of the translation semigroup, solutions of the equations connected with classical phenomena such as heat propagation are described by semigroups, and one-parameter groups and semigroups also describe the dynamics of quantum mechanical systems. Although semigroups occur in many other areas the development and scope of the general theory covered in this chapter is well illustrated by the foregoing examples. Hence we begin with a brief discussion of each of them.

The semigroup of right translations on $C_0(\mathbb{R})$, the continuous functions over the real line which vanish at infinity, is defined by

$$f \in C_0(\mathbb{R}) \mapsto S_t f \in C_0(\mathbb{R}) ,$$

where

$$(S_t f)(x) = f(x-t) .$$

Thus one has the semigroup property

$$S_s S_t = S_{s+t} , \qquad s, t \geq 0$$

and

$$S_0 = I$$

where I is the identity operator. Moreover S is strongly continuous, i.e.,

$$\lim_{t\to 0+} \|S_t f - f\|_\infty = 0 , \qquad f \in C_0(\mathbb{R}) ,$$

where $\|\cdot\|_\infty$ indicates the supremum norm. Infinitesimally the action of this semigroup is left differentiation and globally S corresponds in some sense to the exponential of the differentiation operator, e.g., if f is analytic

$$(S_t f)(x) = \sum_{n\geq 0} \frac{(-t)^n}{n!} \frac{d^n f(x)}{dx^n} .$$

Alternatively, the passage from the infinitesimal action $-d\ dx$ to the semigroup S can be described as integration,

$$(S_t f)(x) = f(x-t) = \int_t^\infty ds\left(-\frac{d}{ds}\right) f(x-s) .$$

Thus this example illustrates how differentiation, integration, and approximation theory, underlie the general theory of one-parameter semigroups.

An alternative way of describing the translates $S_t f$ of a function $f \in C_0(\mathbb{R})$ are as solutions of the first-order partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} = 0 ,$$

in $C_0(\mathbb{R}^2)$, and this is the natural way of viewing the second example, the heat semigroup.

The heat equation

$$\frac{\partial f}{\partial t}(x, t) = \frac{\partial^2 f}{\partial x^2}(x, t)$$

describes the infinitesimal change with time t of the spatial heat distribution of an idealized one-dimensional rod. If $f \in C_0(\mathbb{R})$ describes the initial heat distribution, $f(x) = f(x, 0)$, of the infinitely long rod then at time t it is described by the solution $T_t f \in C_0(\mathbb{R})$ of the above equation,

$$f(x, t) = (T_t f)(x) = (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dy\, e^{-(x-y)^2/4t} f(y) .$$

Again $T = \{T_t\}_{t\geq 0}$ is a strongly continuous semigroup of bounded operators acting on $C_0(\mathbb{R})$, with $T_0 = I$ and once more this semigroup corresponds to exponentiation of the operator $-\partial^2/\partial x^2$ describing the infinitesimal heat flow, e.g., if f is analytic

$$(T_t f)(x) = \sum_{n\geq 0} \frac{t^n}{n!} \frac{\partial^{2n}}{\partial x^{2n}} f(x) .$$

Thus solution of the heat equation can be viewed as construction of the semigroup from its infinitesimal action.

The translation semigroup and heat semigroup may be integrated on other function spaces such as $L^p(\mathbb{R})$, $p \in [1, \infty]$, but there are also interesting evolution equations in more general spaces than function spaces. For example the theory of quantum mechanics can be phrased in terms of observables $A, B, C, \ldots$ which are bounded operators on a Hilbert space $\mathcal{H}$ and the change $(A, t) \mapsto A_t$ of these observables with time is given by the Heisenberg equation of motion

$$\frac{\partial A_t}{\partial t} = i\left(HA_t - A_t H\right) ,$$

where H is a self-adjoint operator, the Hamiltonian, and $A_0 = A$. Formally the solution of this equation is

$$A_t = U_t \, AU_{-t}$$

where U_t describes the solution of the Schrödinger equation

$$\frac{\partial \psi_t}{\partial t} = i \, H\psi_t$$

on the Hilbert space $\mathcal{H}$, i.e., $\psi_t = U_t \psi_0$. Thus the evolution of the quantum mechanical observables is described by a semigroup $A_t = S_t A$ acting on the space of all bounded operators $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. The infinitesimal action of the semigroup is given by

$$A \mapsto \delta(A) = i(HA-AH)$$

and solution of the Heisenberg equations of motion again corresponds to 'exponentiation' of this action.

The general problem of semigroup theory is to study differential equations of the form

$$\frac{\partial a_t}{\partial t} + Ha_t = 0$$

under a variety of circumstances, to establish criteria for existence of solutions, to develop constructive methods of solution, and to analyze stability properties of the solutions. Each of these aspects will be discussed in this chapter. Formally the solution is always $a_t = \exp\{-tH\}a$ and the key problem is to define the exponential of the infinitesimal operator H. But there are also several important subsidiary factors to consider.

The translation semigroup and the quantum-mechanical semigroup, which were briefly sketched above, both extend to one-parameter groups which are isometric, e.g., $\|S_t f\|_\infty = \|f\|_\infty$ for all $f \in C_0(\mathbb{R})$. The heat semigroup cannot be extended in this manner but it is nevertheless contractive, i.e., $\|T_t f\|_\infty \leq \|f\|_\infty$. In the context of dynamics these conditions of isometry and contraction are connected with conservation laws, e.g., the contractive property of the heat equation reflects the fact that no heat is created in the isolated system, but it can dissipate. Continuity properties are also important. The translation semigroup is strongly continuous on any of the spaces $C_0(\mathbb{R})$ or $L^p(\mathbb{R})$ with $p \in [1, \infty\rangle$ but this is certainly not the case on $L^\infty(\mathbb{R})$. Nevertheless one has the residual continuity property

$$\lim_{t\to 0+} \left(S_t f,\ g\right) = \lim_{t\to 0+} \int_{-\infty}^{\infty} dx \left(S_t f\right)(x) g(x)$$

$$= \int_{-\infty}^{\infty} dx\ f(x)g(x) = (f,\ g)$$

for all $f \in L^{\infty}(\mathbb{R})$ and $g \in L^{1}(\mathbb{R})$. Since L^{∞} is the dual of L^{1} this corresponds to weak*-continuity, i.e., weak continuity with respect to the predual. Similarly the heat semigroup is only weak*-continuous on $L^{\infty}(\mathbb{R})$ and the quantum-mechanical semigroup is weak*-continuous on $\mathcal{L}(H)$. Finally each of these semigroups is positive in a natural sense; the translation semigroup and the heat semigroup map positive functions into positive functions, and the quantum-mechanical semigroup maps positive operators into positive operators. Again this form of positivity can often be interpreted in terms of physical conservation laws.

Motivated by these examples we concentrate in this chapter on strongly continuous contraction semigroups and partially describe the theory of weak*-continuous semigroups and groups of isometries. In Chapter 2 we examine positive semigroups.

1.2. Semigroups and Generators.

Let $\mathcal{B}$ be a complex Banach space and $\mathcal{B}^*$ its dual. We denote elements of $\mathcal{B}$ by a, b, c, ... and elements of $\mathcal{B}^*$ by f, g, h, ... Moreover we use (f, a) to denote the value of f on a and $\|\cdot\|$ to denote the norm on $\mathcal{B}$ and also the dual norm on $\mathcal{B}^*$, i.e.,

$$\|f\| = \sup\{|f(a)| \; ; \; \|a\| \leq 1\} .$$

A *semigroup* S on $\mathcal{B}$ is defined to be a family $S \; ; \; t \in \mathbb{R}_+ \mapsto S_t \in \mathcal{L}(\mathcal{B})$ of bounded linear operators on $\mathcal{B}$ which satisfy

1. $S_s S_t = S_{s+t}$, $\quad s, t \geq 0$,

2. $S_0 = I$

where I denotes the identity operator on $\mathcal{B}$.

This notion of semigroup is not of great interest unless one imposes some further hypothesis of continuity. There are a variety of possible forms of continuity. Let us first consider continuity at the origin.

The strongest possible requirement would be uniform continuity, i.e.,

$$\lim_{t \to 0+} \|S_t - I\| = 0 ,$$

where the operator norm is defined in the usual manner

$$\|S_t - I\| = \sup\{\|S_t a - a\| \ ; \ \|a\| \le 1\} \ .$$

But this is a very restrictive assumption. It can be established that a semigroup is uniformly continuous at the origin if, and only if, there exists a bounded operator H such that

$$S_t = I + \sum_{n\ge 1} \frac{(-t)^n}{n!} H^n = \exp\{-tH\}$$

(see Exercise 1.2.1). This is of limited interest in applications. Nevertheless we occasionally use uniformly continuous matrix semigroups for illustrative purposes.

A weaker continuity requirement is strong continuity at the origin, i.e.,

$$\lim_{t\to 0+} \|(S_t - I)a\| = 0$$

for all $a \in \mathcal{B}$. Semigroups with this property are usually called C_0-*semigroups* and we adopt this notation throughout the sequel. The heat semigroup on $C_0(\mathbb{R})$ is a semigroup of this type. Note that if S is a C_0-semigroup then it follows from the principle of uniform boundedness (see Exercise 1.2.2) that

$$\|S_t\| \le Me^{\omega t}$$

for some $M \ge 1$ and some finite $\omega \ge \inf_{t>0} \left(t^{-1} \log\|S_t\|\right)$. In particular this implies that strong continuity of S at the origin is equivalent to strong continuity at all $t \ge 0$. This follows from the easy estimate

$$\|(S_{s+t}-S_s)a\| \le \|S_s\| \, \|(S_t-I)a\|$$
$$\le Me^{\omega s}\|(S_t-I)a\| \; .$$

Moreover it establishes that the analysis of a general C_0-semigroup can be reduced to the analysis of an M-bounded C_0-semigroup, i.e., a semigroup satisfying

$$\|S_t\| \le M \; .$$

This reduction is effected by replacing S_t by $S_t e^{-\omega t}$. The case $M = 1$ is of particular importance.

A C_0-semigroup S for which each S_t is contractive, i.e.,

$$\|S_t\| \le 1 \; , \qquad t \ge 0 \; ,$$

is called a C_0-*semigroup of contractions*. The foregoing discussion of boundedness properties indicates that the theory of contractive semigroups is very close to the general theory. Nevertheless there are some significant differences which lead to complications if $M > 1$ and there are a number of techniques which are only applicable to the contractive case $M = 1$, $\omega = 0$. Consequently for simplicity of exposition and diversity of method we restrict the ensuing discussion to contraction semigroups.

Before proceeding to the detailed discussion of C_0-semigroups we note that there are other weaker forms of continuity which are of interest. One continuity hypothesis, which is natural from the mathematical point of view, is weak continuity at the origin. By this we mean

$$(*) \qquad \lim_{t\to 0+} \left(f, S_t a\right) = (f, a)$$

for all $a \in \mathcal{B}$, and all $f \in \mathcal{B}^*$. But here an unexpected simplification occurs; *every weakly continuous semigroup is automatically strongly continuous* (see Exercise 1.2.3). Alternatively one could make the weaker hypothesis that (*) is valid for all $a \in \mathcal{B}$ and all f in some 'large' subspace of $\mathcal{B}^*$. In particular if $\mathcal{B}$ has a predual, i.e., if $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$, then one could suppose that (*) holds for all $a \in \mathcal{B}$ and all $f \in \mathcal{B}_*$. This hypothesis is referred to as weak*-continuity and a semigroup that satisfies it is called a C_0^**-semigroup*. This notation is appropriate because it follows by duality that each C_0^*-semigroup on $\mathcal{B}$ is the dual of a C_0-semigroup acting on the predual $\mathcal{B}_*$. Hence many facets of the theory of C_0^*-semigroups can be deduced by duality from the C_0-case. The group of translations acting on $L^\infty(\mathbb{R}; dx)$ is an example of a C_0^*-group which is not a C_0-group; it is the dual of the C_0-group of translations acting on $L^1(\mathbb{R}; dx)$. We consider the basic theory of C_0^*-semigroups of contractions in Section 1.6.

The most important concept in the theory of continuous semigroups is that of the (infinitesimal) generator. This generator is defined as the (right) derivative of the semigroup at the origin where the sense in which the derivative is taken is dictated by the continuity hypothesis. In particular the generator of a C_0-semigroup is defined as the strong derivative. The detailed definition is as follows.

If S is a C_0-semigroup on the Banach space $\mathcal{B}$ the *(infinitesimal) generator* of S is defined as the linear operator H on $\mathcal{B}$ whose domain $D(H)$ consists of those $a \in \mathcal{B}$ for which there exists a $b \in \mathcal{B}$ with the property that

$$\lim_{t\to 0+} \left\| \frac{(I-S_t)}{t} a - b \right\| = 0 .$$

If $a \in D(H)$ the action of H is defined by

$$Ha = b .$$

Note that the semigroup property of S automatically implies $S_t D(H) \subseteq D(H)$ and

$$HS_t a = S_t Ha$$

for all $a \in D(H)$ and $t \geq 0$. Moreover one has the differential equation

$$\frac{dS_t}{dt} a = -HS_t a = -S_t Ha$$

for each $a \in D(H)$, where the strong derivative dS_t/dt is defined by

$$\frac{dS_t}{dt} a = \lim_{h\to 0} \frac{(S_{t+h}-S_t)}{h} a$$

whenever the limit exists. It also follows that S and H are connected by the integral equation

$$S_t a - a = - \int_0^t ds\ S_s Ha = - \int_0^t ds\ HS_s a$$

for each $a \in D(H)$. The integrals, both here and throughout the sequel, are understood as $\mathcal{B}$-valued Riemann integrals.

We now derive the basic properties of generators and their resolvents.

Recall that the *resolvent set* $r(H)$ of an operator H on $\mathcal{B}$ is the set $\lambda \in \mathbb{C}$ for which $\lambda I - H$ has a bounded inverse, the spectrum $\sigma(H)$ of H is the complement of $r(H)$ in $\mathbb{C}$, and if $\lambda \in r(H)$ then $(\lambda I - H)^{-1}$ is called the *resolvent* of H .

PROPOSITION 1.2.1. *Let* S *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *with generator* H .

It follows that

1. H *is norm closed, norm densely defined,*

2. *If* $\mathrm{Re}\lambda < 0$ *the range* $R(\lambda I - H)$ *of* $\lambda I - H$ *satisfies*

$$R(\lambda I - H) = \mathcal{B}$$

and for $a \in D(H)$

$$\|(\lambda I - H)a\| \geq |\mathrm{Re}\lambda|\ \|A\| ,$$

3. *If* $\mathrm{Re}\lambda < 0$ *the resolvent of* H *is given by the*

Laplace transform

$$(\lambda I-H)^{-1}a = -\int_0^\infty ds\; e^{\lambda s}S_s a\;, \qquad a \in \mathcal{B}\;.$$

In particular $\sigma(H) \subseteq \{\lambda\;;\; \mathrm{Re}\lambda \geq 0\}$.

Proof. Since $\mathrm{Re}\lambda < 0$ we may define a bounded operator $R_\lambda(H)$ on $\mathcal{B}$ by

$$R_\lambda(H)a = -\int_0^\infty ds\; e^{\lambda s}S_s a\;, \qquad a \in \mathcal{B}\;.$$

Explicitly one has

$$\begin{aligned}\|R_\lambda(H)a\| &\leq \int_0^\infty ds\; e^{-s|\mathrm{Re}\lambda|}\|S_s a\| \\ &\leq \int_0^\infty ds\; e^{-s|\mathrm{Re}\lambda|}\|a\| = |\mathrm{Re}\lambda|^{-1}\|a\|\;.\end{aligned}$$

But for each $a \in \mathcal{B}$ one also has

$$\begin{aligned}t^{-1}(I-S_t)R_\lambda(H)a &= -t^{-1}\int_0^\infty ds\; e^{\lambda s}(S_s-S_{s+t})a \\ &= -t^{-1}\int_0^\infty ds\; e^{\lambda s}(1-e^{-\lambda t})S_s a - t^{-1}\int_0^t ds\; e^{\lambda(s-t)}S_s a \\ &\xrightarrow[t\to 0+]{} \lambda R_\lambda(H)a - a\end{aligned}$$

where both integrals converge in norm. This last conclusion uses the strong continuity of S and the Lebesgue dominated theorem. It follows that $R_\lambda(H)a \in D(H)$ and

$$(\lambda I-H)R_\lambda(H)a = a\;.$$

In particular

$$R(\lambda I-H) = \mathcal{B} .$$

But since

$$S_t R_\lambda(H) = R_\lambda(H) S_t$$

and $R_\lambda(H)$ is bounded one finds that

$$(\lambda I-H)R_\lambda(H)a = R_\lambda(H)(\lambda I-H)a = a$$

for $a \in D(H)$. Hence $\lambda \in r(H)$ and

$$(\lambda I-H)^{-1} = R_\lambda(H) .$$

But boundedness of $(\lambda I-H)^{-1}$ implies that $\lambda I - H$, and hence H , is norm closed. Moreover the explicit estimate for $\|R_\lambda(H)a\|$ derived at the beginning of the proof immediately gives the desired lower bound on $\|(\lambda I-H)a\|$.

Finally $a_n = -nR_n(H)a \in D(H)$ for all $a \in \mathcal{B}$ and $n \geq 1$. But

$$\begin{aligned} a_n - a &= n \int_0^\infty ds\, e^{-ns}\left(S_s - I\right)a \\ &= \int_0^\infty ds\, e^{-s}\left(S_{\frac{s}{n}} - I\right)a \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

by another application of strong continuity and the Lebesgue dominated convergence theorem. Thus $D(H)$ is norm dense. □

This result has two simple implications which we often use in the sequel without further comment. First the proposition implies that for each $\alpha > 0$ generators satisfy

(*) $$\|(I+\alpha H)a\| \geq \|a\| \ , \qquad a \in D(H) \ .$$

But it immediately follows that *the generator* H *of a* C_0*-semigroup* S *has no proper extension satisfying* (*), i.e., generators are in this sense maximal. To deduce this suppose $\hat{H}$ extends H and also satisfies (*) then for $a \in D(\hat{H})$ set $b = (I+\alpha\hat{H})a$. But there is an $a' \in D(H)$ such that $b = (I+\alpha H)a'$, by Condition 2 of Proposition 1.2.1, and hence $(I+\alpha\hat{H})(a-a') = 0$, because $\hat{H}$ extends H. Thus $a = a'$ by (*) and $\hat{H} = H$. The second implication gives a characterization of a core of H. Recall that a subset D of the domain $D(X)$ of an operator X is called a core of X if for each $a \in D(X)$ there is a sequence $a_n \in D$ such that $\|a_n - a\| \to 0$ and $\|Xa_n - Xa\| \to 0$ as $n \to \infty$. In particular if X is closed then D is a core if, and only if, the norm closure $X|_D$, of X restricted to D, is equal to X. It follows that *a subset* $D \subseteq D(H)$ *is a core for the generator* H *if, and only if,* $(\lambda I-H)D$ *is norm dense in* $\mathcal{B}$ *for some* λ *with* $\mathrm{Re}\lambda < 0$, *or for all* λ *with* $\mathrm{Re}\lambda < 0$. Clearly if D is a core $\overline{R(\lambda I-H)D} = \mathcal{B}$ by Proposition 1.2.1. Conversely if $\hat{H}$ denotes the closure of $H|_D$ and $R(\lambda I-\hat{H}) = \mathcal{B}$ one again concludes that $A = H$ by use of (*).

A slight variation of the argument used to prove Proposition 1.2.1 also provides the following slightly less evident criterion for a core of a generator.

COROLLARY 1.2.2. *Let* S *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *with generator* H *and let* D *be a subset of the domain* $D(H)$ *of* H *which is norm dense and invariant under* S, *i.e.,* $S_t a \in D$ *for all* $a \in D$ *and* $t \geq 0$.

It follows that D *is a core for* H.

Proof. Let $\hat{H}$ denote the closure of $H|_D$. By the above remarks it suffices to prove that $R(\lambda I-\hat{H}) = \mathcal{B}$ for some λ with $\mathrm{Re}\lambda < 0$. But for $a \in D$ one can choose Riemann approximants

$$\Sigma_N(a) = -\sum_{i=1}^{N} e^{\lambda s_i} S_{s_i} a(s_{i+1}-s_i)$$

$$\Sigma_N((\lambda I-H)a) = -\sum_{i=1}^{N} e^{\lambda s_i} S_{s_i} (\lambda I-H)a(s_{i+1}-s_i)$$

which converge simultaneously to $(\lambda I-H)^{-1}a$ and a. Now $\Sigma_N(a) \in D$ because of the invariance of D under S and

$$(\lambda I-H)\,\Sigma_N(a) = \Sigma_N((\lambda I-H)a) .$$

Thus $\Sigma_N(a) \to (\lambda I-H)^{-1}a$ and $(\lambda I-H)\,\Sigma_N(a) \to a$. Therefore $D \subseteq R(\lambda I-H)$. But $(\lambda I-H)^{-1}$ is bounded and hence $R(\lambda I-\hat{H})$ is norm closed. Thus $R(\lambda I-\hat{H}) = D$ by the norm density of D. □

Exercises.

1.2.1. Prove that if a semigroup S is uniformly continuous then there exists a bounded operator H such that

$$S_t = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n .$$

Hint: For small $s > 0$ the operator

$$H_s = s^{-1} \int_0^s dt\, S_t$$

is invertible, with bounded inverse, and

$$\left(I-S_t\right)H_s/t = \left(I-S_s\right)H_t/s .$$

1.2.2. Prove that a weakly continuous semigroup S must satisfy

$$\|S_t\| \le Me^{\omega t}$$

for some $M \ge 1$ and some finite $\omega \ge \inf_{t\ge 0}\left(t^{-1} \log\|S_t\|\right)$.

Hint: Use the uniform boundedness principle for small t and the semigroup property for large t .

1.2.3. Verify that if $\operatorname{Re} z > 0$ then

$$t \mapsto \left(S_t f\right)(x) = (4\pi t z)^{-\nu/2} \int d^\nu y \; e^{-(x-y)^2/4tz} f(y)$$

defines a C_0-semigroup on $C_0\left(\mathbb{R}^\nu\right)$ satisfying

$$\|S_t\|_\infty = \left(|z|/\operatorname{Re} z\right)^{\nu/2} .$$

1.2.4. Prove that weak and strong continuity of a semigroup S are equivalent.

Hint: The weak generator H_w of a weakly continuous semigroup S is defined by

$$\left(f, H_w a\right) = \lim_{t\to 0+} \left(f, \left(I-S_t\right)a\right)/t$$

with $D(H_w)$ the set of a for which the limit exists for all $f \in \mathcal{B}^*$. Adapt the argument used in the proof of Proposition 1.2.1 to deduce that $D(H_w)$ is weakly dense and hence, by the Hahn-Banach theorem, strongly dense. Finally use

$$(f, a-S_t a) = \int_0^t ds\ (f, S_s H_w a)$$

to prove strong continuity for all $a \in D(H_w)$.

1.2.5. Prove that the generator H and weak generator H_w of a C_0-semigroup S coincide.

Hint: Adapt the proof of Proposition 1.2.1 to deduce that $H \supseteq H_w$.

1.2.6. If H is the generator of a C_0-semigroup prove that

$$D_\infty(H) = \bigcap_{n\geq 1} D(H^n)$$

is norm dense.

Hint: For each a define a_n by

$$a_n = \int_0^\infty dt\ f(t) S_{t/n}\, a$$

where f is a positive, infinitely often differentiable, function with compact support in $\langle 0, \infty\rangle$ and with total integral one. Then $a_n \in D_\infty(H)$ and $\|a_n - a\| \to 0$ as $n \to \infty$.

1.2.7. Let S denote the heat semigroup on $L^p(\mathbb{R}^\nu)$,

$$(S_t f)(x) = (4\pi t)^{-\nu/2} \int d^\nu y \; e^{-(x-y)^2/4t} f(y) \; .$$

Prove that the generator of S is the closure of the restriction of the Laplacian

$$-\nabla^2 = -\sum_{i=1} \frac{\partial^2}{\partial x_i^{\,2}}$$

to the infinitely often differentiable functions in $L^p(\mathbb{R}^\nu)$.

Hint: Use Corollary 1.2.2.

