The Valuation of Contingent Securities.

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In financial economics a contingent security is an instrument that can be traded now, and which returns at some fixed time T in the future an amount that depends on the value at time T of some underlying security (usually a share). Since the price s at time T of the security is effectively a random variable on $[0,\infty]$, the contingent security may be identified with the function $f(s)$ specifying its return. For example, the holder of a call option has the right, but not the obligation, to buy a share at some agreed strike price s_0 at time T; the associated return function is $f(s) \equiv max[0,s-s_0]$. The problem is to determine the present value (i.e. price) of such a contingent security, or equivalently to identify the pricing functional $V : L \rightarrow \Re$, (L is the set of feasible return functions).

At present option pricing is done via the Black-Scholes formula [4]. The formula is derived by constructing a self-financing portfolio consisting of shares and bonds (a bond is