1.3. Generators and Semigroups.

Proposition 1.2.1 states necessary conditions for an operator to generate a C_0-semigroup of contractions. Next we examine sufficient conditions and also study the construction of a semigroup from its generator.

The problem of characterizing a generator H is equivalent to the problem of proving existence and uniqueness of global solutions of a differential equation

$$\frac{da_t}{dt} + Ha_t = 0 , \qquad a_t = a$$

for all a in a suitable Banach space $\mathcal{B}$. Formally the solution of the differential equation is

$$a_t = \exp\{-tH\}a$$

and the difficulty is to give an appropriate definition of the exponential. Various algorithms and approximation techniques are of use. For example the algorithm

$$\exp\{-tx\} = \lim_{n\to\infty} (1+tx/n)^{-n}$$

for the numerical exponential can be extended to an operator relation if the (pseudo-) resolvent $(I+\alpha H)^{-1}$ has suitable properties for small positive α .

It should perhaps be emphasized that in applications the Banach space $\mathcal{B}$ is not necessarily specified in advance. Typically one might encounter a differential equation of the above type for functions over some measure space but without specification of a particular norm. Thus the problem consists of choosing the norm and reinterpreting the operator H such that an appropriate solution can be found.

The first basic result which characterizes generators is the following:

THEOREM 1.3.1. (Hille-Yosida). *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. H *is the infinitesimal generator of a* C_0*-semigroup of contractions* S ,

2. H *is norm closed, norm densely defined:*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ *(or for one* $\alpha = \alpha_0 > 0$*)* :

$$\|(I+\alpha H)a\| \geq \|a\|$$

for all $a \in D(H)$ *and all* $\alpha > 0$ *(or for all* $\alpha \in \langle 0, \alpha_0]$*)* .

If these conditions are satisfied

$$\lim_{n\to\infty} \|S_t a - (I+tH/n)^{-n} a\| = 0$$

for all $a \in \mathcal{B}$, *uniformly for* t *in any finite interval of* $[0, \infty\rangle$.

Proof. $1 \Rightarrow 2$. This follows from Proposition 1.2.1, it suffices to set $\lambda = -1/\alpha$.

$2 \Rightarrow 1$. Assume $R(I+\alpha_0 H) = \mathcal{B}$ and $\|(I+\alpha H)a\| \geq \|a\|$ for all $a \in D(H)$ and $\alpha \in \langle 0, \alpha_0]$. Thus $(I+\alpha_0 H)^{-1}$ is a bounded operator with norm one. First we extend this conclusion to all $\alpha \in \langle \alpha_0/2, \alpha_0]$ and then by iteration to all $\alpha \in \langle 0, \alpha_0]$.

If $\alpha \in \langle \alpha_0/2, \alpha_0]$ then

$$R_N = \left(\frac{\alpha_0}{\alpha}\right) \sum_{n=0}^{N} \left(\frac{\alpha-\alpha_0}{\alpha}\right)^n (I+\alpha_0 H)^{-n-1}$$

converges in norm to a bounded operator R . But for $a \in D(H)$ one has $R_N a \in D(H)$ and a simple rearrangement argument proves that $\|(I+\alpha H)R_N a - a\| \to 0$ and $\|R_N(I+\alpha H)a - a\| \to 0$ as $N \to \infty$. Since H is norm closed it follows that $R = (I+\alpha H)^{-1}$ and then $\|R\| \leq 1$ by the bound $\|(I+\alpha H)a\| \geq \|a\|$.

The remainder of the proof consists of establishing that the strong limit of the operators

$$r_n(t) = (I+tH/n)^{-n}$$

exists as $n \to \infty$ and that it defines a C_0-semigroup of contractions with generator H . Note that for n sufficiently large $t/n < \alpha_0$ and $\|(I+\alpha H)^{-1}\| \leq 1$ for all α relevant to the remainder of the proof.

As a preliminary to studying the above limit we note that if $a \in D(H)$

$$\|(I+\alpha H)^{-1}a - a\| = \alpha\|(I+\alpha H)^{-1}Ha\|$$
$$\leq \alpha\|Ha\| \underset{\alpha\to 0+}{\longrightarrow} 0 \ .$$

Since $D(H)$ is dense one concludes that $(I+\alpha H)^{-1}$ converges strongly to the identity as $\alpha \to 0+$. This has several implications.

First if for $a \in \mathcal{B}$ one defines

$$a_n = (I+H/n)^{-2}a$$

then $a_n \in D(H^2)$ and a_n is norm convergent to a. Thus $D(H^2)$ is dense. Second if $a \in D(H)$ then a_n converges to a and Ha_n also converges to Ha. Thus $D(H^2)$ is a core for H. Third $r_n(t)$ is strongly convergent to the identity as $t \to 0$.

Next one claculates that $dr_n(t)/dt$ is bounded for $t > 0$ and, more specifically,

$$\frac{dr_n(t)}{dt} = -H(I+tH/n)^{-n-1} \ .$$

Combining these facts one calculates that if $a \in D(H^2)$ then

$$r_n(t)a - r_m(t)a = \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds \frac{d}{ds}\left\{r_n(s)r_m(t-s)a\right\}$$
$$= \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds \left\{r_n'(s)r_m(t-s)a - r_n(s)r_m'(t-s)a\right\}$$
$$= \lim_{\varepsilon\to 0+} \int_\varepsilon^{t-\varepsilon} ds\, r_n(s)r_m(t-s) \left\{-H(I+sH/n)^{-1}a + H(I+(t-s)H/m)^{-1}a\right\}$$
$$= \int_0^t ds \left(\frac{s}{n} - \frac{t-s}{m}\right)(I+sH/n)^{-n-1}(I+(t-s)H/m)^{-m-1}H^2a \ .$$

This immediately yields the estimate

$$\|r_n(t)a - r_m(t)a\| \le \frac{t^2}{2}\left(\frac{1}{n}+\frac{1}{m}\right)\|H^2a\| .$$

Thus $\{r_n(t)a\}_{n\ge1}$ is a Cauchy sequence which is norm convergent, uniformly for t in any finite interval of $[0, \infty\rangle$. But since $D(H^2)$ is norm dense, and $\|r_n(t)\| \le 1$ for all $n = 1, 2, \ldots$, it follows that $\{r_n(t)\}_{n\ge1}$ is strongly convergent, uniformly for t in any finite interval of $[0, \infty\rangle$. If $S = \{S_t\}_{t\ge0}$ denotes the strong limit one readily deduces that $S_0 = I$, $t \in \mathbb{R}_+ \mapsto S_t \in \mathcal{L}(B)$ is strongly continuous, and $\|S_t\| \le 1$. To establish the semigroup property we use the combinatoric identity

$$x^n - y^n = \sum_{m=1}^{n} x^{n-m}(x-y)y^{m-1} .$$

Hence for $a \in D(H^2)$ one calculates that

$$\begin{aligned} r_n(s)r_n(t)a - r_n(s+t)a &= \sum_{m=1}^{n} (I+sH/n)^{-n+m}(I+tH/n)^{-n+m}(I+(s+t)H/n)^{-m+1} \times \\ &\qquad \times \left\{(I+sH/n)^{-1}(I+tH/n)^{-1} - (I+(s+t)H/n)^{-1}\right\}a \\ &= \sum_{m=1}^{n} (I+sH/n)^{-n+m-1}(I+tH/n)^{-n+m-1}(I+(s+t)H/n)^{-m}\frac{st}{n^2}H^2a . \end{aligned}$$

Therefore

$$\|r_n(s)r_n(t)a - r_n(s+t)a\| \le \frac{st}{n}\|H^2a\| .$$

In the limit $n \to \infty$ one finds

$$S_s S_t a = S_{s+t} a$$

and the semigroup property follows from the density of $D(H^2)$ and the contractivity of S.

It remains to identify the generator of S.

Again one calculates for $a \in D(H^2)$

$$t^{-1}\big(r_n(t)-I\big)a + Ha = t^{-1}\sum_{m=0}^{n-1}(I+tH/n)^{-m}\big((I+tH/n)^{-1}-I\big)a + Ha$$

$$= -\frac{1}{n}\sum_{m=1}^{n}\big((I+tH/n)^{-m}-I\big)Ha$$

$$= \frac{t}{n^2}\sum_{m=1}^{n}\sum_{p=1}^{m}(I+tH/n)^{-p}H^2 a$$

Consequently

$$\|t^{-1}\big(r_n(t)-I\big)a + Ha\| \le \frac{t(n+1)}{2n}\|H^2 a\|$$

and in the limit $n \to \infty$

$$\|t^{-1}\big(S_t-I\big)a + Ha\| \le t\|H^2 a\| .$$

Thus if $\hat{H}$ denotes the generator of S then

$$\hat{H}a = Ha$$

for all $a \in D(H^2)$. But $D(H^2)$ is a core for H and hence $\hat{H}$ is an extension of H. This, however implies that $(I+\alpha\hat{H})^{-1}$

is an extension of $(I+\alpha H)^{-1}$ for all small $\alpha > 0$. Since the latter operator is everywhere defined it is not possible that $\hat{H}$ is a strict extension of H . Therefore $\hat{H} = H$. □

There are a number of possible variations of the Hille-Yosida theorem. It follows from Condition 1 of the theorem that H is norm closed but for the implication $2 \Rightarrow 1$ it is not necessary to assume the closedness since it follows from the other hypotheses of Condition 2. For example if $a_n \in D(H)$, $\|a_n - a\| \to 0$, and $\|Ha_n - b\| \to 0$ then there is a c such that

$$(I+\alpha H)c = a + \alpha b ,$$

by the range condition, and consequently

$$\|c - a_n\| \le \|(I+\alpha H)(c-a_n)\| \to 0 ,$$

by the lower bound. Hence $c = a$, $b = Ha$, and H is norm closed. This redundancy will reoccur, without comment, in several of the subsequent statements.

The Hille-Yosida theorem can also be rephrased as a criterion for an operator to be a *pre-generator*, i.e., a closable operator whose closure is a generator.

THEOREM 1.3.2. *Let* H *be a norm densely defined operator on the Banach space* $\mathcal{B}$ *and assume that*

$$\|(I+\alpha H)a\| \ge \|a\|$$

for all $a \in D(H)$ *and all* $\alpha \in \langle 0, \alpha_0]$ *, for some* $\alpha_0 > 0$.

It follows that H *is norm closable and the following conditions are equivalent:*

1. *The closure* $\overline{H}$ *of* H *is the generator of a* C_0*-semigroup of contractions,*

2. $$\overline{R(I+\alpha H)} = \mathcal{B}$$

   for one $\alpha \in \langle 0, \alpha_0]$, *where the bar denotes norm closure.*

Proof. If $a_n \in D(H)$, $\|a_n\| \to 0$, and $\|Ha_n - b\| \to 0$, then H is norm closable if, and only if, $b = 0$. Now suppose $a' \in D(H)$ and $b' = Ha'$ then

$$\|(I+\alpha H)(a_n + \alpha a')\| \geq \|a_n + \alpha a'\|$$

for $\alpha \in \langle 0, \alpha_0]$. Therefore taking the limit over n and subsequently dividing by α one finds

$$\|b + a' + \alpha b'\| \geq \|a'\| .$$

Hence

$$\|a' + b\| \geq \|a'\| .$$

But $D(H)$ is norm dense and so for each $\varepsilon > 0$ one can choose a' such that $\|b + a'\| < \varepsilon$ and $\|a'\| \geq \|b\|$. Therefore $\|b\| < \varepsilon$ and $b = 0$.

Next suppose $a_n \in D(H)$, $\|a_n - a\| \to 0$, and $\|Ha_n - \overline{H}a\| \to 0$ then

$$\|(I+\alpha\overline{H})a\| = \lim_{n\to\infty} \|(I+\alpha H)a_n\|$$

$$\geq \lim_{n\to\infty} \|a_n\| = \|a\| \ .$$

Moreover if $c \in \mathcal{B}$ and one chooses $c_n \in R(I+\alpha H)$ such that $\|c_n - c\| \to 0$ then $c_n = (I+\alpha H)a_n$ for some $a_n \in D(H)$ and

$$\|c_n - c_m\| = \|(I+\alpha H)(a_n - a_m)\|$$

$$\geq \|a_n - a_m\| \ .$$

Therefore a_n must be a convergent sequence. But

$$\|H(a_n-a_m)\| \leq \alpha^{-1}\Big\{\|(I+\alpha H)(a_n-a_m)\| + \|a_n - a_m\|\Big\}$$

$$= \alpha^{-1}\Big\{\|c_n - c_m\| + \|a_n - a_m\|\Big\}$$

and consequently Ha_n is also convergent. Hence if $\|a_n - a\| \to 0$ then $a \in D(\overline{H})$ and $\|Ha_n - \overline{H}a\| \to 0$ because H is norm closable. Thus

$$c = (I+\alpha\overline{H})a$$

and this establishes that

$$R(I+\alpha\overline{H}) = \overline{R(I+\alpha H)} \ .$$

Therefore Conditions 1 and 2 are equivalent by the Hille-Yosida theorem. □

Remark 1.3.3. Results analogous to Theorem 1.3.1 and 1.3.2 are valid for general C_0-semigroups. For example if one replaces the lower bound in Condition 2 of Theorem 1.31 by the set of lower bounds

$$(*)\quad \|(I+\alpha H)^n a\| \geq M^{-1}(1-\alpha\omega)^n\|a\| \ , \qquad a \in D(H^n) \ , \quad n = 1, 2, 3, \ldots$$

for all $\alpha \in \langle 0, \omega]$ and repeats the proof of $2 \Rightarrow 1$ then the new bounds give the estimates

$$\|r_n(t)\| \leq M(1-\alpha\omega)^{-n}$$

and one readily concludes that H generates a C_0-semigroup S satisfying

$$\|S_t\| \leq Me^{\omega t} \ .$$

Conversely if S satisfies these bounds then the lower bounds (*) follow from the Laplace transforms

$$(I+\alpha H)^{-n}a = \frac{1}{n!}\int_0^\infty dt\ t^n e^{-t} S_{\alpha t} a \ .$$

Remark 1.3.4. If S is a C_0-semigroup with generator H it is customary to write

$$S_t = e^{-tH} \ .$$

This is justified by the definition of the generator and also by the construction of Theorem 1.3.1. Moreover if H is bounded S_t coincides with $\exp\{-tH\}$ defined as a uniformly convergent power series.

The Hille-Yosida theorem can be reformulated in a much neater manner: H *is the generator of a* C_0*-semigroup of contractions if, and only if,* $(I+\alpha H)^{-1}$ *is a bounded contraction operator for all sufficiently small positive* α . Nevertheless it is useful to identify explicitly the two pieces of information which are contained in the statement that $(I+\alpha H)^{-1}$ is a bounded contraction operator, the range condition

$$R(I+\alpha H) = \mathcal{B} ,$$

and the lower bounds

$$\|(I+\alpha H)a\| \geq \|a\| , \qquad a \in D(H) .$$

These latter lower bounds can often be re-expressed in quite different terms. They are related to the maximum principle when applied to differential operators and to a spectral property for operators on Hilbert space. In the next section we discuss the interpretation of these bounds as a criterion of dissipation. But for the present we adopt the terminology that *the operator* H *is norm-dissipative if*

$$\|(I+\alpha H)a\| \geq \|a\|$$

for all $a \in D(H)$ *and all small* $\alpha > 0$.

The following example illustrates this concept for elliptic differential operators.

Example 1.3.5. (The Laplace Operator). Let $\mathcal{B} = C_0(\mathbb{R}^\nu)$, the space of continuous functions over $\mathbb{R}^\nu$ which vanish at infinity, equipped with the usual supremum norm. The Laplace operator $-\nabla^2$ is defined on $C_0^2(\mathbb{R}^\nu)$, the twice continuously differentiable functions in $C_0(\mathbb{R}^\nu)$, by

$$-\nabla^2 a = -\sum_{i=1}^{\nu} \frac{\partial^2 a}{\partial x_i^2} ,$$

and one has the obvious identity

$$2|\underline{\nabla} a|^2 - \nabla^2|a|^2 = (-\nabla^2\bar{a})a + \bar{a}(-\nabla^2 a) .$$

Therefore if $\alpha > 0$

$$\begin{aligned} |(1-\alpha\nabla^2)a|^2 &= |a|^2 + \alpha^2|\nabla^2 a|^2 + \alpha(-\nabla^2\bar{a})a + \alpha\bar{a}(-\nabla^2 a) \\ &= |a|^2 + \alpha^2|\nabla^2 a|^2 + 2\alpha|\underline{\nabla} a|^2 - \alpha\nabla^2|a|^2 \\ &\geq |a|^2 - \alpha\nabla^2|a|^2 . \end{aligned}$$

Now if $|a|$ has a maximum at $x = x_0$ then the maximum principle states that $x \mapsto -\nabla^2|a(x)|^2$ is non-negative at $x = x_0$. Therefore the preceding estimate establishes that

$$\begin{aligned} \|(I-\alpha\nabla^2)a\|_\infty^2 &\geq |(1-\alpha\nabla^2)a(x_0)|^2 \\ &\geq |a(x_0)|^2 = \|a\|_\infty^2 , \end{aligned}$$

i.e., the Laplace operator is $\|\cdot\|_\infty$-dissipative. A similar conclusion is true for more general elliptic operators by the same calculation. □

Now let us examine operators on Hilbert space. In this case one has

$$\|(I+\alpha H)a\|^2 = \|a\|^2 + \alpha^2\|Ha\|^2 + \alpha(Ha, a) + \alpha(a, Ha)$$
$$= \|a\|^2 + \alpha^2\|Ha\|^2 + 2\alpha \operatorname{Re}(a, Ha) .$$

Therefore H is norm-dissipative if, and only if,

$$\operatorname{Re}(a, Ha) \geq 0$$

for all $a \in D(H)$. But under certain quite general circumstances these latter conditions are equivalent to a spectral property of H . For example if H is bounded and normal, i.e., if H commutes with its adjoint H^* , these conditions are equivalent to

$$\operatorname{Re} \sigma(H) \geq 0 .$$

This follows by a numerical range argument. Define the numerical range $W(H)$ of H by

$$W(H) = \{(a, Ha) \ ; \ a \in D(H)\} .$$

If H is bounded then the Hausdorff-Stone theorem establishes that $W(H)$ is convex. If, moreover, H is normal then the closure $\overline{W(H)}$ of $W(H)$ coincides with the convex closure of $\sigma(H)$. Therefore in this latter case $\operatorname{Re} W(H) \geq 0$ if, and only if, $\operatorname{Re} \sigma(H) \geq 0$. This conclusion can be extended to unbounded generators of normal semigroups.

Example 1.3.6. (Normal Semigroups). Let $S = \{S_t\}_{t\geq 0}$ be a C_0-semigroup acting on a Hilbert space $\mathcal{H}$. The adjoints

$S^* = \{S_t^*\}_{t\geq 0}$ form a weakly, hence strongly (Exercise 1.2.3), continuous semigroup called the *adjoint semigroup*. The semigroup S is defined to be *normal* if S_s and S_t^* commute for all $s, t > 0$ and *self-adjoint* if $S_t = S_t^*$ for all $t > 0$. Note that $\|S_t\| = \|S_t^*\|$ and hence S and S^* are simultaneously contractive. Moreover if H generates S then the adjoint H^* of H generates S^* (Exercise 1.3.4).

If S *is contractive then* $\mathrm{Re}\,\sigma(H) \geq 0$ *and if* S *is normal it is contractive if, and only if,* $\mathrm{Re}\,\sigma(H) \geq 0$.

The first statement was established in Proposition 1.2.1. Moreover if H is bounded it is normal if, and only if, S is normal and the second statement follows from the discussion preceding the example. The case of unbounded H can now be deduced by an approximation technique based on the functional analysis of generators.

If $\mathrm{Re}\,\sigma(H) \geq 0$ then $(I+\alpha H)^{-1}$ is a well-defined bounded operator for all $\alpha \geq 0$. Consequently the operators

$$H_\alpha = H(I+\alpha H)^{-1} = \alpha^{-1}\left(I-(I+\alpha H)^{-1}\right)$$

are bounded. But if S is normal it follows that H_α is normal and the uniformly continuous semigroups $S_t^\alpha = \exp\{-tH_\alpha\}$ are also normal. Moreover it follows from the identity

$$\lambda(I+\alpha\lambda)^{-1}I - H_\alpha = (I+\alpha\lambda)^{-1}(\lambda I-H)(I+\alpha H)^{-1}$$

that if $\lambda \in r(H)$ then $\lambda(I+\alpha\lambda)^{-1} \in r\left(H_\alpha\right)$, unless $\lambda = -\alpha^{-1}$.

Therefore $r(H_\alpha)$ contains the open left hand plane, $\mathrm{Re}\,\sigma(H_\alpha) \geq 0$, and S^α is contractive by the preceding argument for bounded generators. Finally the formula

$$S_t^\alpha a - S_t a = \int_0^1 d\lambda \, \frac{d}{d\lambda} S_{\lambda t}^\alpha \, S_{(1-\lambda)t} a$$

$$= t \int_0^1 d\lambda \, S_{\lambda t}^\alpha \, S_{(1-\lambda)t} \left(I - (I+\alpha H)^{-1} \right) Ha$$

with $a \in D(H)$, and the fact that $(I+\alpha H)^{-1} \mapsto I$ as $\alpha \to 0$ (see the proof of Theorem 1.3.1), establish that S_t^α converges strongly to S_t. Hence S is contractive.

Note that if $S_t = \exp\{-tH\}$ is contractive then $\mathrm{Re}\,\sigma(H) \geq 0$ but the converse is not necessarily true if S is not normal. For example if

$$H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then $\sigma(H) = 0$ but

$$S_t = e^{-tH} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

and $\|S_t\| > 1$ for all $t \neq 0$. □

Throughout this section we have examined criteria for an operator H on a Banach space $\mathcal{B}$ to be the generator of a C_0-semigroup of contractions. More generally one can ask whether a given operator H has extensions which are generators, and then try to classify all such extensions. Unfortunately the theory of extensions is poorly developed except for specific examples, or for the special case of norm-dissipative operators on Hilbert space. This Hilbert space theory can be briefly described as follows.

Assume H is a norm closed, norm densely defined, norm-dissipative, operator on the Hilbert space $\mathcal{H}$ and for small $\alpha > 0$ define

$$\mathcal{H}_\alpha = R(I+\alpha H) .$$

It follows by a simple variation of the argument used in the proof of Theorem 1.3.2 that $\mathcal{H}_\alpha$ is a closed subspace of $\mathcal{H}$. Now if $\mathcal{H}_\alpha = \mathcal{H}$ then H is the generator of a C_0-contraction semigroup by the Hille-Yosida theorem. Therefore we consider the situation $\mathcal{H}_\alpha \neq \mathcal{H}$ and try to construct extensions of H which generate contraction semigroups.

It is useful to introduce the spaces $D_\alpha = D_\alpha(H) = \mathcal{H}_\alpha^\perp$ which measure the extent to which the range spaces $R(I+\alpha H)$ fail to equal $\mathcal{H}$. The D_α are called *deficiency spaces* and the first key observation is that *the dimension of* D_α *is independent of* α. This dimension is called the *deficiency index* of H. To prove the independence statement one first remarks that

$$\|(\lambda I+H)a\| \geq \lambda\|a\|$$

for all $a \in D(H)$ and all sufficiently large $\lambda > 0$ because H is norm-dissipative. Next define E_α as the orthogonal projection onto D_α and note that

$$\|(I-E_\alpha)b\| = \sup_{a\in D(H)} |(b,\ (I+\alpha H)a)| \Big/ \|(I+\alpha H)a\| \ .$$

Therefore if $b \in D_{1/\lambda}$ then

$$\begin{aligned} \|(I-E_{1/\mu})b\| &= \sup_{a\in D(H)} |(b,\ (\mu I+H)a)| \Big/ \|(\mu I+H)a\| \\ &\leq \sup_{a\in D(H)} \{|(b,\ (\lambda I+H)a)| + |\mu-\lambda|\ |(b,\ a)|\} \Big/ \|(\mu I+H)a\| \\ &\leq \left(|\xi-\lambda| \big/ \mu\right) \|b\| \end{aligned}$$

where the last estimate uses the norm dissipativity of H . Consequently

$$\|(I-E_\alpha)E_\beta\| \leq |\alpha-\beta| \big/ \beta \ .$$

But this implies that

$$\begin{aligned} \|E_\alpha - E_\beta\| &= \|(I-E_\alpha)E_\beta - E_\beta(I-E_\alpha)\| \\ &\leq |\alpha-\beta| \big/ \beta + |\alpha-\beta| \big/ \alpha \ . \end{aligned}$$

Thus if $\alpha > 0$ is in a sufficiently small open interval around β one has $\|E_\alpha - E_\beta\| < 1$ which is equivalent to $E_\alpha = E_\beta$. Therefore D_α and D_β have the same dimension. But since $\beta > 0$ was arbitrary the general independence statement follows immediately.

The second crucial observation is that $D(H) \cap D_\alpha = \{0\}$

for each $\alpha > 0$. This is established by noting that if $a \in D(H) \cap D_\alpha$ then $(a, (I+\alpha H)a) = 0$ and

$$(a, Ha) = -\|a\|^2/\alpha .$$

But H is norm-dissipative, hence $\mathrm{Re}(a, Ha) \geq 0$, and $a = 0$.

Now one can construct generator extensions of H by iteration of the following procedure for the simplest case that the deficiency index is one.

Assume D_α is one-dimensional. Then define H_α by $D(H_\alpha) = D(H) \oplus D_\alpha$ and

$$H_\alpha(a+b) = Ha + b/\alpha$$

for $\alpha \in D(H)$ and $b \in D_\alpha$. If $a + b = 0$ one has $a = 0 = b$ because $D(H) \cap D_\alpha = \{0\}$. Therefore $Ha = 0 = b/\alpha$ and $H_\alpha(a+b) = 0$, i.e., the operator H_α is linear. But

$$\begin{aligned}\mathrm{Re}\big((a+b), H_\alpha(a+b)\big) &= \mathrm{Re}((a+b), (Ha+b/\alpha)) \\ &= \mathrm{Re}(a, Ha) + \|b\|^2/\alpha + \mathrm{Re}(b, (I+\alpha H)a)/\alpha \\ &\geq \mathrm{Re}(a, Ha) \geq 0\end{aligned}$$

where we have used $b \in D_\alpha$. Thus H_α is norm-dissipative. Finally if $c \in R(I+\alpha H_\alpha)^\perp$ then

$$\big(c, (I+\alpha H_\alpha)(a+b)\big) = (c, (I+\alpha H)a) + 2(c, b) = 0$$

for $a \in D(H)$ and $b \in D_\alpha$. But $R(I+\alpha H) = \mathcal{H}_\alpha$ and $b \in \mathcal{H}_\alpha^\perp$. Therefore $c = 0$. Thus to summarize $\alpha > 0 \longmapsto H_\alpha$ is a one-parameter family of norm densely defined, norm-dissipative, operators with $R(I+\alpha H_\alpha) = \mathcal{H}$. Hence the H_α are norm closed and each H_α generates a C_0-semigroup of contractions by the Hille-Yosida theorem.

The above construction generalizes quite easily. If D_α has dimension $n > 1$ one first chooses a one-dimensional subspace $D_\alpha^{(1)} \subset D_\alpha$ and defines H_α by $D(H_\alpha) = D(H) \oplus D_\alpha^{(1)}$ and

$$H_\alpha(a+b) = a + b/\alpha$$

for all $a \in D(H)$ and $b \in D_\alpha^{(1)}$. It then follows as above that H_α is norm-dissipative and the corresponding deficiency space is given by $D_\alpha(H_\alpha) = D_\alpha(H) \backslash D_\alpha^{(1)}$. Thus the deficiency index is reduced by one. Iteration of this procedure then produces a family of extensions of H which generate contraction semigroups. If n is finite, or countably infinite, this iterative procedure is straightforward. In the general case it is necessary to appeal to complete induction.

Although the foregoing method allows the construction of some generator extensions, in the Hilbert space context, it does not give all possible extensions. A complete classification of such extensions is only known for the even more special cases of symmetric operators,

$$\mathrm{Im}(a, Ha) = 0\ , \qquad a \in D(H)\ ,$$

or anti-symmetric operators,

$$\mathrm{Re}(a, Ha) = 0 , \qquad a \in D(H) .$$

These particular cases will be discussed in greater detail in Chapter 2.

Example 1.3.7. Define $H = d^2/dx^2$ on the twice continuously differentiable functions with compact support in $\langle 0, \infty\rangle$. Then H is a symmetric norm-dissipative operator on $L^2(0, \infty)$ because

$$(f, Hf) = \int_0^\infty dx \, |\tfrac{df}{dx}(x)|^2$$

for all $f \in D(H)$. But the deficiency index of H is one because $R(I+\alpha^2 H)^\perp$ consists of multiples of the function f_α where

$$f_\alpha(x) = \exp\{-x/\alpha\} .$$

Hence the above construction gives a one-parameter family of norm-dissipative extensions H_α of H satisfying the range condition $\overline{R(I+\alpha^2 H_\alpha)} = L^2(0, \infty)$. But

$$\left(\partial f_\alpha(x) - \alpha^{-1} f_\alpha(x)\right)\Big|_{x=0} = 0 ,$$

where ∂ denotes the right derivative. Therefore the family of extensions of $-d^2/dx^2$ to the twice differentiable functions $f \in L^2(0, \infty)$ with

$$\left(\partial f(x) - \alpha^{-1} f(x)\right)\Big|_{x=0} = 0$$

must also satisfy the range condition. But these extensions, which we also denote by H_α, are also symmetric and norm dissipative because

$$\left(f,\ H_\alpha f\right) = \alpha^{-1}|f(0)|^2 + \int_0^\infty dx\ \left|\frac{df}{dx}(x)\right|^2$$

for all $f \in D\left(H_\alpha\right)$. Hence the H_α are pre-generators of contraction semigroups S^α. This construction omits, however, two extensions which formally correspond to the values $\alpha = 0$ and $\alpha = \infty$; the first is related to *Dirichlet boundary conditions* $f(0) = 0$ and the second to *Neumann boundary conditions* $\partial f(0) = 0$.

Exercises.

1.3.1. If H generates the C_0-semigroup S prove that

$$\lim_{\alpha \to 0+} \|S_t a - \exp\{-tH(I+\alpha H)^{-1}\}a\| = 0$$

and

$$\lim_{S \to 0+} \|S_t a - \exp\{-t\left(I-S_s\right)/s\}a\| = 0 .$$

Hint: See Example 1.3.6.

1.3.2. Complete the proof of Remark 1.3.3 that a norm closed, norm densely defined, operator generates a C_0-semigroup S satisfying $\|S_t\| \leq M \exp\{\omega t\}$ if, and only if,

$$R(I+\alpha H) = \mathcal{B}$$

and

$$\|(I+\alpha H)^n a\| \geq M^{-1}(1-\alpha\omega)^n \|a\| \ , \qquad a \in D(H^n) \ ,$$

for all small $\alpha > 0$.

1.3.3. Let Λ be a bounded open subset of $\mathbb{R}^\nu$. Define the Laplace operator $H = -\nabla^2$ on the twice continuously differentiable functions with compact support in Λ . Prove that H is norm-dissipative on $L^p(\Lambda)$ for all $p \in [1, \infty\rangle$.

1.3.4. Let $S = \exp\{-tH\}$ be a C_0-semigroup on a reflexive Banach space, i.e., $\mathcal{B} = (\mathcal{B}^*)^*$. Prove that the adjoints $S^* = \{S_t^*\}_{t\geq 0}$ define a C_0-semigroup, the adjoint semigroup, with generator H^* , the adjoint of H .

Hint: Use Exercises 1.2.4 and 1.2.5 together with the definition

$$D(H^*) = \{f \ ; \ f \in \mathcal{B}^* \ , \ |(f, Ha)| \leq c_f \|a\| \ , \ a \in D(H)\}$$

$$(H^*f, a) = (f, Ha) \quad \text{for} \quad f \in D(H^*) \ , \qquad a \in D(H) \ .$$

1.3.5. Consider the Laplacian $H = -\nabla^2$ defined on the infinitely often differentiable functions in $L^2(\mathbb{R}^\nu)$ which vanish in a neighbourhood of the origin. Prove that the deficiency index $d(H)$ of H satisfies

$$d(H) = 2 \quad \text{if} \quad \nu = 1$$

$$d(H) = 1 \quad \text{if} \quad \nu = 2 \ , \ 3$$

$$d(H) = 0 \quad \text{if} \quad \nu \geq 4 \ .$$

1.4. Norm-dissipative Operators.

The Hille-Yosida theorem establishes that norm-dissipativity of a generator H, i.e., the condition

$$\|(I+\alpha H)a\| \geq \|a\| , \qquad a \in D(H) ,$$

for small $\alpha > 0$, is an infinitesimal reflection of contractivity of the associated semigroup. Next we discuss a reformulation of dissipativity which corresponds to a more geometric interpretation of contractivity. This reformulation is the Banach space analogue of the condition

$$\mathrm{Re}(a, Ha) \geq 0 , \qquad a \in D(H) ,$$

which characterizes dissipative operators H on Hilbert space.

The semigroup S is contractive if, and only if, it maps the unit sphere, $\{a ; \|a\| = 1\}$, into the unit ball, $\mathcal{B}_1 = \{a ; \|a\| \leq 1\}$. Thus the change $S_t a - a$ of an element a must be toward the interior of the ball of radius $\|a\|$. To describe this last geometric idea in a quantitative manner it is necessary to introduce the notion of a tangent functional.

An element $f_a \in \mathcal{B}^*$ is defined to be a norm-tangent functional at a if

$$\|b\| \geq \|a\| + \mathrm{Re}(f_a, b-a)$$

for all $b \in \mathcal{B}$. Geometrically each such functional describes a hyperplane tangent to the graph of $b \in \mathcal{B} \mapsto \|b\| \geq 0$ at the point a. The functional f_a divides the space into two sets

$E_a = \{b\ ;\ \mathrm{Re}(f_a, b) \geq 0\}$ and $I_a = \{b\ ;\ \mathrm{Re}(f_a, b) \leq 0\}$. The first set can be interpreted as the b which are directed toward the exterior of the ball $\{b\ ;\ \|b\| \leq \|a\|\}$ and the second set the b which are directed toward the interior. Hence the geometric rephrasing of contractivity of S given in the last paragraph can be quantitatively expressed as

$$\mathrm{Re}(f_a,\ S_t a - a) \leq 0\ ,$$

i.e., the change $S_t a - a$ of a is toward the interior of the ball. Indeed this property follows directly from the definition of the tangent functional f_a ,

$$\mathrm{Re}(f_a,\ S_t a - a) \leq \|S_t a\| - \|a\| \leq 0\ .$$

Thus if H is the generator of the C_0-contraction semigroup S one concludes that

$$\mathrm{Re}(f_a,\ Ha) = \lim_{t\to 0+} \mathrm{Re}(f_a,\ a - S_t a)\big/t \geq 0$$

for all $a \in D(H)$ and all norm-tangent functionals f_a at a . This is the alternative reformulation of norm-dissipativity of H ; equivalence with the original formulation is provided by the following.

THEOREM 1.4.1. *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. (1') $\|(I+\alpha H)a\| \geq \|a\|$

for all $a \in D(H)$ *and all* $\alpha > 0$ *(for all small* $\alpha > 0$*)*,

2. $$\mathrm{Re}\left(f_a, Ha\right) \geq 0$$

for one non-zero norm-tangent functional at each $a \in D(H)$.

Moreover if H *is norm densely defined these conditions are equivalent to the following:*

3. $$\mathrm{Re}\left(f_a, Ha\right) \geq 0$$

for all norm-tangent functionals f_a *at each* $a \in D(H)$.

The proof uses an alternative characterization of norm-tangent functional which can be used to establish the existence of such functionals.

LEMMA 1.4.2. *For* $f \in \mathcal{B}^*$ *the following conditions are equivalent:*

1. f *is a norm-tangent functional at* a ,

2. $$|(f, b)| \leq \|b\| , \qquad b \in \mathcal{B} ,$$

and

$$(f, a) = \|a\| .$$

Hence for each $a \in \mathcal{B} \setminus \{0\}$ *there exists a non-zero norm-tangent functional.*

Proof. $1 \Rightarrow 2$. Condition 1 states that

$$(*) \qquad \|b\| \geq \|a\| + \mathrm{Re}(f, b-a) .$$

Thus replacing b by $\lambda e^{i\theta}b$ one finds

$$\|b\| \geq \lim_{\lambda\to\infty} \{\|a\|/_\lambda + \mathrm{Re}\, e^{i\theta}(f, b-a/_\lambda)\}$$

$$= \mathrm{Re}\, e^{i\theta}(f, b) .$$

Hence $|(f, b)| \leq \|b\|$. But setting $b = 0$ in (*) one also obtains $(f, a) \geq \|a\|$ and therefore $(f, a) = \|a\|$.

$2 \Rightarrow 1$. Successively applying the two relations of Condition 2 one has

$$\begin{aligned} \|b\| &\geq \mathrm{Re}(f, b) \\ &= \mathrm{Re}(f, a) + \mathrm{Re}(f, b-a) \\ &= \|a\| + \mathrm{Re}(f, b-a) . \end{aligned}$$

Finally the Hahn-Banach theorem states that *if* p *is a real-valued function over* $\mathcal{B}$ *satisfying*

$$p(a+b) \leq p(a) + p(b) , \qquad a, b \in \mathcal{B} ,$$

$$p(\lambda a) = \lambda p(a) \qquad , \qquad \lambda \geq 0 , \quad a \in \mathcal{B}$$

and f *is a linear functional over a subspace* $C \subseteq \mathcal{B}$ *such that* $|(f, c)| \leq p(c)$ *for* $c \in C$ *then there exists a linear extension* F *of* f *to* $\mathcal{B}$ *such that* $|F(a)| \leq p(a)$ *for all* $a \in \mathcal{B}$. Therefore choosing $p(\cdot) = \|\cdot\|$, $C = \{\lambda a ; \lambda \in \mathbb{C}\}$, and setting $(f, \lambda a) = \lambda\|a\|$, one can find a linear extension F to $\mathcal{B}$ satisfying $|F(b)| \leq \|b\|$ and $F(a) = (f, a) = \|a\|$. Hence F is a non-zero norm-tangent functional at a by Condition 2 of the lemma. □

Proof of Theorem 1.4.1. $1' \Rightarrow 2$. Set $b = Ha$ and for each sufficiently small α choose a norm-tangent functional g_α at the point $a + \alpha b$. Then from Condition 1

$$\begin{aligned} \|a\| &\leq \|a + \alpha b\| \\ &= \mathrm{Re}\left(g_\alpha, a+\alpha b\right) \\ &= \mathrm{Re}\left(g_\alpha, a\right) + \alpha\, \mathrm{Re}\left(g_\alpha, b\right) \\ &\leq \mathrm{Re}\left(g_\alpha, a\right) + \alpha\|b\| . \end{aligned}$$

Now the unit ball of $\mathcal{B}^*$ is weakly* compact by the Alaoglu-Birkhoff theorem, i.e., for every net $f_\alpha \in \mathcal{B}^*$ with $\|f_\alpha\| \leq 1$ there is a subset $f_{\alpha'}$ which converges to an $f \in \mathcal{B}^*$ in the sense that $\left(f_{\alpha'}, a\right) \to (f, a)$ for all $a \in \mathcal{B}$. Hence one deduces from the foregoing inequality that

$$\begin{aligned} \|a\| &\leq \lim_{\alpha'\to 0} \left\{\mathrm{Re}\left(g_{\alpha'}, a\right) + \alpha'\|b\|\right\} \\ &= \mathrm{Re}(g, a) \end{aligned}$$

where g is the weak* limit of the subset $g_{\alpha'}$. Now since $\|g_\alpha\| = 1$ one has $\|g\| \leq 1$ and then

$$\mathrm{Re}(g, a) \leq \|g\|\ \|a\| \leq \|a\| .$$

Hence

$$(g, a) = \|a\| .$$

This proves that g is a norm-tangent functional at a . But one also has

$$\|a\| \leq \text{Re}(g_\alpha, a) + \alpha \text{ Re}(g_\alpha, b)$$

$$\leq \|a\| + \alpha \text{ Re}(g_\alpha, b)$$

and hence in the limit $\alpha' \to 0$ one obtains

$$0 \leq \text{Re}(g, b) = \text{Re}(g, Ha) ,$$

i.e., Condition 2 is satisfied.

$2 \Rightarrow 1$. Let f be a norm-tangent functional at $a \in D(H)$ satisfying

$$\text{Re}(f, Ha) \geq 0 .$$

Then

$$\|a\| = \text{Re}(f, a)$$

$$\leq \text{Re}(f, a+\alpha Ha)$$

$$\leq \|(I+\alpha H)a\|$$

for all $\alpha > 0$.

$1 \Rightarrow 1'$. This is evident.

Finally 3 ⇒ 2 and it remains to prove 1 ⇒ 3 under the assumption that $D(H)$ is norm dense.

Now if $a, b \in D(H)$ and f is a non-zero norm-tangent functional at a one has

$$\|(I-\alpha H)a\| \geq \|a\| - \alpha \operatorname{Re}(f, Ha) .$$

Therefore

$$\operatorname{Re}(f, Ha) \geq \limsup_{\alpha \to 0+} \left(\|a\| - \|(I-\alpha H)a\|\right)\Big/\alpha$$

But

$$\begin{aligned} \|(I-\alpha H)a\| &\leq \|a + \alpha b\| + \alpha\|b + Ha\| \\ &\leq \|(I+\alpha H)(a+\alpha b)\| + \alpha\|b + Ha\| \\ &\leq \|a\| + 2\alpha\|b + Ha\| + \alpha^2\|Hb\| \end{aligned}$$

for all sufficiently small $\alpha > 0$ by Condition 1. Therefore by combination of these results

$$\operatorname{Re}(f, Ha) \geq -2\|b + Ha\| .$$

But since $D(H)$ is norm dense we may choose b arbitrarily close to $-Ha$ and deduce that

$$\operatorname{Re}(f, Ha) \geq 0 ,$$

i.e., Condition 3 is satisfied. □

Example 1.4.3. Let H be a Hilbert space and hence identifiable with its dual. If $a, b \in H$ then

$$|(a,\ b)| \le \|a\|\ \|b\|$$

with equality if, and only if, $a = \lambda b$ for some $\lambda \in \mathbb{C}$. Therefore $a/\|a\|$ is the unique norm-tangent functional at $a \in \mathcal{H}$ and Theorem 1.4.1 states that an operator H is norm-dissipative if, and only if,

$$\mathrm{Re}(a,\ Ha) \ge 0$$

for all $a \in D(H)$. This is the characterization used in Section 1.3. □

Example 1.4.4. If $\mathcal{B} = L^p(X;\ d\mu)$ with $p \in \langle 1, \infty\rangle$ then there is a unique norm-tangent functional at each $f \in \mathcal{B}$ given by $\left(|f|^{p-1}\ \arg f\right)\ \|f\|_p^{p-1}$ where $\arg f(x) = f(x)/|f(x)|$ if $|f(x)| \ne 0$ and $\arg f(x) = 0$ if $|f(x)| = 0$. If $p = 1$ this gives the tangent functional $\arg f$, but this is not unique if $f = 0$ on a set Y of non-zero measure. In this case $g + \arg f$, where g has support in Y and $|g| \le 1$, is also a tangent functional. □

Theorem 1.4.1 allows an immediate reformulation of the Hille-Yosida theorem which is often more convenient for applications.

THEOREM 1.4.5. **(Lumer and Phillips).** *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. H *is the generator of a* C_0*-contraction semigroup* S,

2. H *is (norm closed), norm densely defined*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ *(or for an* $\alpha > 0$*) and*

$$\mathrm{Re}(f_a, Ha) \geq 0$$

for one norm-tangent functional f_a *at each* $a \in D(H)$.

The alternative characterization of norm-dissipativity provided by Theorem 1.4.1 also allows an easy proof of a version of the Hille-Yosida theorem in which the range condition $R(I+\alpha H) = \mathcal{B}$ does not occur explicitly.

THEOREM 1.4.6. *Let* H *be an operator on the Banach space* $\mathcal{B}$ *and consider the following conditions:*

1. H *is norm densely defined with norm densely defined adjoint* H^* *and both* H *and* H^* *are norm dissipative,*

2. H *is norm closable and its closure* $\overline{H}$ *generates a* C_0*-contraction semigroup.*

Then $1 \Rightarrow 2$ *and if* $\mathcal{B}$ *is reflexive* $2 \Rightarrow 1$.

Proof. $1 \Rightarrow 2$. Suppose $R(I+H)$ is not norm dense in $\mathcal{B}$. The Hahn-Banach theorem then implies the existence of a non-zero $f \in \mathcal{B}^*$ such that $(f, (I+H)a) = 0$ for all $a \in D(H)$. Therefore

$$|(f, Ha)| = |(f, a)| \leq \|f\| \, \|a\|$$

and hence $f \in D(H^*)$. Moreover since $D(H)$ is norm dense $(I+H^*)f = 0$. Thus if $b \in \mathcal{B}^{**}$ is a norm-tangent functional at $f \in \mathcal{B}^*$ one has

$$(b, H^*f) = -(b, f) = -\|f\|$$

which contradicts the norm-dissipativity of H^*. Hence $R(I+H)$ is norm dense and the desired implication follows from Theorem 1.3. .

Next assume $\mathcal{B}$ is reflexive and consider the converse.

2 ⇒ 1. If $\overline{H}$ generates the C_0-contraction semigroup S then $\overline{H}^*$ generates the C_0-contraction semigroup S^* (see Exercise 1.3.4). Hence Condition 1 follows from the Hille-Yosida theorem applied to S and S^*. □

Of course the drawback of this criterion is that one has to specifically identify the adjoint H^* before it is applicable.

Finally we illustrate the notion of norm-dissipativity with two examples of matrices acting on finite-dimensional spaces.

Example 1.4.7. (Matrix Semigroups). Let $x = (x_1, x_2, \ldots, x_n)$ denote an element of the finite-dimensional space $\mathbb{C}^n$. Further let $H = (H_{ij})$ be a complex-valued $n \times n$ matrix acting on $\mathbb{C}^n$ and $S_t = \exp\{-tH\}$, $t \geq 0$, the corresponding matrix semigroup.

The space $\mathbb{C}^n$ can be equipped with various norms which are all equivalent in the topological sense. But S can be contractive with respect to one norm without being contractive with respect to an equivalent norm. Nevertheless if a norm is given then S is contractive if, and only if, H is dissipative. Dissipativity with respect to the ℓ^∞- and ℓ^1-norms is particularly easy to describe because of the simple geometry of the corresponding balls. We will not pursue, however, the geometric aspects but proceed analytically.

Define the ℓ^∞-norm on $\mathbb{C}^n$ by

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| .$$

It follows that $S_t = \exp\{-tH\}$ *is* ℓ^∞*-contractive if, and only if,*

$$(*) \qquad \mathrm{Re}H_{ii} - \sum_{j\ne i} |H_{ij}| \ge 0$$

for all $i = 1, 2, \ldots, n$. This is established as follows. For i fixed choose x such that $x_i = 1$, $x_j = -\overline{H}_{ij}\big/|H_{ij}|$ if $j \ne i$ and $H_{ij} \ne 0$, and $x_j = 0$ if $j \ne i$ and $H_{ij} = 0$. Next choose $f = (f_1, \ldots, f_n)$ such that $f_i = 1$ and $f_j = 0$ if $j \ne i$. Then f is a norm-tangent functional at x and

$$\mathrm{Re}(f, Hx) = \mathrm{Re}\ H_{ii} - \sum_{j\ne i} |H_{ij}| .$$

Thus (*) is necessary for S to be ℓ^∞-contractive. Conversely let x be a non-zero element of $\mathbb{C}^n$ and choose i such that $|x_i| \ge |x_j|$ for all $j \ne i$. Set $f_i = x_i\big/|x_i|$ and $f_j = 0$ if $j \ne i$. It follows that f is a norm-tangent functional at x and

$$\begin{aligned}\mathrm{Re}(f, Hx) &= |x_i|\ \mathrm{Re}\ H_{ii} + \mathrm{Re} \sum_{j\ne i} H_{ij}\overline{x}_i x_j \Big/ |x_i| \\ &\ge |x_i| \left(\mathrm{Re}\ H_{ii} - \sum_{j\ne i} |H_{ij}|\right).\end{aligned}$$

Thus (*) is sufficient for H to be ℓ^∞-dissipative and S to be

ℓ^∞-contractive.

Next if

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

denotes the ℓ^1-norm it follows by duality that $S_t = \exp\{-tH\}$ is ℓ^1-contractive if, and only if, the adjoint semigroup $S_t^* = \exp\{-tH^*\}$ is ℓ^∞-contractive. Thus $S_t = \exp\{-tH\}$ *is ℓ^1-contractive if, and only if,*

$$\mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ji}| \geq 0$$

for all $i = 1, 2, \ldots, n$.

Finally one can equip $\mathbb{C}^n$ with the ℓ^p-norms

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

for $1 < p < \infty$ and consider ℓ^p-contractivity. If S is both ℓ^1- and ℓ^∞-contractive it follows by abstract interpolation that S is ℓ^p-contractive for all $p \in [1, \infty]$. This conclusion can, however, be reached by explicit estimate. For example if $p = 2$ then $x/\|x\|_2$ is the unique tangent functional at x and

$$\begin{aligned}\mathrm{Re}(x, Hx) &= \sum_{i=1}^{n} \left\{|x_i|^2 \,\mathrm{Re}\, H_{ii} + \sum_{j\neq i} H_{ij}\bar{x}_i x_j\right\} \\ &\geq \sum_{i=1}^{n} \left\{|x_i|^2 \,\mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ij}|\left(|x_i|^2+|x_j|^2\right)/2\right\}\end{aligned}$$

$$= \sum_{i=1}^{n} \left(|x_i|^2 \left(\mathrm{Re}\, H_{ii} - \sum_{j\neq i} \left(|H_{ij}|+|H_{ji}|\right)\Big/2\right)\right.$$

where we have used the Cauchy-Schwarz inequality. Thus combination of the conditions for ℓ^1- and ℓ^∞-contractivity imply that H is ℓ^2-dissipative. A similar argument using the Minkowski inequality establishes that ℓ^1- and ℓ^∞-contractivity imply that H is ℓ^p-dissipative.

If $p \neq 1$ or ∞ the ℓ^p-dissipative conditions cannot be expressed in any particularly practical terms of the matrix element H_{ij}. Nevertheless if H is self-adjoint, i.e., if $H = H^*$, then ℓ^∞-contractivity of S implies ℓ^1-contractivity by duality and ℓ^p-contractivity, $p \in \langle 1, \infty\rangle$, by interpolation. Thus *a self-adjoint matrix semigroup is* ℓ^p*-contractive for all* $p \in [1, \infty]$ *if, and only if,* (*) *is valid.* More generally if H is normal, i.e., if $HH^* = H^*H$, then ℓ^2-dissipativity is implied by ℓ^1- or ℓ^∞-dissipativity. This will be established in the next example. □

Example 1.4.8. **(Normal matrix semigroups).** Let $S_t = \exp\{-tH\}$ denote the matrix semigroup of Exmaple 1.4.7. We first argue that if the conditions

$$(*) \qquad \mathrm{Re}\, H_{ii} - \sum_{j\neq i} |H_{ij}| \geq 0 , \qquad i = 1, 2, \ldots, n$$

for ℓ^∞-contractivity are valid then $\mathrm{Re}\,\lambda \geq 0$ for all eigenvalues λ of H. This follows by noting that if $(H-\lambda I)x = 0$ then

$$|H_{ii} - \lambda|\ |x_i| = \Big|\sum_{j\neq i} - H_{ij}x_j\Big|$$

$$\leq \sum_{j\neq i} |H_{ij}|\ |x_i|$$

where i has been chosen such that $|x_i| \geq |x_j|$ for $j \neq i$. Thus if x is non-zero the conditions (*) imply that

$$|H_{ii} - \lambda| \leq \mathrm{Re}\, H_{ii}$$

and hence $\mathrm{Re}\,\lambda \geq 0$. Consequently Example 1.3.6 implies that *if* S *is a normal matrix semigroup then* ℓ^∞*-contractivity implies* ℓ^2*-contractivity* and hence, by interpolation or by explicit estimation, it implies ℓ^p-contractivity for all $p \in [2, \infty]$. Similarly if S is a normal matrix semigroup then ℓ^1-contractivity implies ℓ^2-contractivity, and hence ℓ^p-contractivity for all $p \in [1, 2]$. □

Exercises.

1.4.1. Let H be the generator of a C_0-semigroup of contractions. Prove that the operators $H_\alpha = H(I+\alpha H)^{-1}$, $\alpha \geq 0$, are norm-dissipative.

1.4.2. Prove that if H is an invertible norm-dissipative operator on a Hilbert space then H^{-1} is norm-dissipative.

1.4.3. Prove that the closure of a norm densely defined, norm-dissipative, operator is norm-dissipative.

1.5. C_0^*-semigroups.

If the Banach space $\mathcal{B}$ is the dual of a Banach space $\mathcal{B}_*$, the pre-dual of $\mathcal{B}$, then it is of interest to study families of bounded operators $S = \{S_t\}_{t\geq 0}$ with the semigroup property $S_sS_t = S_{s+t}$ which are weak*-continuous in the sense that

1. $$\lim_{t\to 0+} (S_tf, a) = (f, a)$$

   for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$,

2. $$\lim_{\alpha} (S_tf_\alpha, a) = (S_tf, a)$$

   for all $t > 0$, all $a \in \mathcal{B}_+$, and all families f_α such that

   $$\lim_{\alpha} (f_\alpha, a) = (f, a) .$$

Such families are called C_0^*-semigroups. The simplest example is translations on $L^\infty(\mathbb{R})$ which has pre-dual $L^1(\mathbb{R})$.

Our first aim is to show that if S is a C_0^*-semigroup there exists an adjoint semigroup S_* on $\mathcal{B}_*$ such that

$$(S_tf, a) = (f, S_{*t}a) .$$

The weak*-continuity of S then implies the weak, and hence strong, continuity of S_*, i.e., the C_0^*-semigroup S is the adjoint of a C_0-semigroup S_*. This explains the name C_0^*-semigroup. In the sequel we demonstrate that much of the foregoing theory of C_0-semigroups can be carried over to the C_0^*-semigroups by duality arguments.

We begin by recalling a number of standard definitions.

A family $f_\alpha \in \mathcal{B}$ is weak*-convergent if there is an $f \in \mathcal{B}$ such that

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

for all $a \in \mathcal{B}_*$, and a set $\mathcal{D} \subseteq \mathcal{B}$ is weak*-closed if each weak*-convergent family $f_\alpha \in \mathcal{D}$ has a limit $f \in D$. Alternatively a set $\mathcal{D} \subseteq \mathcal{B}$ is weak*-dense if each $f \in \mathcal{B}$ can be approximated by $f_\alpha \in \mathcal{D}$ in the weak*-sense, i.e.,

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

for all $a \in \mathcal{B}_*$.

Next an operator H on $\mathcal{B}$ is weak*-densely defined if its domain $D(H)$ is weak*-dense in $\mathcal{B}$ and it is weak*-weak*-closed if $f_\alpha \in D(H)$ and

$$\lim_\alpha \left(f_\alpha, a\right) = (f, a)$$

$$\lim_\alpha \left(Hf_\alpha, a\right) = (g, a) ,$$

for all $a \in \mathcal{B}_*$, imply that $f \in D(H)$ and $g = Hf$. Moreover H is weak*-weak*-closable if it has a weak*-weak*-closed extension or, equivalently, if $f_\alpha \in D(H)$ and $\left(f_\alpha, a\right) \to 0$, $\left(Hf_\alpha, a\right) \to (g, a)$, for all $a \in \mathcal{B}_*$, imply that $g = 0$.

The basic duality properties of operators rely upon two versions of the bipolar theorem. Specifically *if* A *is a weak*-closed sbuspace of* $\mathcal{B}$ *and one defines*

$$A^{\perp} = \{a \in \mathcal{B}_* \; ; \; (f, a) = 0 \text{ for all } f \in A\} ,$$

$$A^{\perp\perp} = \{f \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } a \in A^{\perp}\}$$

then $A = A^{\perp\perp}$. Similarly *if* A_* *is a closed subspace of* $\mathcal{B}_*$ *and*

$$A_*^{\perp} = \{f \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } a \in A_*\} ,$$

$$A_*^{\perp\perp} = \{a \in \mathcal{B} \; ; \; (f, a) = 0 \text{ for all } f \in A_*^{\perp}\}$$

then $A_* = A_*^{\perp\perp}$. Both these statements are a consequence of the Hahn-Banach theorem. Consider, for example, the second statement.

It follows by definition that $A_* \subseteq A_*^{\perp\perp}$. Next define p over $\mathcal{B}_*$ by

$$p(a) = \inf\{\|a - c\| \; ; \; c \in A_*\} ,$$

then $p(a) = 0$ for all $a \in A_*$ but $p(a) \neq 0$ for $a \notin A_*$. Moreover p satisfies the hypotheses of the Hahn-Banach theorem cited in Section 1.4. Hence for $a \in A_*$ and $b \notin A_*$ one has

$$p(a+\lambda b) = \pm\lambda p(b \pm a/\lambda) = |\lambda| p(b)$$

where the $+$ and $-$ signs correspond to positive and negative λ respectively. Next introduce C as the subspace spanned by A_*

and b and define a linear functional f over $\mathcal{C}$ by

$$(f, a+\lambda b) = \lambda p(b)$$

for $a \in \mathcal{A}_*$. One has $|(f, c)| = p(c)$ for $c \in \mathcal{C}$ and hence, by the Hahn-Banach theorem, there exists a linear extension F of f to $\mathcal{B}_*$ satisfying $|F(a)| \le p(a)$ for all $a \in \mathcal{B}_*$. Since $p(a) \le \|a\|$ it follows that $F \in \mathcal{B}$ and since $F(a) = 0$ for all $a \in \mathcal{A}_*$ one also concludes that $F \in \mathcal{A}_*^{\perp}$. Finally $F(b) = (f, b) = p(b) \neq 0$ and hence $b \notin \mathcal{A}_*^{\perp\perp}$. Thus $\mathcal{A}_*^{\perp\perp} \subseteq \mathcal{A}_*$. and the two sets must be identical.

LEMMA 1.5.1. *Let* $\mathcal{B}$ *be a Banach space with a predual* $\mathcal{B}_*$ *and* H *an operator on* $\mathcal{B}$.

The following conditions are equivalent:

1. H *is weak*-densely defined and weak*-weak*-closed,*

2. H *is the adjoint of a norm densely defined, norm closed, operator* H_* *on* $\mathcal{B}_*$.

If these conditions are fulfilled and H *is bounded then* $\|H\| = \|H_*\|$.

Proof. 1 ⇒ 2. Consider $\mathcal{B} \times \mathcal{B}$ equipped with the norm $\|(f, g)\| = \left(\|f\|^2+\|g\|^2\right)^{\frac{1}{2}}$ and $\mathcal{B}_* \times \mathcal{B}_*$ with the norm $\|(a, b)\| = \left(\|a\|^2+\|b\|^2\right)^{\frac{1}{2}}$. These two spaces are then in duality through the relation

$$\big((f, g), (a, b)\big) = (f, a) + (g, b) .$$

Next introduce the graph $G(H)$ of H in $\mathcal{B} \times \mathcal{B}$ as the subspace

$$G(H) = \{(f, Hf) \; ; \; f \in D(H)\} .$$

Thus the orthogonal complement $G(H)^{\perp}$ of $G(H)$ in $\mathcal{B}_* \times \mathcal{B}_*$ consists of the pairs (a, b) which satisfy

$$(f, a) + (Hf, b) = 0$$

for all $f \in D(H)$. Now define

$$G = \{(-b, a) \; ; \; (a, b) \in G(H)^{\perp}\} .$$

Then G is the graph of an operator H_* on $\mathcal{B}_*$. This follows because if $(0, a) \in G$ the orthogonality relation gives

$$(f, a) = 0 ,$$

for all $f \in D(H)$, and $a = 0$ because $D(H)$ is weak*-dense. But $G(H)^{\perp}$, and G, are norm closed by definition and hence H_* is norm closed. Finally, if H_* is not norm densely defined there must exist a non-zero element of $G^{\perp}$ of the form $(-f, 0)$. Thus $(0, f) \in G(H)^{\perp\perp}$. But since H is weak*-weak*-closed $G(H)$ is a weak*-closed subspace and $G(H)^{\perp\perp} = G(H)$, by the first version of the bipolar theorem cited above. Hence $(0, f) \in G(H)$. This, however, contradicts the linearity of H and consequently $D(H_*)$ must be norm dense.

$2 \Rightarrow 1$. The proof is identical but $\mathcal{B}_*$ replaces $\mathcal{B}$, H_* replaces H, etc., and one uses the second version of the bipolar theorem.

Finally the equality of the norms for bounded operators follows because

$$\|H\| = \sup\{|(Hf, a)| \; ; \; f \in \mathcal{B} \; , \; a \in \mathcal{B}_*\}$$
$$= \sup\{|(f, H_* a)| \; ; \; f \in \mathcal{B} \; , \; a \in \mathcal{B}_*\} = \|H_*\| \; . \qquad \square$$

If $S = \{S_t\}_{t\geq 0}$ is a C_0^*-semigroup on $\mathcal{B}$ then the S_t are everywhere defined and weak*-weak*-closed, by the second continuity hypothesis. Hence Lemma 1.5.1 establishes the existence of an adjoint semigroup $S_* = \{S_{*t}\}_{t\geq 0}$ on $\mathcal{B}_*$ such that

$$(S_t f, a) = (f, S_{*t} a) \; ,$$

for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$. Moreover,

$$\|S_t\| = \|S_{*t}\| \; .$$

But weak*-continuity of S is equivalent to weak, and hence strong, continuity of S_* . Thus S_* is a C_0-semigroup and in general satisfies bounds of the form $\|S_{*t}\| \leq M \exp\{\omega t\}$. Hence the C_0^*-semigroup S satisfies similar bounds. Now by exploiting the Hille-Yosida theorem for the C_0-semigroup S_* and the duality properties of Lemma 1.5.1 one can obtain a Hille-Yosida theorem for the C_0^*-semigroup S . But first we must define the generator of S .

If S is a C_0^*-semigroup its generator H is defined as the weak*-derivative of S at the origin. Explicitly $D(H)$ consists of those $f \in \mathcal{B}$ for which there is a $g \in \mathcal{B}$ such that the limits

$$(g, a) = \lim_{t\to 0+} \big((I-S_t)f, a\big) \big/ t$$

exist for all $a \in \mathcal{B}_*$ and the action of H is then given by $Hf = g$. Note that if K is the generator of the C_0-semigroup S_* on $\mathcal{B}_*$, which is adjoint to S, then

$$(Hf, a) = \lim_{t\to 0+} \big((I-S_t)f, a\big) \big/ t$$

$$= \lim_{t\to 0+} \big(f, (I-S_{*t})a\big) \big/ t = (f, Ka)$$

for all $f \in D(H)$ and $a \in D(K)$. This demonstrates that the adjoint K^*, of K, extends H but part of the proof of the following result is to show that in fact $K^* = H$.

THEOREM 1.5.2. *Let $\mathcal{B}$ be a Banach space with a predual $\mathcal{B}_*$ and H an operator on $\mathcal{B}$. The following conditions are equivalent:*

1. *H is the infinitesimal generator of a C_0^*-semigroup of contractions,*

2. *H is weak*-densely defined, weak*-weak*-closed,*

$$R(I+\alpha H) = \mathcal{B}$$

for all $\alpha > 0$ (or for one $\alpha = \alpha_0 > 0$), and

$$\|(I+\alpha H)f\| \geq \|f\|$$

for all $f \in D(H)$ and all $\alpha > 0$ (or for all $\alpha \in \langle 0, \alpha_0]$).

Proof. $1 \Rightarrow 2$. The proof of this implication follows the reasoning used to establish Proposition 1.2.1.

First for $\alpha > 0$ one can define a bounded operator $R_\alpha(H)$ on $\mathcal{B}$ by

$$\left(R_\alpha(H)f,\ a\right) = \int_0^\infty dt\ e^{-t}\left(S_{\alpha t}f,\ a\right)$$

and since S is contractive one has the bound

$$\|R_\alpha(H)\| \le 1 \ .$$

But a weak*-version of the calculation used in the proof of Proposition 1.2.1 demonstrates that

$$R_\alpha(H) = (I+\alpha H)^{-1} \ .$$

Hence

$$R(I+\alpha H) = \mathcal{B}$$

and

$$\|(I+\alpha H)f\| \ge \|f\|$$

for all $f \in D(H)$. But $R_\alpha(H)f \in D(H)$ for all $f \in \mathcal{B}$ and

$$\lim_{\alpha\to 0+} \left(R_\alpha(H)f,\ a\right) = \lim_{\alpha\to 0+} \int_0^\infty dt\ e^{-t}\left(S_{\alpha t}f,\ a\right)$$

$$= (f,\ a)$$

for all $a \in \mathcal{B}_*$ by weak*-continuity of S and the Lebesgue dominated convergence theorem. Thus $D(H)$ is weak*-dense. Finally suppose $f_\beta \in D(H)$ and

$$\lim_\beta (f_\beta, a) = (f, a)$$

$$\lim_\beta ((I+\alpha H)f_\beta, a) = (g, a)$$

for all $a \in \mathcal{B}_*$. Then

$$(f, a) = \lim_\beta (R_\alpha(H)(I+\alpha H)f_\beta, a)$$

$$= (R_\alpha(H)g, a)$$

for all $a \in \mathcal{B}_*$ by another application of the Lebesgue dominated convergence theorem. Thus $(I+\alpha H)$, and hence H, is weak*-weak*-closed.

$2 \Rightarrow 1.$ It follows from Lemma 1.5.1 that H is the adjoint of a norm densely defined, norm closed, operator H_* on $\mathcal{B}_*$. But for $\alpha > 0$ and $a \in D(H)$

$$\|(I+\alpha H_*)a\| = \sup\{|(f, (I+\alpha H_*)a)| \; ; \; f \in D(H) \; , \; \|f\| \le 1\}$$

$$= \sup\{|((I+\alpha H)f, a) \; ; \; f \in D(H) \; , \; \|f\| \le 1\} \; .$$

Thus since $\|(I+\alpha H)f\| \ge \|f\|$ and $R(I+\alpha H) = \mathcal{B}$ one concludes that

$$\|(I+\alpha H_*)a\| \ge \sup\{|(g, a)| \; ; \; \|g\| \le 1\}$$

$$\|a\| \; ,$$

i.e., H_* is norm-dissipative.

Next suppose there is an $f \in \mathcal{B}$ such that

$$(f,\ (I+\alpha H_*)a) = 0$$

for all $a \in D(H_*)$. But then

$$(f,\ H_* a) = -(f,\ a)/\alpha$$

is continuous in a . Hence $f \in D(H)$ and

$$((I+\alpha H)f,\ a) = 0$$

for all $a \in D(H_*)$. Since $D(H_*)$ is norm dense it follows that $(I+\alpha H)f = 0$ and then $f = 0$ because H is norm-dissipative. Hence $R(I+\alpha H_*) = \mathcal{B}$.

Finally we can apply the Hille-Yosida theorem to deduce that H_* generates a C_0-semigroup of contractions S_* on $\mathcal{B}_*$. Then the adjoint semigroup S on $\mathcal{B}$ is a C_0^*-semigroup of contractions. But if K denotes the generator of this latter semigroup then by Laplace transformation

$$\left((I+\alpha K)^{-1}f,\ a\right) = \left(f,\ (I+\alpha H_*)^{-1}a\right)$$

for all $f \in \mathcal{B}$ and $a \in \mathcal{B}_*$. Thus $(I+\alpha K)^{-1} = (I+\alpha H)^{-1}$, i.e., $K = H$ is the generator of S . $\square$

There is also a pre-generator version of the foregoing theorem. If H is weak*-densely defined and weak*-weak*-closable then its weak*-closure $\overline{H}$ generates a C_0^*-semigroup of contractions if, and only if, H is norm-dissipative and $R(I+\alpha H)$ is weak*-dense in $\mathcal{B}$ for all sufficiently small $\alpha > 0$.

Finally we remark that a result analogous to Theorem

1.5.2 can be obtained for a general C_0^*-semigroup. The norm-dissipativity which is characteristic of contraction semigroups is replaced by a family of lower bounds of the type described in Remark 1.3.3.

Exercises.

1.5.1. Let $\mathcal{L}(H)$ denote the algebra of all bounded operators on the Hilbert space H and $\mathcal{T}(H)$ the Banach space of trace class operators, with the norm

$$T \in \mathcal{T}(H) \mapsto \|T\|_{tr} = \mathrm{Tr}\left((T^*T)^{\frac{1}{2}}\right) .$$

Prove that $\mathcal{L}(H)$ is the dual of $\mathcal{T}(H)$ with the duality

$$(T, B) \mapsto \mathrm{Tr}(TB) .$$

1.5.2. Let S be a C_0^*-semigroup on the Banach space $\mathcal{B}$ with generator H . Prove that $f \in D(H)$ if, and only if,

$$\sup_{0<t<1} \|(I-S_t)f\|/t < +\infty .$$

Hint: The unit ball of $\mathcal{B}$ is weakly*-compact by the Alaoglu-Birkhoff theorem.

1.5.3. Let S be a C_0^*-semigroup with generator H and define $\mathcal{B}_0 \subseteq \mathcal{B}$ as the norm closure of $D(H)$. Prove that $S\mathcal{B}_0 \subseteq \mathcal{B}_0$ and that the restriction of S to $\mathcal{B}_0$ is a C_0-semigroup.

1.6. Analytic Vectors.

In the previous sections we examined various methods of constructing a contraction semigroup from the resolvent of its generator. Next we analyze the possibility of a direct construction based on an operator extension of the numerical algorithms

$$\exp\{-tx\} = \sum_{n\geq 0} \frac{(-t)^n}{n!} x^n$$

$$= \lim_{n\to\infty} \left(1-\frac{t}{n}x\right)^n .$$

The problem with this new construction is that it is not applicable to all C_0-semigroups, or contraction semigroups, although it is applicable to all C_0-groups. The basic new concept is that of an analytic element.

If H is an operator on a Banach space $\mathcal{B}$ an element $a \in \mathcal{B}$ is defined to be an *(entire) analytic element for* H if

$$a \in \bigcap_{n\geq 1} D\left(H^n\right)$$

and the function

$$t \geq 0 \longmapsto \sum_{n\geq 0} \frac{t^n}{n!} \|H^n a\|$$

has a non-zero (infinite) radius of convergence. It is not at all evident that an operator possesses analytic elements but this is indeed the case if H is the generator of a strongly continuous group (a C_0-group). In fact one can explicitly construct a norm dense set of entire analytic elements by the following regularization procedure.

Let $S = \{S_t\}_{t\in\mathbb{R}}$ be a C_0-group with generator H and to each $a \in \mathcal{B}$ associate the sequence a_n defined by

$$a_n = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\ e^{-t^2} S_{t/n} a\ .$$

Since $\|S_t\| \le M \exp\{\omega|t|\}$ for some $M \ge 1$ and $\omega \ge 0$ the integral is well defined. Moreover

$$a_n - a = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\ e^{-t^2} \left(S_{t/n} a - a\right)$$

and it follows from strong continuity and the Lebesgue dominated convergence theorem that a_n converges uniformly to a. But since H is norm closed one may argue recursively that $a_n \in D\left(H^m\right)$ for all $m = 1, 2, \ldots$ and

$$H^m a_n = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt \left\{(-n)^m \frac{d^m}{dt^m} e^{-t^2}\right\} S_{t/n} a$$

$$= (-n)^m \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} dt\ H_m(t) e^{-t^2} S_{t/n} a$$

where H_m is the usual Hermite function. Thus

$$\|H^m a_n\|^2 \le n^{2m} \pi^{-1} M^2 \left(\int_{-\infty}^{\infty} dt\ H_m(t) e^{-t^2} e^{\omega|t|}\right)^2 \|a\|^2$$

$$\le n^{2m} \pi^{-1} M^2 \int_{-\infty}^{\infty} dt\ e^{2\omega|t|} e^{-t^2} \int_{-\infty}^{\infty} dt |H_m(t)|^2 e^{-t^2} \|a\|^2$$

where we have used the Cauchy-Schwarz inequality. Using the normalization properties of the Hermite functions,

$$\int_{-\infty}^{\infty} dt \, |H_m(t)|^2 e^{-t^2} = 2^m m! \pi^{\frac{1}{2}} ,$$

one finally deduces that

$$\|H^m a_n\|^2 \leq M^2 e^{\omega^2} 2^{m+1} n^{2m} m! .$$

Hence a_n is an entire analytic element for H and the set of such elements is norm dense.

Despite this positive result the generator of the semigroup of left translations on $C_0[0, \infty\rangle)$ has no non-zero analytic elements. The action of this semigroup is given by $(S_t f)(x) = f(x-t)$ if $x \geq t$, and 0 if $x < t$. It follows that for f to be an analytic element it must vanish with all its (right) derivatives at the origin but it must also be analytic in a strip about the right half axis. Thus $f = 0$. Nevertheless the translation group acting on $C_0(\mathbb{R})$ does have dense sets of analytic elements and a function is analytic for this group if, and only if, it is an analytic function in the usual sense.

Now we consider the construction of a semigroup through analytic elements and for simplicity we again restrict the discussion to contraction semigroups.

PROPOSITION 1.6.1. *Let* H *be a norm closed operator on a Banach space* $\mathcal{B}$. *Suppose that*

1. H *possesses a norm dense set of analytic elements,*

2. H *is norm-dissipative.*

It follows that H *is the generator of a* C_0*-semigroup of contractions.*

Proof. Let a be an analytic element for H. Thus there is a $t_a > 0$ such that

$$S_t a = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n a$$

converges uniformly for $|t| < t_a$. Moreover for t fixed in this range $S_t a$ is again an analytic element for H and one can define $S_s(S_t a)$ for suitably small s. Calculation with norm convergent power series then establishes that

$$S_t(S_t a) = S_{s+t} a$$

for all s, t satisfying $|s| + |t| < t_a$. Next we examine properties of the function $t \in \langle -t_a, t_a \rangle \mapsto \|S_t a\|$.

First one has

$$\big| \|S_t a\| - \|S_s a\| \big| \leq \|S_t a - S_s a\|$$

by the triangle inequality. But another power series estimation of the right hand side then establishes that $t \mapsto \|S_t a\|$ is continuous. Second for $0 < h < t < t_a$ one has

$$\|S_{t-h}a\| = \left\|S_{-h}\left(S_t a\right)\right\|$$

$$= \lim_{n\to\infty} \left\|\left(I+\frac{h}{n}H\right)^n \left(S_t a\right)\right\|$$

$$\geq \|S_t a\|$$

where we have used the assumed norm-dissipativity of H . But this estimate implies that $t \in \langle 0, t_a\rangle \mapsto \|S_t a\|$ is decreasing and hence

$$\|S_t a\| \leq \|a\|$$

for $0 \leq t < t_a$. This contractive estimate now allows one to extend the definition of $S_t a$ to all $t \geq 0$.

Since H is closed $S_t a \in D(H)$ and

$$HS_t a = S_t(Ha) .$$

Therefore

$$\|HS_t a\| = \|S_t(Ha)\| \leq \|Ha\|$$

for $0 < t < t_a$. Iteration of this argument establishes that if $0 < t < t_a$ then $S_t a$ is an analytic element for S with associated radius of convergence equal to t_a . Thus it is possible to iterate the definition of S_t

$$S_{t+s}a = S_t\left(S_s a\right) = \sum_{n\geq 0} \frac{(-t)^n}{n!} H^n\left(S_s a\right)$$

for $0 < s$, $t < t_a$ and consequently deduce that $\|S_t a\| \le \|a\|$ for all $0 < t < 2t_a$. Repeating this argument one defines $S_t a$ for all $t \ge 0$ by

$$S_t a = \left(S_{t/n}\right)^n a$$

where n is chosen so that $n > t/t_a$. It is then easy to establish that this definition is independent of the choice of n ,

$$S_s\left(S_t a\right) = S_{s+t} a \ ,$$

for all $s, t > 0$,

$$\|S_t a\| \le \|a\|$$

for all $t > 0$, and

$$\lim_{t\to 0} \|S_t a - a\| = 0 \ .$$

Therefore, since the analytic elements are assumed to be norm dense, S extends by continuity to a C_0-semigroup of contractions on $\mathcal{B}$. □

The foregoing result readily extends to C_0-groups of contractions. But if $S = \{S_t\}_{t\in\mathbb{R}}$ is a group of contractions with $S_0 = I$ then S is automatically isometric because

$$\|a\| = \|S_{-t}S_t a\| \le \|S_t a\| \le \|a\| \ .$$

Second if S is also strongly continuous then $S_\pm = \{S_{\pm t}\}_{t\ge 0}$ are both C_0-semigroups of isometries. But

$$\left\|\frac{(I-S_t)}{t}a - b\right\| = \left\|S_t\frac{(I-S_{-t})}{t}a + b\right\|$$

and hence the generator of S_+ is minus the generator of S_- . Combining these observations with Proposition 1.4.1 and the construction of analytic elements described prior to the propositon one obtains the following.

THEOREM 1.6.2. *Let* H *be an operator on the Banach space* $\mathcal{B}$. *The following conditions are equivalent:*

1. H *is the infinitesimal generator of a* C_0*-group of isometries of* $\mathcal{B}$.

2. H *is norm closed;* H *possesses a norm dense set of analytic elements* ±H *are both norm-dissipative.*

Proof. **1 ⇒ 2.** The entire analytic elements for H are dense by the construction preceding Proposition 1.6.1. The rest of the properties of H follow from the Hille-Yosida theorem.

2 ⇒ 1. This follows by successively applying Proposition 1.6.1 to ±H and then using the above observation that a group $S_t = \exp\{-tH\}$ of contractions is automatically isometric. □

One can also give a C_0^*-version of Proposition 1.6.1 and then deduce a weak*-version of Theorem 1.6.2. Since the second result is deduced by the same argument given above we will merely prove the analogue of Proposition 1.6.2.

PROPOSITION 1.6.3. *Let $\mathcal{B}$ be a Banach space with a predual $\mathcal{B}_*$ and* H *a weak*-weak*-closed operator on $\mathcal{B}_*$. Suppose*

1. *the unit ball of the set of analytic elements for* H *is weak*-dense in the unit ball of $\mathcal{B}$,*

2. H *is norm-dissipative.*

It follows that H *is the generator of a C_0^*-semigroup of contractions.*

Proof. Let $\mathcal{B}_a \subseteq \mathcal{B}$ denote the norm closure of the subspace of all analytic elements for H and let H_a denote the restriction of H to $\mathcal{B}_a$. It follows immediately that H_a is norm closed and hence by Proposition 1.6.1 it generates a C_0-semigroup S of contractions on $\mathcal{B}_a$. In particular H_a is norm-dissipative and $R(I+\alpha H_a) = \mathcal{B}_a$ for all $\alpha > 0$.

Now by Condition 1 we may choose for each $f \in \mathcal{B}$ a family $f_\beta \in \mathcal{B}_a$ such that f_β converges to f in the weak*-sense and $\|f_\beta\| \le \|f\|$. But it follows from the foregoing argument that there exist $g_\beta \in D(H_a) \subseteq D(H)$ such that $f_\beta = (I+\alpha H)g_\beta$ and

$$\|g_\beta\| \le \|(I+\alpha H_a)g_\beta\| = \|f_\beta\| \le \|f\| .$$

Thus $\{\|g_\beta\|\}$ is uniformly bounded. But the unit ball in $\mathcal{B}$ is weak*-compact, by the Alaoglu-Birkhoff theorem, and hence one may choose a weak*-convergent subfamily $g_{\beta'}$ of g_β. Let g denote its limit. Then $g_{\beta'} \to g$ and $f_{\beta'} = (I+\alpha H_a)g_{\beta'} \to f$ where both limits are in the weak*-sense.

But H is weak*-weak*-closed and so H_a is weak*-weak*-closable and its closure $\overline{H}_a$ is both norm-dissipative and satisfies the range condition $R(I+\alpha\overline{H}_a) = \mathcal{B}$ for $\alpha > 0$. Therefore $\overline{H}_a$ generates a C_0^*-semigroup S by Theorem 1.5.2. But H is a norm dissipative extension of $\overline{H}_a$ and since the latter is a generator one must have $H = \overline{H}_a$. □

We conclude this section with a Hilbert space example.

Example 1.6.4. Consider the criteria of Theorems 1.3.1 and 1.6.2 for a C_0-group of isometries on a Hilbert space $\mathcal{H}$. Norm-dissipativity of $\pm H$ is equivalent to

$$\mathrm{Re}(a, Ha) = 0$$

for all $a \in D(H)$. Setting $H = iK$ this becomes

$$(a, Ka) = (Ka, a)$$

for all $a \in D(K)$, i.e., K must be a symmetric operator. Thus Theorems 1.3.1 and 1.6.2 state that H is the generator of a C_0-group of isometries if, and only if, $H = iK$ where K is a densely defined, closed, symmetric operator satisfying

either $R(I+i\alpha K) = \mathcal{H}$, $\alpha \in \mathbb{R}\setminus\{0\}$

or K possesses a dense set of analytic elements.

The first of these conditions is the usual criterion for self-adjointness of K. Hence one can conclude from this argument that a densely defined, closed, symmetric opperator is self-adjoint if, and only if, it possesses a dense set of analytic elements.

If these conditions are satisfied then the associated operators $S_t = \exp\{-iKt\}$ form a unitary group, e.g., $S_t^* = S_{-t}$. Both the unitary group and the generator can be represented by spectral theory as direct integrals of multiplication operators. In particular there exists a family of projection valued probability measures E over $\mathbb{R}$ such that

$$\left(a, S_t b\right) = \int_{-\infty}^{\infty} d(a, E(\lambda)b)e^{-i\lambda t}$$

for all $a, b \in H$ and

$$(a, Kb) = \int_{-\infty}^{\infty} d(a, E(\lambda)b)\lambda$$

for all $a \in H$, and $b \in D(K)$, where the domain of K is defined by

$$D(K) = \left\{b \; ; \int_{-\infty}^{\infty} d(b, E(\lambda)b)\lambda^2 < +\infty\right\} . \qquad \square$$

In the Hilbert space context one can further elaborate the extension theory mentioned at the end of Section 1.3. Thus given a symmetric operator K one tries to construct self-adjoint extensions. This construction is a repetition of the procedure outlined in Section 1.3. Both $\pm iK$ must be extended to generators iK_1, $-iK_2$, of contraction semigroup, $S^{\pm}$. But these semigroups determine a C_0-group of isometries, by $S_t = S_t^+$ if $t \geq 0$ and

$S_t = S_t^-$ if $t \le 0$, if, and only if, $K_1 + K_2 = 0$. To obtain this latter relation it is imperative that the deficiency indices of $\pm K$ are identical.

Exercises.

1.6.1. An element $a \in \mathcal{B}$ is defined to be bounded for H if $a \in D(H^n)$ for all $n \ge 1$ and

$$\|H^n a\| \le r^n \|a\|$$

for some $r \ge 0$. Prove that if H is the generator of the C_0-semigroup S and a is bounded for H then

$$Ha = t^{-1} \sum_{n\ge1} (I-S_t)^n a/n$$

for $rt \le 1$.

Hint: Use $(I-S_t)a = \int_0^t ds\, S_s Ha$.

1.7. Holomorphic Semigroups.

Among the many semigroups which occur in applications one class is very common, the holomorphic semigroups. Roughly speaking these are the semigroups $t \geq 0 \mapsto S_t \in \mathcal{L}(\mathcal{B})$ which can be continued holomorphically into a sector of the complex plane containing the positive axis. Among these semigroups one can also identify a subclass analogous to the M-bounded semigroups, i.e., the semigroups satisfying a bound of the form $\|S_t\| \leq M$. This subclass consists of holomorphic semigroups which are uniformly bounded within appropriate subsectors of the sector of holomorphy. For example if H is a positive self-adjoint operator on the Hilbert space $\mathcal{H}$ and $S_t = \exp\{-tH\}$ is the corresponding semigroup then $a \in \mathcal{H} \mapsto S_t a \in \mathcal{H}$ extends to a vector valued function holomorphic in the right half plane satisfying

$$\|S_z a\| = \|S_{\mathrm{Re}\ z} a\| \leq \|a\|$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}\ z \geq 0$. Thus S is a bounded holomorphic semigroup with the right half plane as region of holomorphy.

The general definition of these semigroups is as follows.

DEFINITION 1.7.1. *A* C_0*-semigroup* S *on the Banach space* $\mathcal{B}$ *is called a holomorphic semigroup if for some* $\theta \in \langle 0, \pi/2]$ *one has the following properties:*

1. $t \geq 0 \mapsto S_t$ *is the restriction to the positive real axis of a holomorphic operator function*

$z \in \Delta_\theta \mapsto S_z \in \mathcal{L}(B)$ *where* $\Delta_\theta = \{z \; ; \; |\text{Arg } z| < \theta\}$,

2.
$$S_{z_1} S_{z_2} = S_{z_1+z_2}$$

for all $z_1, z_2 \in \Delta_\theta$,

3.
$$\lim_{z\in\Delta_\theta, z\to 0} \|S_z a - a\| = 0$$

for all $a \in B$.

If additionally S *is uniformly bounded in* Δ_{θ_1} *for each* $0 < \theta_1 < \theta$ *then* S *is called a bounded holomorphic semigroup.*

There are a variety of ways of characterizing holomorphic semigroups and the following theorem presents two characterizations in terms of the derivative of $t \mapsto S_t$ and the derivatives of the powers $(I+\alpha H)^{-n}$ of the resolvent $(I+\alpha H)^{-1}$.

THEOREM 1.7.2. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup on the Banach space* B . *The following conditions are equivalent:*

1. S *is a (bounded) holomorphic semigroup,*

2. *there is a* $C > 0$ *such that*

$$\|HS_t\| < Ct^{-1}$$

for all $0 < t \leq 1$ *(for all* $t \geq 0$*)* ,

3. *there is a* $C > 0$ *such that*

$$\|H(I+\alpha H)^{-(n+1)}\| \le C(\alpha n)^{-1}$$

for $0 < \alpha \le 1$ *,* $n\alpha \le 1$ *, and* $n = 1, 2, \ldots$

(for $\alpha > 0$ *and* $n = 1, 2, \ldots$ *.)*

N.B. In the above formulation the parenthetic conditions should be read simultaneously to give a characterization of bounded holomorphic semigroups. Their omission covers the general case.

Proof. 1 ⇒ 2. Assume S has a holomorphic extension to $\Delta_\theta = \{z \; ; \; |\mathrm{Arg}\, z| < \theta\}$. Since S is continuous it follows from the principle of uniform boundedness that there exists an M_1 such that $\|S_z\| \le M_1$ for all $z \in \Delta_{\theta_1} \cap \{z \; ; \; |z| \le 2\}$ where $0 < \theta_1 < \theta$. But by Cauchy's integral representation

$$HS_t = \frac{-d}{dt} S_t = (2\pi i)^{-1} \int_{C_1} dz \, \frac{S_z}{(z-t)^2}$$

with $C_1 = \{z \; ; \; |z - t| = \mathrm{Sin}\, \theta_1 t\}$. Consequently

$$\|HS_t\| \le \frac{M_1}{\mathrm{Sin}\, \theta_1} \frac{1}{t}$$

for all $0 < t \le 1$. Moreover if $\|S_z\|$ is uniformly bounded in Δ_{θ_1} the same argument establishes the estimate for all $t > 0$.

2 ⇒ 3. Since S is a C_0-semigroup there exist constants $M \ge 1$ and $\omega \ge 0$ such that

$$(*) \qquad \|HS_t\| < \frac{C_1 e^{\omega_1 t}}{t} .$$

But

$$H(I+\alpha H)^{-(n+1)} = (n!)^{-1} \int_0^\infty dt\ t^n e^{-t} HS_{\alpha t}$$

and hence

$$\|H(I+\alpha H)^{-(n+1)}\| \le (n!)^{-1} \int_0^\infty dt\ t^{n-1}\alpha^{-1}C_1 e^{-t(1-\alpha\omega_1)}$$

$$= \left(\frac{C_1}{n\alpha}\right)\left(\frac{1}{1-\alpha\omega_1}\right)^n , \qquad 0 < \alpha\omega_1 < 1$$

$$\le \left(\frac{C_1}{n\alpha}\right)\left(\frac{1}{1-\omega_1/n}\right)^n$$

$$\le \left(\frac{C_1}{n\alpha}\right) \frac{1}{1-\omega_1} .$$

Where the second inequality follows from $n\alpha \le 1$ and the third follows because $x \mapsto \left(1-\omega_1/x\right)^{-x}$ is decreasing.

Note that in the bounded case (*) is valid with $\omega_1 = 0$ and then the required bound follows for all $\alpha > 0$.

3 ⇒ 2. It follows directly from Condition 3 and Remark 1.3.3 that

$$\|HS_t\| = \lim_{n\to\infty} \|H\left(I + \frac{t}{n} H\right)^{-n}\| \le Ct^{-1} .$$

2 ⇒ 1. This implication can be established by a variety of arguments which begin with a power series definition. We will briefly sketch the sequence of ideas.

First let $z = t + is$ with $|s| < t/Ce$ and $0 < t \leq 1$. Then one can define S_z by the norm convergent power series

$$S_z = \sum_{n\geq 0} \frac{(-is)^n}{n!} \left(HS_{t/n}\right)^n .$$

Second one calculates that $S_z D(H) \subseteq D(H)$ and

$$\frac{d}{dz} S_z a = -HS_z a = -S_z Ha$$

for all $a \in D(H)$. Thus

$$\|(S_z - I)a\| \leq |z| \; \|Ha\|$$

and consequently

$$\lim_{z\to 0} \|(S_z - I)a\| = 0$$

for all $a \in D(H)$. But then the same conclusion is valid for all $a \in \mathcal{B}$ because $D(H)$ is norm dense.

Third if $0 < t \leq 1$, $a \in D(H)$, and $z_1, z_2, z_1 + z_2$ are in the domain of definition of S_z, the foregoing identification of the derivative gives

$$\frac{d}{dt} \left\{S_{tz_1} S_{tz_2} - S_{t(z_1+z_2)}\right\} a = 0 .$$

Thus integrating and using strong continuity at the origin one finds

$$\left(S_{z_1}S_{z_2}-S_{z_1+z_2}\right)a = 0 .$$

But $D(H)$ is norm dense and hence

$$S_{z_1}S_{z_2} = S_{z_1+z_2} .$$

Finally one must extend the definition of S_z to the region $\Delta_\theta = \{z ; \ \mathrm{Re}\, z > 0 \ |\mathrm{Arg}\, z| < \theta\}$ where $\mathrm{Tan}\,\theta = 1/Ce$. This is achieved by first remarking that each $z \in \Delta_\theta$ can be decomposed in the form $z = z_1 + z_2 + \ldots + z_n$ with $z_i \in \Delta_\theta$ and $\mathrm{Re}\, z_i \leq 1$. Then one defines

$$S = S_{z_1} S_{z_2} \ldots S_{z_n} .$$

There is, however, a problem of consistency since the decomposition of z is clearly not unique. But consistency is easily established by use of the semigroup property in the restricted region. The semigroup property for the larger region then follows by definition.

In the bounded case this last argument is superfluous because S_z can be defined for all $z \in \Delta_\theta$ by the power series expansion and this also establishes that $\|S_z\|$ is uniformly bounded in Δ_{θ_1} for each $0 < \theta_1 < \theta$. □

There are alternative characterizations of holomorphic semigroups in terms of spectral properties of the generator and resolvent bounds. Typically one has the following criterion for a bounded holomorphic semigroup.

THEOREM 1.7.3. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup on the Banach space* $\mathcal{B}$ *.*

The following conditions are equivalent:

1. S *is a bounded holomorphic semigroup,*

2. *there is a* $\theta > 0$ *such that*

$$\sigma(H) \subseteq \overline{\Delta}_{\frac{\pi}{2}-\theta} = \{z \; ; \; |\text{Arg } z| \le \tfrac{\pi}{2} - \theta\}$$

where $\sigma(H)$ *denotes the spectrum of* H *. Moreover*

$$\|(zI-H)^{-1}\| \le M_1 \big/ d_{\theta_1}(z)$$

for all $z \in \mathbb{C}\backslash\overline{\Delta}_{\frac{\pi}{2}-\theta_1}$ *, where* $0 \le \theta_1 < \theta$ *,*

$$d_{\theta_1}(z) = \inf\{|w-z| \; ; \; w \in \Delta_{\frac{\pi}{2}-\theta_1}\}$$

and M_1 *can depend on* θ_1 *.*

Proof. $1 \Rightarrow 2$. Suppose $z \mapsto S_z$ is holomorphic in the sector $\Delta_\theta = \{z \; ; \; |\text{Arg } z| < \theta\}$. Next consider the C_0-semigroups $S_t^w = \exp\{-twH\}$ where $w = \exp\{i\alpha\}$ and $0 \le |\alpha| < \theta$. The generator of S^w is wH and hence $\sigma(wH) \subseteq \{z \; ; \; \text{Re } z \ge 0\}$, by Proposition 1.2.1. Therefore $\sigma(H) \subseteq \{z \; ; \; |\text{Arg } z| \le \frac{\pi}{2} - \theta\}$. Moreover, since there is an M_1 such that $\|S_t^w\| \le M_1$ for

$w \in \Delta_{\theta_1}$ where $0 \le \theta_1 < \theta$, one must have

$$\|(\lambda I - wH)^{-1}\| = \|\int_0^\infty dt\ e^{\lambda t} S_t^w\| \le M_1 \Big/ |\text{Re}\ \lambda|$$

whenever $\text{Re}\ \lambda < 0$. Consequently

$$\|(zI-H)^{-1}\| \le M_1 \Big/ d_{\theta_1}(z) .$$

$2 \Rightarrow 1$. The detailed proof of this implication is rather protracted, although completely straightforward. Again we only sketch the outlines.

First let Γ be a wedge shaped contour lying in the resolvent set $r(H)$ of H with asymptotes $\text{Arg}\ z = \pm\left(\frac{\pi}{2}-\theta_2\right)$ where $0 \le \theta_2 < \theta_1$ and for $z \in \Delta_\theta$ define S by

$$S_z = (2\pi i)^{-1} \int_\Gamma d\lambda\ e^{\lambda z}(\lambda I - H)^{-1} .$$

By Cauchy's theorem the integral is independent of the particular contour chosen and one can use this freedom of choice, together with the resolvent bounds, to deduce that $z \in \Delta_\theta \mapsto \|S_z\|$ is uniformly bounded.

Second one calculates that S satisfies the semigroup property $S_{z_1} S_{z_2} = S_{z_1+z_2}$ by choosing Γ_2 outside Γ_1 and noting that

$$S_{z_1}S_{z_2} = (2\pi i)^{-2} \int_{\Gamma_1} \int_{\Gamma_2} d\lambda_1 d\lambda_2 \; e^{\lambda_1 z_1 + \lambda_2 z_2} (\lambda_1 I - H)^{-1} (\lambda_2 I - H)^{-1}$$

$$= (2\pi i)^{-2} \int_{\Gamma} \int_{\Gamma} d\lambda_1 d\lambda_2 \; \frac{e^{\lambda_1 z_1 + \lambda_2 z_2}}{\lambda_2 - \lambda_1} \left\{ (\lambda_1 I - H)^{-1} - (\lambda_2 I - H)^{-1} \right\}$$

$$= (2\pi i)^{-1} \int_{\Gamma} d\lambda \; e^{\lambda(z_1 + z_2)} (\lambda I - H)^{-1} .$$

Here we have used the obvious resolvent identity, Cauchy's theorem, and Fubini's theorem.

Third one notes that if $a \in D(H)$

$$\left(I - S_z\right)a = (2\pi i)^{-1} \int_{\Gamma} d\lambda \; e^{\lambda z} \left\{ \lambda^{-1} I - (\lambda I - H)^{-1} \right\} a$$

$$= -(2\pi i)^{-1} \int_{\Gamma} d\lambda \; e^{\lambda z} \lambda^{-1} (\lambda I - H)^{-1} Ha$$

$$\xrightarrow[z \to 0]{} 0$$

when the last conclusion follows from the resolvent bound and the Lebesgue dominated convergence theorem.

Finally one identifies H as the generator of S by careful calculation of the derivative of S. This again requires Cauchy's theorem. $\square$

One simple explicit example of a bounded holomorphic semigroup is the semigroup S generated by the Laplacian on $L^p(\mathbb{R}^\nu)$. This semigroup is holomorphic in the sector $\Delta_{\pi/2}$ and its action is given by

$$\left(S_z a\right)(x) = (4\pi z)^{-\nu/2} \int d^\nu y \; e^{-(x-y)^2/4z} a(y) .$$

Note that if $p = 2$ then

$$\|S_2 a\|_2 = \|S_{\mathrm{Re}\ z} a\|_2 \leq \|a\|_2$$

since $S_z = \exp\{-zH\}$ where H is self-adjoint. Moreover S has a boundary value as $\mathrm{Re}\ z \to 0$ because

$$\lim_{s \to 0} \|S_{s+it} a - e^{-itH} a\| = 0 \ .$$

But if $p = 1$

$$\|S_z\| = \int d^{\nu}y \, |(4\pi z)^{-\nu/2} e^{-y^2/4z}| = \left(|z| \ \mathrm{Re}\ z\right)^{\nu/2}$$

for $\mathrm{Re}\ z > 0$, and a similar result is true for $p = \infty$. Thus in these latter cases $\|S_z\| \to \infty$ as z approaches the imaginary axis, away from the origin, and S does not have a boundary value.

Exercises 1.7.1.

1. Let S be a self-adjoint contraction semigroup on a Hilbert space $\mathcal{H}$. Prove that S is holomorphic for $\mathrm{Re}\ z > 0$ and that $\|S_z\| \leq 1$ in this sector.

1.8. Convergence of Semigroups

In the preceding sections we examined the existence and construction of various classes of semigroup and next we analyze their stability properties. First we consider convergence properties and use these to extend the foregoing results on semigroup construction.

Let $S^{(n)}$ be a sequence of C_0-semigroups on a Banach space $\mathcal{B}$ and assume that $S_t^{(n)}$ converges strongly to S_t, for each $t \geq 0$. Since the product of strongly convergent sequences is strongly convergent the S_t must satisfy the semigroup property $S_sS_t = S_{s+t}$ for all $s, t \geq 0$, and of course $S_0 = I$. Nevertheless $S = \{S_t\}_{t\geq 0}$ is not necessarily a C_0-semigroup because of a possible lack of continuity. The simplest example of this phenomenon is given by the numerical semigroups $S_t^{(n)} = e^{-nt}$ acting on $\mathbb{C}$. The limit S satisfies $S_0 = I$, and $S_t = 0$ if $t > 0$; it is clearly discontinuous. Thus it is of interest to establish conditions for stability of C_0-semigroups under strong convergence and to identify stability criteria in terms of the generators.

Although the strong limit S of the sequence $S^{(n)}$ often fails to be a C_0-semigroup on the whole Banach space $\mathcal{B}$ it is possible that its restriction to a Banach subspace $\mathcal{B}_0$ is a C_0-semigroup. For example if $\mathcal{B} = \mathcal{B}_0 \oplus \mathbb{C}$ and $S_t^{(n)} = T_t \oplus e^{-nt}$, where T is a fixed C_0-semigroup on $\mathcal{B}_0$, then the limit S is discontinuous for a rather trivial reason; on the subspace $\mathcal{B}_0$ one has continuity, because $S = T$, and the discontinuity only occurs in the extra dimension. Thus it is of some interest to broaden the discussion of stability of convergence by attempting to identify subspaces of continuity for the limit semigroup.

We begin the analysis by first establishing that semigroup convergence is equivalent to convergence of the resolvents of the generators. For simplicity we consider contraction semigroups.

PROPOSITION 1.8.1. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be a sequence of* C_0*-semigroups of contractions on the Banach space* $\mathcal{B}$ *and let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions acting on a Banach subspace* $\mathcal{B}_0 \subseteq \mathcal{B}$.

The following four conditions are equivalent:

1(1')
$$\lim_{n\to\infty} \left\|\left(S_t^{(n)}-S_t\right)a\right\| = 0$$

for all $a \in \mathcal{B}_0$ *and all* $t \geq 0$ *(uniformly for* t *in finite intervals of* $[0, \infty\rangle$*)* ,

2(2')
$$\lim_{n\to\infty} \left\|\left((I+\alpha H_n)^{-1}-(I+\alpha H)^{-1}\right)a\right\| = 0$$

for all $a \in \mathcal{B}_0$ *and for some* $\alpha > 0$ *(uniformly for* α *in finite intervals of* $[0, \infty\rangle$*)* .

Clearly 1' ⇒ 1 and 2' ⇒ 2. The proof that 2 ⇒ 2' involves two arguments. First one uses the Neumann series

$$\left(I+\alpha H_n\right)^{-1} = \left(\frac{\alpha_0}{\alpha}\right) \sum_{n=0}^{\infty} \left(\frac{\alpha-\alpha_0}{\alpha}\right)^n \left(I+\alpha_0 H_n\right)^{-n-1} ,$$

which is convergent for $\alpha > \alpha_0/2$, to prove that resolvent convergence for $\alpha = \alpha_0$ implies resolvent convergence for all $\alpha > \alpha_0/2$, and hence by iteration for all $\alpha > 0$. Second one extimates from the Laplace transform relation

$$(I+\alpha H_n)^{-1}a = \alpha^{-1}\int_0^\infty dt\ e^{-\alpha^{-1}t}S_t^{(n)}a$$

that

$$\left\|\left\{(I+\alpha_1 H_n)^{-1}-(I+\alpha_2 H_n)^{-1}\right\}a\right\| \le 2(\alpha_1-\alpha_2)\alpha_2^{-1}\|a\|$$

for $\alpha_1 > \alpha_2 > 0$. Hence the convergence is uniform for α in finite intervals by a standard equicontinuity argument.

Next we argue that $1 \Rightarrow 2 \Rightarrow 1'$.

$1 \Rightarrow 2.$ By Laplace transformation one has

$$\begin{aligned}\left\|\left\{(I+\alpha H_n)^{-1} - (I+\alpha H)^{-1}\right\}a\right\| &= \left\|\int_0^\infty dt\ e^{-t}\left(S_{\alpha t}^{(n)}-S_{\alpha t}\right)a\right\| \\ &\le \int_0^\infty dt\ e^{-t}\left\|\left(S_{\alpha t}^{(n)}-S_{\alpha t}\right)a\right\| \\ &\xrightarrow[n=\infty]{} 0\end{aligned}$$

where the last conclusion follows from the Lebesgue dominated convergence theorem.

$2 \Rightarrow 1'.$ Since the semigroups under discussion are all contractive on $\mathcal{B}_0$ it suffices to prove their convergence on a norm dense subspace of $\mathcal{B}_0$. We will repeat the tactic used in the construction of S in Theorem 1.3.1 and work on the norm dense subspace $D(H^2)$. Now $D(H^2) = R((I+\alpha H)^{-2})$ for $\alpha > 0$. Moreover if $a \in \mathcal{B}_0$ then

$$\left(S_t^{(n)} - S_t\right)(I+\alpha H)^{-2}a = A_t^{(n)} + B_t^{(n)} + C_t^{(n)}$$

where

$$A_t^{(n)} = S_t^{(n)}\left\{(I+\alpha H)^{-1}-\left(I+\alpha H_n\right)^{-1}\right\}(I+\alpha H)^{-1}a \ ,$$

$$B_t^{(n)} = \left(I+\alpha H_n\right)^{-1}\left(S^{(n)}-S_t\right)(I+\alpha H)^{-1}a \ ,$$

and

$$C_t^{(n)} = \left\{\left(I+\alpha H_n\right)^{-1}-(I+\alpha H)^{-1}\right\}(I+\alpha H)^{-1}S_t a \ .$$

Let us estimate each of these terms. First

$$\left\|A_t^{(n)}\right\| \le \left\|\left\{\left(I+\alpha H_n\right)^{-1}-(I+\alpha H)^{-1}\right\}(I+\alpha H)^{-1}a\right\| \xrightarrow[n=\infty]{} 0 \ .$$

Second

$$\left\|C_t^{(n)}\right\| = \left\|\left\{\left(I+\alpha H_n\right)^{-1}-(I+\alpha H)^{-1}\right\}(I+\alpha H)^{-1}S_t a\right\| \xrightarrow[n=\infty]{} 0 \ .$$

But using $\|\left(I+\alpha H_n\right)^{-1}\| \le 1$ and $\|(I+\alpha H)^{-1}\| \leqq 1$ one readily derives the equicontinuity relation

$$\left\|C_{t_1}^{(n)} - C_{t_2}^{(n)}\right\| \le 2\left\|\left(S_{t_1}-S_{t_2}\right)a\right\|$$

and hence $A^{(n)}$ and $C^{(n)}$ converge to zero uniformly for t in any finite interval of $[0, \infty\rangle$. It remains to examine $B^{(n)}$. For this we use the integral representation

$$B_t^{(n)} = \int_0^t ds \, \frac{d}{ds} \, S_s^{(n)} \left(I+\alpha H_n\right)^{-1} S_{t-s} (I+\alpha H)^{-1} a$$

$$= \alpha^{-1} \int_0^t ds \, S_s^{(n)} \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} S_{t-s} a \, .$$

Thus

$$\left\| B_t^{(n)} \right\| \leq \alpha^{-1} \int_0^t ds \, \left\| \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} S_{t-s} a \right\|$$

$$\xrightarrow[n=\infty]{} 0$$

where the last conclusion follows from the Lebesgue dominated convergence theorem. But for $t_1 \geq t_2$ one has

$$B_{t_1}^{(n)} - B_{t_2}^{(n)} = \alpha^{-1} \int_{t_2}^{t_1} ds \, S_s^{(n)} \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} S_{t_1 - s} a$$

$$+ \alpha^{-1} \int_0^{t_2} ds \, S_s^{(n)} \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} S_{t_2 - s} \left(S_{t_1 - t_2} - I \right) a \, .$$

Therefore

$$\left\| \left(B_{t_1}^{(n)} - B_{t_2}^{(n)} \right) \right\| \leq 2\alpha^{-1} \left\{ \left(t_1 - t_2\right) \|a\| + \left\| \left(S_{t_1 - t_2} - I \right) a \right\| \right\}$$

and the convergence is again uniform for t in any finite interval of $[0, \infty\rangle$.

Combining these conclusions we see that

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t \right) b \right\| = 0$$

for all $b \in D\left(H^2\right)$, and consequently for all $b \in \mathcal{B}_0$. Moreover the convergence is uniform for t in finite intervals of $[0, \infty\rangle$. □

Although Proposition 1.8.1 could be viewed as a criterion for strong convergence of semigroups it does have two distinct drawbacks. First it gives an indirect link between semigroup convergence and convergence of the generators, because it concerns convergence of their resolvents. Second it assumes that the limit of the resolvents is the resolvent of a generator of a C_0-semigroup, at least on a subspace. The next theorem avoids both these disadvantages and relates semigroup convergence directly to graph convergence of the generators. This latter notion is introduced as follows.

If H_n is a sequence of operators on the Banach space $\mathcal{B}$ then the *graphs* $G(H_n)$ of H_n are defined as subspaces of $\mathcal{B} \times \mathcal{B}$ by

$$G(H_n) = \Big\{\{a, H_n a\} \; ; \; a \in D(H_n)\Big\} .$$

Now consider all sequences $a_n \in D(H_n)$ such that

$$\lim_{n\to\infty} \|a_n - a\| = 0 \; , \quad \lim_{n\to\infty} \|H_n a_n - b\| = 0$$

for some pair $\{a, b\} \in \mathcal{B} \times \mathcal{B}$. The pairs $\{a, b\}$ obtained in this way form a subspace G of $\mathcal{B} \times \mathcal{B}$ and we introduce the notation $D(G)$ for the set of a such that $\{a, b\} \in G$ for some b. Similarly $R(G)$ is the set of b such that $\{a, b\} \in G$ for some a. Moreover we write

$$G = \lim_{n\to\infty} G(H_n) .$$

In general G is not the graph of an operator but if there exists an operator H on the Banach space $\mathcal{B}$, or on a Banach subspace $\mathcal{B}_0$, such that $G = G(H)$ then H is called the *graph limit* of the H_n. Clearly in this case $D(G) = D(H)$ and $R(G) = R(H)$.

The next result demonstrates that this kind of convergence is appropriate for the characterization of semigroup convergence.

THEOREM 1.8.2. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be a sequence of* C_0-*semigroups of contractions on the Banach space* $\mathcal{B}$ *and define the subspaces* $G_\alpha \subseteq \mathcal{B} \times \mathcal{B}$

$$G_\alpha = \lim_{n\to\infty} G(I+\alpha H_n) .$$

The following conditions are equivalent:

1. *There exists a Banach subspace* $\mathcal{B}_0$ *of* $\mathcal{B}$ *and a* C_0-*semigroup* S *on* $\mathcal{B}_0$ *such that*

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t\right) a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *and* $t > 0$, *uniformly for* t *in any finite interval of* $[0, \infty\rangle$,

2. *There exists a Banach subspace* $\mathcal{B}_0$ *of* $\mathcal{B}$ *such that*

$$\mathcal{B}_0 = \overline{\{a \; ; \; \{a, b\} \in G_\alpha \;\; \textit{for some} \;\; b \in \mathcal{B}_0\}}$$

$$\mathcal{B}_0 = \overline{\{b \; ; \; \{a, b\} \in G_\alpha \;\; \textit{for some} \;\; a \in \mathcal{B}_0\}}$$

for some $\alpha > 0$ *, where the bar denotes norm closure.*

If these conditions are satisfied then S *is a contraction semigroup on* $\mathcal{B}_0$ *,* G_α *is the graph of* $I + \alpha H$ *where* H *is the generator of* S *, and* $\mathcal{B}_0 = \overline{D(G_\alpha)} = R(G_\alpha)$.

Proof. 1 ⇒ 2. It follows from strong convergence that

$$\|S_t a\| = \lim_{n\to\infty} \|S_t^{(n)} a\| \le \|a\|$$

for all $a \in \mathcal{B}_0$ and hence S is a contraction semigroup on $\mathcal{B}_0$. Let H denote the generator of S then

$$\lim_{n\to\infty} \left\|\left(I+\alpha H_n\right)^{-1} a - (I+\alpha H)^{-1} a\right\| = 0$$

for all $a \in \mathcal{B}_0$ by Proposition 1.8.1. Thus if $a_n = \left(I+\alpha H_n\right)^{-1} a$ one has

$$\lim_{n\to\infty} \left\|a_n - (I+\alpha H)^{-1} a\right\| = 0$$

and

$$\left(I+\alpha H_n\right) a_n = a \; .$$

Consequently $\{(I+\alpha H)^{-1} a, a\} \in G$. This demonstrates Condition 2 and gives the identification of G_α , $D(G_\alpha)$, and $R(G_\alpha)$.

$2 \Rightarrow 1$. Define G to be the set of pairs $\{a, b\} \in \mathcal{B}_0 \times \mathcal{B}_0$ such that there exists a sequence $a_n \in D(H_n)$ with the property that $a_n \to a$, and $H_n a_n \to b$, as $n \to \infty$. To prove that G is the graph of an operator on $\mathcal{B}_0$ we must demonstrate that $a = 0$ implies $b = 0$. But suppose $a = 0$ and for an arbitrary pair $\{a', b'\} \in G$ choose $a_n' \in D(H_n)$ such that $a_n' \to a'$, and $H_n a_n' \to b'$, as $n \to \infty$. Then

$$\|\alpha(a'+b) + \alpha^2 b'\| = \lim_{n\to\infty} \|(I+\alpha H_n)(a_n+\alpha a_n')\|$$

$$\geq \lim_{n\to\infty} \|a_n + \alpha a_n'\| = \alpha\|a'\| .$$

Dividing by α and taking the limit $\alpha \to 0$ one obtains

$$\|b + a'\| \geq \|a'\| .$$

But this inequality is true for all $a' \in D(G)$. Moreover $D(G) = D(G_\alpha)$ and hence $D(G)$ is norm dense in $\mathcal{B}_0$, by assumption. Therefore one must have $b = 0$ and consequently G is the graph of a norm densely defined operator H on $\mathcal{B}_0$.

Now $G_\alpha = G(I+\alpha H)$ and it follows by limiting that $\|(I+\alpha H)a\| \geq \|a\|$ for all $a \in D(H) = D(G_\alpha)$. The same inequality then extends to the closure $\overline{H}$ of H and it readily follows that $R(I+\alpha\overline{H})$ is norm closed. But $R(I+\alpha\overline{H}) = \overline{R(G_\alpha)} = \mathcal{B}_0$ and hence $\overline{H}$ is the generator of a C_0-semigroup of contractions S by the Hille-Yosida theorem. Now if $a_n \to a$ and $b_n = (I+\alpha H_n)a_n \to (I+\alpha H)a = b$ then

$$\left\|\left\{(I+\alpha H_n)^{-1} - (I+\alpha\overline{H})^{-1}\right\}b\right\| = \left\|(I+\alpha H_n)^{-1}(b-b_n) + (a_n-a)\right\|$$

$$\leq \|b - b_n\| + \|a_n - a\| .$$

Since $R(I+\alpha\overline{H}) = \mathcal{B}_0$ it follows that $(I+\alpha H_n)^{-1}a \to (I+\alpha\overline{H})^{-1}a$ for all $a \in \mathcal{B}$ and $S^{(n)}$ converges to S by Proposition 1.8.1. But the resolvent convergence also implies that G_α is closed and hence $H = \overline{H}$. □

In Theorem 1.8.2 there is not necessarily any unique or natural subspace $\mathcal{B}_0$ of convergence, e.g., if $S^{(n)}$ converges to S on $\mathcal{B}_0$ and $\mathcal{B}_1 \subset \mathcal{B}_0$ is an S-invariant subspace of $\mathcal{B}_0$ then $S^{(n)}$ converges to $S|_{\mathcal{B}_1}$ on $\mathcal{B}_1$. Of course the largest possible subspace of convergence is determined by the closure of $D(G_\alpha)$. If $S^{(n)}$ converges strongly to S on $\mathcal{B}_0$ it follows from the argument used to prove $1 \Rightarrow 2$ in Theorem 1.8.2 that $G_\alpha = G(I+\alpha H)$ where H is the generator of S. Thus $D(G_\alpha) = D(H)$ and in particular one has the following.

COROLLARY 1.8.3. *Adopt the assumption of Theorem 1.8.2. The following conditions are equivalent*

1. *There exists a C_0-semigroup S on $\mathcal{B}$ such that $\left\|\left(S_t^{(n)}-S_t\right)a\right\| \to 0$ for all $a \in \mathcal{B}$, uniformly for t in any finite interval of $[0, \infty\rangle$.*

2. *$D(G_\alpha)$ and $R(G_\alpha)$ are norm dense in $\mathcal{B}$ for some $\alpha > 0$.*

Proposition 1.8.1, Theorem 1.8.2, and Corollary 1.8.3, have a variety of uses. The latter results give a clear delineation of the infinitesimal properties which characterize semigroup convergence. But unfortunately these properties are often difficult to verify in particular examples. There is, however, one situation in which the first result is easily applicable.

PROPOSITION 1.8.4. *Let* $S_t^{(n)} = \exp\{-tH_n\}$ *be* C_0*-semigroups of contractions on the Banach space* $\mathcal{B}$ *and* $S_t = \exp\{-tH\}$ *a similar semigroup on the Banach subspace* $\mathcal{B}_0$.

If there exists a core D *of* H *such that* $D \subseteq D(H_n)$ *for all* n *or, more generally,*

$$D \subseteq \bigcup_m \left\{ \bigcap_{n \geq m} D(H_n) \right\}$$

and if

$$\lim_{n\to\infty} \| (H_n - H) a \| = 0$$

for all $a \in D$ *, then*

$$\lim_{n\to\infty} \left\| \left(S_t^{(n)} - S_t \right) a \right\| = 0$$

for all $a \in \mathcal{B}_0$ *, uniformly for* t *in finite intervals of* $[0, \infty\rangle$. *In particular* H *is the graph limit of* H_n .

Proof. If $\alpha > 0$ the set $R_\alpha = \{(I+\alpha H)a \; ; \; a \in D\}$ is norm dense in $\mathcal{B}_0$ because D is a core of H and $R(I+\alpha H) = \mathcal{B}_0$. But for $b = (I+\alpha H)a$ with $a \in D$ one has

$$\left\| \left\{ \left(I+\alpha H_n\right)^{-1} - (I+\alpha H)^{-1} \right\} b \right\| = \left\| \left(I+\alpha H_n\right)^{-1} \left(H-H_n\right) a \right\|$$

$$\leq \| \left(H-H_n\right) a \|$$

and hence $(I+\alpha H_n)^{-1}b$ converges strongly to $(I+\alpha H)^{-1}b$ for all $b \in \mathcal{B}_0$. The convergence of $S^{(n)}b$ to Sb now follows from Proposition 1.8.1 . The identification of H as the graph limit of the H_n is a consequence of Corollary 1.8.3. □

There are two general corollaries of this last result which are useful throughout semigroup theory and which we have already partly exploited in Example 1.3.6. These corollaries concern the approximation of a given semigroup by a family of semigroups with bounded generators.

First let

$$H_s = \left(1-e^{-sH}\right)/s$$

where $S_s = \exp\{-sH\}$ is assumed to be a contraction semigroup. It is evident that the H_s are bounded but they also generate contraction semigroups because

$$\left\|e^{-tH_s}\right\| = e^{-t/s}\left\|e^{tS_s/s}\right\| \le 1 \; .$$

Moreover

$$\lim_{s\to 0}\left\|\left(H_s-H\right)a\right\| = 0$$

for all $a \in D(H)$ by definition. Hence Proposition 1.8.4 implies that

$$(*) \qquad \lim_{s\to 0}\left\|\left(e^{-tH}-e^{-t\left(1-e^{-sH}\right)/s}\right)a\right\| = 0$$

for all $a \in \mathcal{B}$ uniformly for t in compact intervals of $[0, \infty\rangle$.

Second let

$$H_s = H(I+sH)^{-1}$$

$$= \{I - (I+sH)^{-1}\}/s \ .$$

Again H_s is bounded and $\exp\{-tH_s\}$ is a contraction semigroup because

$$\|e^{-tH_s}\| = e^{-t/s}\|e^{t(I+sH)^{-1}/s}\| \leq 1$$

where the last estimate uses $\|(I+sH)^{-1}\| \leq 1$. But

$$\lim_{s \to 0} \|(H_s - H)a\| = 0$$

for all $a \in D(H)$ because $(I+sH)^{-1}$ converges strongly to the identity as s tends to zero. This was established in the proof of Theorem 1.3.1. Thus the assumptions of Proposition 1.8.4 are satisfied by $\exp\{-tH\}$ and $\exp\{-tH_s\}$ and hence

$$(**) \qquad \lim_{s \to 0} \|(e^{-tH} - e^{-tH(1+sH)^{-1}})a\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$.

The algorithms (*) and (**) give two methods of approximating a given C_0-semigroup of contractions. The first of these was proposed by Hille and the second by Yosida. Consequently we refer to the semigroups $\exp\{-t(I-S_s)/s\}$ as the *Hille approximants* and $\exp\{-tH(I+sH)^{-1}/s\}$ as the *Yosida approximants* of $S_t = \exp\{-tH\}$.

The Hille and Yosida approximants have many applications. The following example describes the connection between Taylor's series expansion, the Stone-Weierstrass theorem, and the Hille approximants of the semigroup of right translations.

Example 1.8.5. Let $\mathcal{B} = C_0(0, \infty)$, the continuous functions on $[0, \infty\rangle$ which vanish at infinity, equipped with the supremum norm and let S denote the C_0-semigroup of right translations

$$(S_t f)(x) = f(x+t) .$$

If $S_t^{(h)} = \exp\{-t(I-S_h)/h\}$ denotes the Hille approximants then

$$\begin{aligned} f(x+t) &= \lim_{h\to 0} \left(S_t^{(h)} f\right)(x+t) \\ &= \lim_{h\to 0} \sum_{n\geq 0} \frac{1}{n!} (t/h)^n \left(\Delta_h^{(n)} f\right)(x) \end{aligned}$$

where

$$\begin{aligned} \left\{\Delta_h^{(n)} f\right\}(x) &= \left\{(S_h-I)^n f\right\}(x) \\ &= \sum_{k=0}^{n} (-1)^{n-k}\, {}^nC_k f(x+kh) \end{aligned}$$

and the limit is uniform for $x \in [0, \infty\rangle$ and t in any finite interval of $[0, \infty\rangle$. This is a generalization of Taylor's theorem. But setting $x = 0$ one also deduces that for each $\varepsilon > 0$ one can choose N such that

$$\left|f(t) - \sum_{n=0}^{N} \frac{1}{n!} (t/h)^n \left(\Delta_h^{(n)} f\right)(0)\right| < \varepsilon$$

uniformly for t in any finite interval $[0, t_0]$. This is an explicit version of the Stone-Weierstrass theorem. □

Alternatively these approximation techniques can be applied in a variety of ways to differential operators. The following example illustrates a typical problem of statistical mechanics, the independence of the thermodynamic limit from the choice of boundary conditions. In statistical mechanics one describes systems confined to a finite region of the appropriate phase space, e.g., a bounded subset $\Lambda \subset \mathbb{R}^3$, and then attempts to calculate bulk properties, e.g., properties such as the specific heat per unit volume. For sufficiently large systems these properties should be insensitive to the size and any boundary effects.

Example 1.8.6. Let $\mathcal{B} = L^2(\mathbb{R}^\nu)$ and let H denote the usual self-adjoint Laplacian. Thus

$$D(H) = \left\{f \; ; \quad f \in L^2(\mathbb{R}^\nu) \; , \; \int d^\nu p \; p^4 |\tilde{f}(p)|^2 < +\infty\right\}$$

and

$$(Hf)(x) = -\nabla_x^2 f(x) = (2\pi)^{-\nu/2} \int d^\nu p \; p^2 \tilde{f}(p) e^{ipx}$$

where $\tilde{f}$ denotes the Fourier transform. The operator H generates the semigroup of contractions which solves the heat equation

$$\frac{\partial f(x,t)}{\partial t} = -\nabla_x^2 f(x, t)$$

on $L^2(\mathbb{R}^\nu)$. The action of this semigroup is given by

$$(S_t f)(x) = (4\pi t)^{-\nu/2} \int d^\nu y \, e^{-(x-y)^2/4t} f(y) .$$

It is well known, and easily verified, that the space of infinitely often differentiable functions with compact support forms a core D of H .

Next for each bounded open set $\Lambda \subset \mathbb{R}^\nu$ let H_Λ denote any positive self-adjoint extension of H restricted to the infinitely often differentiable functions with support in Λ . There are many such extensions corresponding to different choices of boundary conditions for the Laplace operator on the boundary $\partial\Lambda$ of Λ . Some of these will be discussed explicitly in Section 1.11. But if Λ_n is any increasing sequence such that any open bounded set Λ is contained in Λ_n for n sufficiently large then

$$D \subset \bigcup_m \Big(\bigcap_{n \geq m} D\big(H_{\Lambda_m}\big)\Big)$$

by definition. Hence

$$\lim_{n\to\infty} \left\| \left(e^{-tH_{\Lambda_n}} - e^{-tH} \right) f \right\| = 0$$

for all $f \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$, by a direct application of Proposition 1.8.4. Consequently the net of contraction semigroups $\Lambda \mapsto \exp\{-tH_\Lambda\}$ converges strongly to $S_t = \exp\{-tH\}$. □

The Hille and Yosida approximants are just particular examples of a much broader class. If $t \in \mathbb{R}_+ \mapsto F(t) \in \mathcal{L}(\mathcal{B})$ is a family of contraction operators satisfying

$$\lim_{t\to 0} \|\{(I-F(t))/t - H\}a\| = 0$$

for all a in a core D of H then $t \to \exp\{-t(I-F(s))/s\}$ is a family of contraction semigroups and

$$\lim_{s\to 0} \|(S_t-\exp\{-t(I-F(s))/s\})a\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals of $[0, \infty\rangle$. This is again a direct corollary of Proposition 1.8.4. Next we examine an alternative set of approximations of S by powers $F(t/n)^n$. The first basic estimate which relates the power approximations to the foregoing exponential approximations is provided by the following lemma.

LEMMA 1.8.7. *Let* A *be a bounded operator on the Banach space* $\mathcal{B}$ *with* $\|A\| \le 1$.

It follows that

$$\|(e^{-n(I-A)}-A^n)a\| \le \sqrt{n}\|(I-A)a\|$$

for all $n = 1, 2, 3, \ldots$.

Proof. One estimates

$$\|e^{-n(I-A)}-A^n\, a\| \le e^{-n} \sum_{m\ge 0} \frac{n^m}{m!} \|(A^m-A^n)a\|$$

$$\le e^{-n} \sum_{m\ge 0} \frac{n^m}{m!} \|(A^{|n-m|}-I)a\|$$

$$\le \|(I-A)a\| e^{-n} \sum_{m\ge 0} \frac{n^m}{m!} |n-m|$$

$$\le \sqrt{n}\|(I-A)a\|$$

where the last estimate follows from a straightforward application of the Cauchy Schwarz inequality. □

Combination of this estimate with the previous convergence theorems then leads to the following product formula, which generalizes the construction of the Hille-Yosida theorem.

THEOREM 1.8.8. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions and* $t \in \mathbb{R}_+ \mapsto F(t) \in \mathcal{L}(\mathcal{B})$ *a family of contractions operators on the Banach space* $\mathcal{B}$ *. Further assume that*

$$\lim_{t\to 0+} \|\{(I-F(t))/t - H\}a\| = 0$$

for all a *in a core* D *of* H *.*

It follows that

$$\lim_{n\to\infty} \|\{e^{-tH} - F(t/n)^n\}a\| = 0$$

for all $a \in \mathcal{B}$ *uniformly for* t *in finite intervals of* $[0, \infty\rangle$ *.*

Proof. First it follows from Proposition 1.8.4 that

$$\lim_{s\to 0+} \|(e^{-tH}-e^{-t(I-F(s))/s})a\| = 0$$

for all $a \in \mathcal{B}$, uniformly for t in finite intervals. Therefore

$$\lim_{n\to\infty} \|(e^{-tH}-e^{-n(I-F(t/n))})a\| = 0$$

uniformly for t in finite intervals. But Lemma 1.8.7 gives the estimate

$$\|(e^{-n(I-F(t/n))} - F(t/n)^n)a\| \le (t/\sqrt{n})\|(I-F(t/n))a\|/(t/n)$$

and for $a \in D$, the core of H, the right hand side converges to zero uniformly for t in finite intervals. Since $D(H)$ is norm dense the desired result follows from combination of these two estimates. □

Product formulae of the type described by the theorem have a wide variety of applications. As a first illustration we again consider the semigroup of right translations and the Stone-Weierstrass theorem.

Example 1.8.9. Adopt the notation and assumptions of Example 1.8.5. Next for $0 < \lambda < 1$ set

$$F(t) = (1-\lambda)I + \lambda S_{t/\lambda}$$

in Theorem 1.8.8. Clearly the hypotheses of the theorem are valid and one has

$$\lim_{n\to\infty} \|(S_t-((1-\lambda)I+\lambda S_{t/n\lambda})^n)a\| = 0 .$$

Therefore if $f \in C_0(0, \infty)$

$$f(x+t) = \lim_{n\to\infty} \sum_{m=0}^{n} {}^{m}C_n (1-\lambda)^{n-m} \lambda^m f\left(x + \frac{mt}{n\lambda}\right)$$

and the limit is uniform for $x \in [0, \infty\rangle$ and t in any finite interval of $[0, \infty\rangle$. Thus for $f \in C_b(0, 1)$ one deduces that

$$f(t) = \lim_{n\to\infty} \sum_{m=0}^{n} {}^{n}C_m (1-t)^m t^{n-m} f(m/n)$$

uniformly for $t \in [0, 1]$. This is Bernstein's version of the Stone-Weierstrass theorem. □

As a second, completely different, application of the product formula we derive an approximation procedure for the semigroup generated by the Dirichlet Laplacian. This example is of some importance because it provides the operator theoretic structure behind the Wiener integral, i.e., the functional integration description of the heat equation.

Example 1.8.10. Let S denote the C_0-semigroup generated by the Laplacian $H = -\nabla^2$ on $L^2(\mathbb{R}^\nu)$. Furthermore identify $L^2(\Lambda)$ as the subspace of $L^2(\mathbb{R}^\nu)$ formed by the functions with support in the bounded open set $\Lambda \subset \mathbb{R}^\nu$, Now define H_Λ as the restriction of H to the twice continuously differentiable with support in the interior of Λ . Since H is norm closed in $L^2(\mathbb{R}^\nu)$ its restriction H_Λ is norm closable in $L^2(\mathbb{R}^\nu)$ and we also use H_Λ to denote the closure. One can establish that H_Λ is a positive self-adjoint operator on $L^2(\Lambda)$ and it corresponds to the Laplacian with Dirichlet boundary conditions, i.e., $f \in D(H)$ implies that $f = 0$ on the boundary of Λ at least in some distributional sense. Further details about this Laplacian and others corresponding to different boundary conditions will be given in Section 1.11.

Now consider the family of operators on $L^2(\Lambda)$ defined by

$$F(t) = \chi_\Lambda S_t$$

where χ_Λ denotes multiplication by the characteristic function of the set Λ. If $f \in L^2(\Lambda)$ is twice continuously differentiable with support in Λ one has

$$\lim_{t\to 0+} \left\|\{(I-F(t))/t - H_\Lambda\}f\right\| = 0 .$$

But these f form a core for H_Λ and hence Theorem 1.8.8 is applicable. Thus

$$\lim_{n\to\infty} \left\|\{e^{-tH_\Lambda} - (\chi_\Lambda e^{-tH/n})^n\}f\right\| = 0$$

for all $f \in L^2(\Lambda)$ uniformly for t in finite intervals of $[0, \infty\rangle$. Note that from the explicit form of S one has

$$((\chi_\Lambda e^{-tH/n})^n f)(x) = \left(\frac{4\pi t}{n}\right)^{-n\nu/2}$$

$$\int_\Lambda dy_1 \ldots \int_\Lambda dy_n \, e^{-n\{(x-y_1)^2+(y_1-y_2)^2+\ldots+(y_{n-1}-y_n)^2\}/4t} f(y_n) .$$

These results extend directly to the corresponding semigroups on on $L^p(\Lambda)$, and $L^p(\mathbb{R}^\nu)$, and they provide a proof that the Dirichlet semigroup is positive, i.e., it maps positive functions into positive functions. $\square$

Theorem 1.8.8 can also be applied to semigroups whose generators are sums of generators.

Let $S_t = \exp\{-tH\}$ and $T_t = \exp\{-tK\}$ be two C_0-semigroups of contractions and assume that $H + K$ is a norm closable operator whose closure $\overline{H + K}$ generates a C_0-semigroup U . A slight extension of the argument preceding Proposition 1.3.4 demonstrates that $H + K$ is dissipative and then the closure $\overline{H + K}$ is also dissipative. Thus U is contractive.

Now we choose $F_t = S_t T_t$ and $D = D(H+K)$. One readily checks that the assumptions of Theorem 1.8.8 are satisfied for F and U . Consequently

$$\lim_{n\to\infty} \left(U_t - \left(S_{t/n} T_{t/n}\right)^n\right) a = 0$$

for all $a \in \mathcal{B}$. This relation is called the *Trotter product formula*. A second possible choice of F is

$$F(t) = (I+tH)^{-1}(I+tK)^{-1}$$

and this leads to the product formula

$$\lim_{n\to\infty} \left\| \left(U_t - (I+tH/n)^{-1}(I+tK/n)^{-1}\right) a \right\| = 0 .$$

Exercises.

1.8.1. Let H_n be a uniformly bounded sequence of bounded operators. Prove that the graph limit of H_n exists if, and only if, H_n converges strongly.

1.8.2. Let H_n be a sequence of operators for which the graph limit H exists and P_n a sequence of bounded operators which converges strongly to P . Prove that $H + P$ is the graph limit of $H_n + P_n$.

1.9. Perturbation Theory

The next aspect of stability that we describe is stability of a semigroup under perturbations of its generator. Let H be the generator of a C_0-semigroup of contractions on the Banach space $\mathcal{B}$ and P a linear operator on $\mathcal{B}$. Our aim is to describe conditions on P which ensure that H + P also generates a C_0-semigroup of contractions. In applications the perturbation P is often an unbounded operator and the notion of relatively bounded operator is useful.

Let H and P be linear operators on a Banach space. Then P is defined to be *relatively bounded* with respect to H, or H-*relatively bounded*, if the following two conditions are satisfied:

1. $D(P) \supseteq D(H)$

2. $\|Pa\| \leq \alpha\|a\| + \beta\|Ha\|$

for all $a \in D(H)$ and some $\alpha, \beta > 0$.

The greatest lower bound of the β for which this last relation is valid is called the *relative bound* of P with respect to H, or the H-*bound*.

The key result concerning relative bounded perturbations of generators of contraction semigroups is the following:

THEOREM 1.9.1. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *and assume* P *is* H*-relatively bounded with*

H-*bound* $\beta_0 < 1$.

If P , *or* $H + P$, *is norm-dissipative then* $H + P$ *generates a* C_0*-semigroup of contractions.*

Proof. First note that it follows from Theorems 1.3.1 and 1.4.1 that $D(H)$ is norm dense and $\mathrm{Re}(f_a, Ha) \geq 0$ for all tangent functionals f_a at $a \in D(H)$. Second since $D(H) \subseteq D(P)$ the latter set is norm dense. Hence if P is norm-dissipative $\mathrm{Re}(f_a, Pa) \geq 0$ for all tangent functionals at $a \in D(H)$ by Theorem 1.4.1. Therefore $\mathrm{Re}(f_a, (H+\lambda P)a) \geq 0$ for all $\lambda \geq 0$ and $H + \lambda P$ is norm-dissipative. Alternatively if $H + P$ is norm-dissipative then $\mathrm{Re}(f_a (H+P)a) \geq 0$ and

$$\begin{aligned}\mathrm{Re}(f_a, (H+\lambda P)a) &= (1-\lambda)\mathrm{Re}(f_a, Ha) + \lambda\mathrm{Re}(f_a, (H+P)a)\\ &\geq 0\end{aligned}$$

for $0 \leq \lambda \leq 1$. Thus in both cases $H + \lambda P$ is norm-dissipative for $0 \leq \lambda \leq 1$.

Next we exploit the relative bound.

Let us assume that

$$\|Pa\| \leq \alpha\|a\| + \beta\|Ha\|$$

for all $a \in D(H)$ where $\alpha > 0$ and $\beta < 1$. Therefore

$$\begin{aligned}\|\lambda P(I+\lambda H)^{-1}a\| &\leq \alpha\|\lambda(I+\lambda H)^{-1}a\| + \beta\|(I-(I+\lambda H)^{-1})a\|\\ &\leq (\alpha\lambda+2\beta)\|a\|\end{aligned}$$

where we have used $\|(I+\lambda H)^{-1}\| \le 1$. Thus if $0 \le \lambda_1 \le (2\beta)^{-1}$ one may choose $\lambda_0 > 0$ such that $\lambda_1(\alpha\lambda+2\beta) < 1$ for $0 < \lambda < \lambda_0$ and then the operator $P_\lambda = \lambda_1\lambda P(I+\lambda H)^{-1}$ is bounded with $\|P_\lambda\| < 1$. Hence $I + P_\lambda$ has a bounded inverse. But

$$\left(I+\lambda\left(H+\lambda_1 P\right)\right) = \left(I+P_\lambda\right)(I+\lambda H)$$

and since $R(I+\lambda H) = \mathcal{B}$ one has

$$\begin{aligned} R\left(I+\lambda\left(H+\lambda_1 P\right)\right) &= R\left(I+P_\lambda\right) \\ &= D\left(\left(I+P_\lambda\right)^{-1}\right) = \mathcal{B} . \end{aligned}$$

Therefore $H + \lambda_1 P$ is the generator of a C_0-semigroup of contractions by Theorem 1.3.5.

To continue the proof we remark that

$$\|Pa\| \le \|a\| + \beta\left\|\left(H+\lambda_1 P\right)a\right\| + \beta\lambda_1\|Pa\|$$

and since $\lambda_1 \le (2\beta)^{-1}$ one has

$$\|Pa\| \le 2\alpha\|a\| + 2\beta\left\|\left(H+\lambda_1 P\right)a\right\| .$$

We may now choose $0 \le \lambda_2 \le (4\beta)^{-1}$ and repeat the above argument to deduce that $H + \left(\lambda_1+\lambda_2\right)P$ is the generator of a C_0-semigroup of contractions. Iteration of this argument n times proves that $H + \lambda P$ is a generator for all $0 \le \lambda < \left(1-2^{-n}\right)/\beta$. Choosing n sufficiently large, but finite, one obtains the desired result. □

Next we examine a more restricted class of perturbations. If $S_t = \exp\{-tH\}$ is a C_0-semigroup and P is a linear operator, on the Banach space $\mathcal{B}$, then P is called a *Phillips perturbation* of S if the following three conditions are satisfied:

1. P is closed.

2. For each $t > 0$ one has $S_t\mathcal{B} \subseteq D(P)$ and PS_t has bounded closure.

3. $\int_0^1 dt \, \|PS_t\| < +\infty$.

Note that if S is a group then each Phillips perturbation P of S is automatically bounded because $P = (PS_t)S_{-t}$ for each $t > 0$. More generally, for semigroups, P is relatively bounded. To see this consider the case that S is a contraction semigroup. Consequently

$$(\lambda I+H)^{-1}a = \int_0^\infty dt \; e^{-\lambda t} S_t a$$

for each $a \in \mathcal{B}$ and $\lambda > 0$. But one also has

$$\left\|\int_0^\infty dt \; e^{-\lambda t} PS_t a\right\| \le \|a\| \left(\int_0^\delta dt \; \|PS_t\| + \|PS_\delta\|/\lambda \right)$$

for any $0 < \delta < 1$. Since P is closed a simple Riemann approximation argument establishes that $(\lambda I+H)^{-1}a \in D(P)$, i.e., $D(H) \subseteq D(P)$, and

$$P(\lambda I+H)^{-1}a = \int_0^\infty dt \; e^{-\lambda t} PS_t a \; .$$

Therefore setting $b = (\lambda I+H)^{-1}a$ and using the foregoing estimate one finds

$$\begin{aligned}\|Pb\| &= \left\|\int_0^\infty dt\; e^{-\lambda t} PS_t a\right\| \\ &\le \|(\lambda I+H)b\| \left(\int_0^\delta dt\; \|PS_t\| + \|PS_\delta\|/\lambda\right) \\ &\le (\lambda\|b\|+\|Hb\|)\left(\int_0^\delta dt\; \|PS_t\| + \|PS_\delta\|/\lambda\right) .\end{aligned}$$

Thus P is H-relatively bounded. Moreover choosing δ to be small and λ to be large one sees that P has H-bound zero. The same conclusion is indeed valid for a general C_0-semigroup but one must use the bound $\|S_t\| \le M \exp\{\omega t\}$ and take $\lambda > \omega$.

Theorem 1.9.1 can now be strengthened for the class of Phillips perturbations.

THEOREM 1.9.2. *Let* $S_t = \exp\{-tH\}$ *be a* C_0*-semigroup of contractions on the Banach space* $\mathcal{B}$ *and* P *a Phillips perturbation of* S .

If P , *or* $H + P$, *is norm-dissipative then* $H + P$ *generates a* C_0*-semigroup of contractions* S^P . *Moreover*

$$S^P_t a = S_t a + (-1)^n \sum_{n\ge 1} \int_{0\le t_n \le t_{n-1} \le \ldots \le t_1 \le t} dt_1 \ldots dt_n \; S_{t-t_1} PS_{t_1-t_2} P \ldots PS_{t_n} a$$

for all $a \in \mathcal{B}$, *where the integrals exist in the norm topology and define a series of bounded operators which converges in norm* *uniformly for* t *in any finite interval of the form* $\langle \varepsilon, 1/\varepsilon \rangle$ *where* $0 < \varepsilon < 1$.

Proof. The first statement of the theorem follows from Theorem 1.8.1 and the foregoing observation that a Phillips perturbation P of H is H-relatively bounded with H-bound zero.

Now consider the perturbation series for S^P . It follows from the definition of a Phillips perturbation that each term is well defined as a bounded operator and is strongly continuous for $t > 0$. But if $S_t^{(n)}$ denotes the n-th term then

$$S_t^{(0)} = S_t \ , \qquad S_t^{(n)} = (-1) \int_0^t ds\ S_{t-s} P S_s^{(n-1)} \ .$$

Hence, by iteration,

$$\left\| S_t^{(n)} \right\| \le g * f^{n*}(t)$$

where

$$g(t) = \left\| S_t \right\|, \qquad f(t) = \left\| P S_t \right\| \ ,$$

the $*$ denotes the convolution product, and f^{n*} denotes the n-fold convolution of f with itself.

Now let us examine bounds on f .

Since S is contractive f is non-increasing and the integral

$$I_\lambda = \int_0^\infty dt\ e^{-\lambda t} f(t)$$

is finite for each $\lambda > 0$. Moreover $I_\lambda \to 0$ as $\lambda \to \infty$. But for $0 < s < t$ one has $f(t) \leq f(t-s)$ and hence

$$2\left[e^{-\lambda t}f(t)\right] \leq \left(e^{-\lambda(t-s)}f(t-s)+e^{-\lambda s}\right)^2 .$$

Therefore

$$t\left[e^{-\ t}f(t)\right]^{\frac{1}{2}} \leq \int_0^{t/2} ds\ \left(e^{-\lambda(t-s)}f(t-s)+e^{-\lambda s}\right)$$

$$\leq I_\lambda + 1/\lambda .$$

Consequently for λ sufficiently large

$$\int_0^\infty dt\ e^{-\lambda t}f(t) \leq 1/2^4$$

and

$$f(t) \leq e^{\lambda t}/2^4t^2 .$$

Moreover since S is contractive there is an $M > 0$ such that

$$\int_0^\infty dt\ e^{-\lambda t}g(t) \leq M , \qquad g(t) \leq Me^{\lambda t}/t^2$$

for this same range of large λ .

Next we examine the propagation of these bounds.

Suppose two positive integrable functions f_1 , f_2 , on $[0, \infty\rangle$ satisfy

$$\int_0^\infty dt\ e^{-\lambda t}f_i(t) \leq M_i , \qquad f_i(t) \leq M_ie^{\lambda t}/t^2 .$$

Then

$$\int_0^\infty dt\ e^{\lambda t}(f_1 * f_2)(t) = \int_0^\infty dt\ e^{-\lambda t} f_1(t) \int_0^\infty ds\ e^{-\lambda s} f_2(s)$$

$$\leq M_1 M_2 \leq 8M_1 M_2 \ .$$

Moreover

$$(f_1 * f_2)(t) \leq e^{\lambda t} \int_0^t ds\ \left(e^{\lambda(t-s)} f_1(t-s)\right)\left(e^{-\lambda s} f_2(s)\right)$$

$$\leq e^{\lambda t} \int_0^{t/2} ds\ \frac{M_1}{(t-s)^2}\ e^{-\lambda s} f\ (s) + e^{\lambda t} \int_{t/2}^t ds\ e^{-\lambda(t-s)} f_1(t-s)\ \frac{M_2}{s^2}$$

$$\leq 8M_1 M_2 e^{\lambda t}/t^2 \ .$$

Thus the bounds propagate.

Combining the foregoing estimates one concludes that

$$\int_0^\infty dt\ e^{-\lambda t} g * f^{n*}(t) \leq M/2^n$$

and

$$g * f^{n*}(t) \leq Me^{\lambda t}/2^n t^2 \ .$$

Consequently the perturbation series for S^P is majorized in norm by the series

$$\sum_{n\geq 0} Me^{\lambda t}/2^n t^2 = 2Me^{\lambda t}/t^2$$

and this immediately implies the convergence statements for the perturbation series.

It remains to prove that S^P is a C_0-semigroup with generator $H + P$.

First strong continuity at the origin follows from the integrability of $t \mapsto \|PS_t\|$ at the origin and the straightforward estimate

$$\|(S_t^P - S_t)a\| \le \sum_{n\ge 1} \int_{0\le t_n\le \ldots \le t_1 \le t} dt_1 \ldots dt_n \, \|PS_{t_1-t_2}\| \, \|PS_{t_2-t_3}\| \ldots \|PS_n\| \|a\|$$

$$\le \sum_{n\ge 1} \left\{ \int_0^t ds \, \|PS_s\| \right\}^n \|a\| \; .$$

Second note that S^P satisfies the integral equation

$$S_t^P = S_t - \int_0^t ds \, S_{t-s} P S_s^P$$

and hence

$$S_{t_1}^P S_{t_2}^P = S_{t_1} S_{t_2}^P - \int_0^{t_1} ds \, S_{t_1-s} P S_s^P S_{t_2}^P$$

$$= S_{t_1+t_2} - \int_0^{t_2} ds \, S_{t_1+t_2-s} P S_s^P - \int_0^{t_1} ds \, S_{t_1-s} P S_s^P S_{t_2}^P$$

$$= S_{t_1+t_2}^P + \int_0^{t_1} ds \, S_{t_1-s} P \left\{ S_{s+t_2}^P - S_s^P S_{t_2}^P \right\} .$$

Thus the family of operator-valued functions

$$\lambda \in \mathbb{C} \mapsto F_{t_1}(\lambda) = S_{t_1}^{\lambda P} S_{t_2}^{\lambda P} - S_{t_1+t_2}^{\lambda P}$$

is entire analytic, in the norm topology, and satisfies the homogeneous integral equations

$$F_t(\lambda) = \lambda \int_0^t ds\, S_{t-s} P F_s(\lambda) .$$

It then follows from Taylor's series that $F_t(\lambda) = 0$, i.e., the semigroup property

$$S^P_{t_1} S^P_{t_2} = S^P_{t_1+t_2}$$

is valid.

Finally let K denote the generator of S^P . For λ sufficiently large one has

$$(\lambda I+K)^{-1} = \int_0^\infty dt\, e^{-\lambda t} S^P_t .$$

But using the integral equation for S^P one finds

$$\begin{aligned}(\lambda I+K)^{-1} &= \int_0^\infty dt\, e^{-\lambda t} S_t - \int_0^\infty dt \int_0^t ds\, e^{-\lambda t} S_{t-s} P S^P_s \\ &= (\lambda I+H)^{-1} - \int_0^\infty dt\, e^{-\lambda t} S_t\, P \int_0^\infty ds\, e^{-\lambda s} S^P_s \\ &= (\lambda I+H)^{-1} - (\lambda I+H)^{-1} P (\lambda I+K)^{-1} .\end{aligned}$$

This establishes that

$$(\lambda I+H+P)(\lambda I+K)^{-1} = I .$$

But

$$(\lambda I+H+P) = \left(I+P(\lambda I+H)^{-1}\right)(\lambda I+H)$$

and

$$\|P(\lambda I+H)^{-1}\| \le \int_0^\infty dt\ e^{-\lambda t}\|PS_t\| < 1$$

for λ sufficiently large. Therefore $(\lambda I+H+P)$ is invertible with bounded inverse. Consequently

$$(\lambda I+H+P)^{-1} = (\lambda I+K)^{-1}$$

and

$$K = H + P . \qquad \square$$

Remark 1.9.3. One can obtain an analogue of Theorem 1.9.2 without assuming that S is contractive or P norm-dissipative. *If* $S_t = \exp\{-tH\}$ *is a* C_0*-semigroup and* P *a Phillips perturbation of* S *then* $H + P$ *generates a* C_0*-semigroup* S^P *which can be defined by the perturbation series of Theorem 1.9.2.* The proof of this generalization is very similar to the above proof but the estimates necessary for the convergence of the series are slightly more onerous because of the growth of $\|S_t\|$.

Example 1.9.4. Let $\mathcal{B} = L^p(\mathbb{R}^\nu)$ and let S be the semigroup generated by the Laplacian, i.e.,

$$\left(S_t f\right)(x) = \left(\mu_t * f\right)(x)$$

where

$$\mu_t(x) = (4\pi t)^{-\nu/2}\exp\{\frac{-x^2}{4t}\} .$$

Next let V be a multiplication operator

$$(Vf)(x) = V(x)f(x)$$

where $V \in L^q(\mathbb{R}^\nu)$ and $q > \nu/2$, $q \geq p$. Then by successively applying Hölder's and Young's inequalities

$$\begin{aligned}\|VS_tf\|_p &\leq \|V\|_q\|\mu_t * f\|_r \\ &\leq \|V\|_q\|\mu_t\|_s\|f\|_p\end{aligned}$$

where $p^{-1} = q^{-1} + r^{-1}$, $r^{-1} + 1 = s^{-1} + p^{-1}$, and $1 \leq p, q, r, s \leq \infty$. But

$$\|\mu_t\|_s \leq ct^{(\nu/2)(s^{-1}-1)} = ct^{-\nu/2q} .$$

Thus $\|VS_t\|_p$ is integrable at the origin and V is a Phillips perturbation of S .

Exercises.

1.9.1. Let P be relatively bounded with respect to H with H-bound less than one. Prove that $H + P$ is closable if, and only if, H is closable and in this case the closures have the same domain.

1.9.2. If P is relatively bounded with respect to H with H-bound $\beta < 1$ prove that P is relatively bounded with respect $H + P$ with H+P-bound $\beta(1-\beta)^{-1}$.

1.9.3. Let H be the generator of a C_0-contraction semigroup on a Banach space $\mathcal{B}$ and suppose P is relatively bounded with respect to H. Prove that if $\lambda > 0$ then

$$\|P(\lambda I+H)^{-1}\| \leq \alpha\lambda^{-1} + 2\beta .$$

Moreover if $\mathcal{B}$ is a Hilbert space

$$\|P(\lambda I+H)^{-1}\| \leq \alpha\lambda^{-1} + \beta .$$

Hint: In the Hilbert space case use norm-dissipativity to prove that

$$\|(\lambda I+H)a\|^2 \geq \lambda^2\|a\|^2 + \|Ha\|^2 .$$

1.9.4. If $a \in L^2(\mathbb{R}^3)$ has partial derivatives in $L^2(\mathbb{R}^3)$ prove that

$$\int d^3x \; |a(x)|^2 / |x|^2 \leq 4 \int d^3x \; |\nabla a(x)|^2 .$$

Hint: Calculate $\nabla |x|^{\frac{1}{2}} a(x)$.

1.9.5. Let H denote the Laplacian on $L^p(\mathbb{R}^\nu)$ and χ_Λ the operator of multiplication by the characteristic function of the open bounded set $\Lambda \subset \mathbb{R}^\nu$. Define $S^{(n)}$ to be the C_0-semigroup generated by the perturbed Laplacian $H_n = H + n(I-\chi_\Lambda)$. Prove that $S^{(n)}$ converges strongly on $L^p(\Lambda)$ to the semigroup generated by the Laplacian with Dirichlet boundary conditions, as $n \to \infty$.

1.10. Comparison of Semigroups.

In perturbation theory one starts from a semigroup S and an operator P , which is "small" with respect to the generator H of S , and then constructs a perturbed semigroup S^P , with generator $H + P$, which is "close" to S. The notions of "smallness" of the perturbation and "closeness" of the semigroups are intimately related. In particular one can estimate from the identity

$$S_t - S_t^P = \int_0^t ds \frac{d}{ds} \left(S_{t-s}^P S_s\right)$$

$$= \int_0^t ds\, S_{t-s}^P P\, S_s$$

that

$$\left\|S_t - S_t^P\right\| = 0(t) ,$$

as $t \to 0$, if P is bounded, or

$$\left\|\left(S_t - S_t^P\right)a\right\| = 0(t)$$

for all $a \in D(H)$, as $t \to 0$, if P is relatively bounded with respect to H . Our aim is to prove converses to these statements.

We now begin with two semigroups satisfying the estimate (*) , or (**) , and attempt to prove that the corresponding generators differ by a bounded, or a relatively bounded, pertúrbation. The difficulty is that these converse statements are not valid for general C_0-semigroups. Nevertheless they are valid for C_0^*-semigroups, with some slight qualification, and hence for C_0-semigroups on reflexive Banach spaces. In general another phenomenon of intertwining of generators has to be taken into account. We will discuss this after describing the basic results on C_0^*-semigroups, and their corollaries.

THEOREM 1.10.1. *Let* S *and* T *be two* C_0^**-semigroups on the Banach space* $\mathcal{B}$ *with generators* H *and* K *, respectively. The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t)$ *as* $t \to 0+$ *,*

2. $D(H) = D(K)$ *and* $K = H + P$ *where* P *is a bounded operator from the norm closure* $\overline{D(H)}$ *of* $D(H)$ *to* $\mathcal{B}$ *.*

Proof. 1 ⇒ 2. Condition 1 states that there are constants N , $\delta > 0$ such that

$$\|S_t - T_t\| \leq Nt$$

for $0 \leq t < \delta$. Now for $f \in D(H)$ consider the one-parameter family $f_t = (S_t - T_t)f/t \in \mathcal{B}$. One has $\|f_t\| \leq N\|f\|$ for $0 \leq t < \delta$. But the unit ball of $\mathcal{B}$ is compact in the weak*-topology, by the Alaoglu-Birkhoff theorem, and hence there exists a subnet f_{t_α} which is weak*-convergent, as $t_\alpha \to 0+$, to a limit g . Now if K_* and T_{*t} denote the adjoints of K and T_t , on $\mathcal{B}_*$, which exist by Lemma 1.5.1, one has

$$f(K_* a) = \lim_\alpha f\left(\left(I-T_{*t_\alpha}\right)a\right)/t_\alpha$$

$$= \lim_\alpha \left(\left(I-S_{t_\alpha}\right)f\right)(a)/t_\alpha + \lim_\alpha f_{t_\alpha}(a)$$

$$= (Hf)(a) + g(a)$$

for all $a \in D(K_*)$ and $f \in D(H)$. Since the right hand side is continuous in a, and since $D(K_*)$ is norm-dense in $\mathcal{B}_*$, one concludes that $f \in D(K)$ and hence $D(H) \subseteq D(K)$. But reversing the roles of S and T in this argument gives $D(K) \subseteq D(H)$ and hence $D(H) = D(K)$. Furthermore the foregoing identity gives

$$Kf = Hf + g .$$

But $\|g\| \le N\|f\|$ and hence $K - H$ extends by closure to a bounded operator P from $\overline{D(H)}$ to $\mathcal{B}$, with $\|P\| \le N$.

2 ⇒ 1. If $f \in D(H)$ then $S_s f \in D(H)$ and

$$\left(\left(S_t - T_t\right)f\right)(a) = \int_0^t ds \left(T_{t-s} P S_s f\right)(a) .$$

Therefore

$$\|S_t - T_t\| \le t\|P\| \sup\left\{\|T_{t-s}\| \, \|S_s\| \; ; \quad 0 \le s \le t\right\}$$

$$= O(t)$$

as $t \to 0+$ because $\|S_t\|, \|T_t\| \le M \exp\{\omega t\}$ for suitable $M, \omega \ge 0$. □

Note that in the proof of 1 ⇒ 2 one establishes that the perturbation P satisfies the estimate

$$\|P\| \leq \sup_{t>0} \|S_t - T_t\|/t \ .$$

But in the proof of 2 ⇒ 1 one has the converse estimate

$$\sup_{t>0} \|S_t - T_t\|/t \leq \|P\|M^2 \ .$$

Thus if S and T are contraction semigroups, or more generally if $\|S_t\|$, $\|T_t\| \leq \exp\{\omega t\}$ for some $\omega \geq 0$, then

$$\|P\| = \sup_{t\geq 0} \|S_t - T_t\|/t \ .$$

The magnitude of the perturbation is measured by the "distance" $\|S_t - T_t\|/t$ between the semigroups for small t .

The difficulty in interpreting Condition 2 of Theorem 1.10.1 as a perturbation result is that the perturbation $P = K - H$ is only defined on the weak*-dense domain $D(H)$. Although it is bounded as an operator from the norm closure of $D(H)$ to $\mathcal{B}$ it is not clear that it has a bounded extension from $\mathcal{B}$ to $\mathcal{B}$. This is the case, however, if $D(H)$ is norm dense. In particular this follows if $\mathcal{B}$ is reflexive because then the norm topology and weak*-topology coincide. Therefore Theorem 1.10.1 has the following corollary.

COROLLARY 1.10.2. *Let* S *and* T *be two* C_0*-semigroups on the reflexive Banach space* $\mathcal{B}$ *with generators* H *and* K *respectively. The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t)$, *as* $t \to 0+$,

2. *There is a bounded operator* P *on* $\mathcal{B}$ *such that* $K = H - P$.

Reflexivity of $\mathcal{B}$ means that $\mathcal{B}^* = \mathcal{B}_*$ and hence weak*-continuity is equivalent to weak, or strong continuity. Therefore C_0^*-semigroups are C_0-semigroups and this result follows from Theorem 1.10.1. But it is not generally true without reflexivity. Before giving a counterexample and discussing the new phenomenon which arises we will, however, describe the relative boundedness version of Theorem 1.10.1.

THEOREM 1.10.3. *Let* S *and* T *be two* C_0^**-semigroups on the Banach space* $\mathcal{B}$ *with generators* H *and* K *, respectively. The following conditions are equivalent:*

1. $\|(S_t-T_t)f\| = O(t)$ *as* $t \to 0+$, *for all* $f \in D(H)$,

2. $$K \supseteq H + P$$

where $D(P) = D(H)$ *and*

$$\|Pf\| \le a\|f\| + b\|Hf\|$$

for all $f \in D(H)$ *and some* $a, b \ge 0$.

Proof. 1 ⇒ 2. First remark that $f \in D(H)$ if, and only if, $\|(I-S_t)f\| = O(t)$ at $t \to 0+$, by Exercise 1.5.2. But then since

$$\|(I-T_t)f\| \le \|(I-S_t)f\| + \|(S_t-T_t)f\| = O(t)$$

one must have $D(H) \subseteq D(K)$ and one can define P by $D(P) = D(H)$

and $P = K - H$.

Next note that H and K are both weak*-closed, and hence strongly closed, and consider the graph $G(H) = \{(f, Hf) ; f \in D(H)\}$ equipped with the norm

$$\|(f, Hf)\| = \|f\| + \|Hf\| .$$

The graph $G(H)$ is a closed subspace of $\mathcal{B} \times \mathcal{B}$ and the mapping $(f, Hf) \longmapsto Kf$ is a linear operator from $G(H)$ into $\mathcal{B}$. But this operator is closed, because if (f_n, Hf_n) converges in $G(H)$ and Kf_n converges in $\mathcal{B}$ then $\|f_n - f\| \to 0$, and $\|Kf_n - g\| \to 0$, for some $f, g \in \mathcal{B}$ and $g = Kf$ since K is closed. Therefore the mapping is bounded by the closed graph theorem, i.e., there is a constant $c > 0$ such that

$$\|Kf\| \le c\big(\|f\|+\|Hf\|\big) .$$

Consequently

$$\begin{aligned}\|Pf\| &= \|(K-H)f\| \\ &\le c\|f\| + (c+1)\|Hf\| .\end{aligned}$$

$2 \Rightarrow 1$. If $f \in D(H)$ then $S_t f \in D(H) \subseteq D(K)$ and

$$\big((S_t-T_t)f\big)(a) = \int_0^t ds\ \big(T_{t-s}PS_sf\big)(a)$$

for all $a \in \mathcal{B}_*$. Therefore

$$\begin{aligned}\|(S_t-T_t)f\| &\le t \sup_{0\le s\le t} \|T_{t-s}\|\big(a\|S_sf\|+b\|HS_sf\|\big) \\ &\le t \sup_{0\le s\le t} \|T_{t-s}\|\ \|S_s\|\big(a\|f\|+b\|Hf\|\big) .\end{aligned}$$

But $\|S_t\|, \|T_t\| \leq M \exp\{\omega t\}$ for suitable $M, \omega \geq 0$ and hence

$$\|(S_t - T_t)f\| = O(t)$$

as $t \to 0+$ for each $f \in D(H)$. □

The analogues of Theorems 1.10.1 and 1.10.3 are not true for general C_0-semigroups because of another effect which is illustrated by the following example.

Example 1.10.4. Let $\mathcal{B} = C_0(\mathbb{R})$ be the continuous functions on the real line which vanish at infinity, equipped with the usual supremum norm, and let S denote the C_0-group of translations,

$$(S_t f)(x) = f(x-t)$$

for $f \in \mathcal{B}$ and $t \in \mathbb{R}$. Thus the generator H of S is the operator of differentiation with domain the differentiable functions $f \in C_0(\mathbb{R})$ whose derivatives f' are also in $C_0(\mathbb{R})$. Next let M be the operator of multiplication by a bounded function m which is non-differentiable at some points but is uniformly Hölder continuous in the sense that

$$|m(x)-m(y)| \leq c|x-y|$$

for some c . Define W by $W = \exp\{iM\}$ and T by $T_t = WS_tW^{-1}$. Since

$$((S_t - T_t)f)(x) = (1-\exp\{i(m(x)-m(x-t))\})f(x-t)$$

one has the estimate

$$\|S_t - T_t\| \le c|t| \ .$$

But the generator K of T is given by $K = WHW^{-1}$ and $D(K) \neq D(H)$ because m is chosen to be non-differentiable. In fact one can choose m to be non-differentiable at a dense set of points and then one obtains the extreme case $D(H) \cap D(K) = \{0\}$.

Note that the same construction on $L^{\infty}(\mathbb{R})$ does not lead to a similar conclusion because the domain of the differentiation operator which generates translations is much larger and contains functions which are not continuously differentiable. □

The infinitesimal comparison of C_0-groups which are close together can be explained by a combination of a perturbation and a twist of the type occurring in Example 1.10.4. It is possible that this is also true for C_0-semigroups but the following proof does use the group property in an essential way. It also broadens the comparison criterion.

THEOREM 1.10.5. *Let* S *and* T *be two* C_0- *or* C_0^*-*groups on the Banach space* $\mathcal{B}$ *with generators* H *and* K, *respectively and let* $0 < \alpha \le 1$. *The following conditions are equivalent:*

1. $\|S_t - T_t\| = O(t^{\alpha})$ *as* $t \to 0$,
2. *there exist bounded operators* P , W , *such that* W *has a bounded inverse,*

$$K = W(H+P)W^{-1}$$

and

$$\left\|S_t - WS_tW^{-1}\right\| = 0\left(t^{\alpha}\right) \quad as \quad t \to 0 \ .$$

Proof. 1 ⇒ 2. Define

$$W = \frac{1}{r} \int_0^r ds \ T_sS_{-s}$$

where r is chosen sufficiently small that $\|I - W\| < 1$, and hence W has a bounded inverse. This is possible by Condition 1. Next introduce

$$U_t = W^{-1}T_tWS_{-t} \ .$$

One then has the identity

$$\left(I-U_h\right)/h = (rhW)^{-1} \int_0^h ds \ T_sS_{-s} - (rhW)^{-1} \int_r^{r+h} ds \ T_sS_{-s}$$

which implies the existence of the strong, or weak*-, limit

$$P = \lim_{h\to 0} \left(I-U_h\right)/h$$

and gives the identification

$$P = W^{-1}\left(I-T_rS_{-r}\right)/r \ .$$

Thus P is bounded. Next remark that

$$\left(I-T_t\right)Wa/t = W\left(I-S_t\right)a/t + W\left(I-U_t\right)S_ta/t \ .$$

But the right hand side converges for all $a \in D(H)$ in the limit

$t \to 0$. Hence $Wa \in D(K)$ and

$$KWa = W(H+P)a \ .$$

Similarly if $a \in D(K)$ then $W^{-1}a \in D(H)$ and

$$W^{-1}Ka = (H+P)W^{-1}a \ .$$

Thus $D(K) = WD(H)$ and

$$K = W(H+P)W^{-1} \ .$$

Finally one has

$$S_t - WS_tW^{-1} = \left(S_t - T_t\right) + \left(T_t - WS_tW^{-1}\right) \ .$$

But $t \mapsto WS_tW^{-1}$ is the group with generator

$$WHW^{-1} = K - WPW^{-1}$$

and hence

$$\left\|T_t - WS_tW^{-1}\right\| = O(t)$$

by perturbation theory, e.g., by Theorem 1.9.2. Thus

$$\begin{aligned}\left\|S_t - WS_tW^{-1}\right\| &\leq \left\|S_t - T_t\right\| + \left\|T_t - WS_tW^{-1}\right\| \\ &= O\left(t^{\alpha}\right) + O(t) = O\left(t^{\alpha}\right) \ .\end{aligned}$$

$2 \Rightarrow 1$. Define $Q = -WPW^{-1}$ then $H = W^{-1}(K+Q)W$ and WSW^{-1} is the group generated by $K + Q$. Thus

$$\|T_t - WS_tW^{-1}\| = O(t)$$

as $t \to 0$ by another application of perturbation theory. But

$$S_t - T_t = \left(S_t - WS_tW^{-1}\right) + \left(WS_tW^{-1} - T_t\right)$$

and hence

$$\|S_t - T_t\| \le \|S_t - WS_tW^{-1}\| + \|WS_tW^{-1} - T_t\|$$

$$= O\left(t^{\alpha}\right) + O(t) = O\left(t^{\alpha}\right) .$$ □

Exercises.

1.10.1. Prove that if S and T are two C_0-, or C_0^*-, semigroups with

$$\|S_t - T_t\| = o(t)$$

as $t \to 0+$ then $S = T$.

1.10.2. If S is a C_0- or C_0^*-semigroup prove that S is uniformly continuous if, and only if, there exist $\varepsilon, \delta > 0$ such that

$$\|I - S_t\| \le 1 - \varepsilon , \qquad 0 < t < \delta .$$

1.10.3. If S and T are two C_0- or C_0^*-groups with generators H and K prove that there exist $\varepsilon_1, \delta_1 > 0$ such that

$$\|S_t T_{-t} - I\| \le 1 - \varepsilon_1 , \qquad 0 < t < \delta_1 ,$$

if, and only if, there exist $\varepsilon_2, \delta_2 > 0$ and bounded operators P , W , such that W has a bounded inverse $K = W(H+P)W^{-1}$ and

$$\|S_t W S_{-t} W^{-1} - I\| \le 1 - \varepsilon_2 , \qquad 0 < t < \delta_2 .$$

Hint: Follow the proof of Theorem 1.10.5. Note that Exercise 1.10.2 follows by setting $T = I$.

BIBLIOGRAPHY

There is an ever broadening literature on one-parameter semigroups and their applications. The subject up to 1957 is comprehensively summarized in the book by E. Hille and R.S. Phillips

- *Functional Analysis and Semigroups,* American Mathematical Society Colloquium Publications No. 31, American Mathematical Society, Providence, Rhode Island (1957).

A recent update of the subject is contained in the book by E.B. Davies

- *One-parameter Semigroups,* Academic Press, London (1980).

Each of the following books also contains additional information on aspects of the theory

- Bratteli, O. and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, I.,* Springer-Verlag, New York-Heidelberg-Berlin (1979).

- Butzer, P.L. and H. Berens, *Semigroups of Operators and Approximation Theory,* Springer-Verlag, New York-Heidelberg-Berlin (1967).

- Kato, T., *Perturbation Theory for Linear Operators,* Springer-Verlag, New York-Heidelberg-Berlin (1976).

- Reed, M. and B. Simon, *Methods of Modern Mathematical Physics, Vol. I : Functional Analysis,* Academic Press, New York-San Francisco-London (1975).

- Riesz, F. and B. Sz-Nagy, *Lecons d'Analyse Fonctionelle,* Gauthier-Villars, Paris (1972).

- Yosida, K., *Functional Analysis,* Springer-Verlag, Berlin-Heidelberg-New York (1908).

A variety of applications of semigroups are described in the book by A. Belleni-Morante

- *Applied Semigroups and Evolution Equations,* Oxford (1979).

Each of these books contains a comprehensive list of references to the original research literature and to further books on related topics.

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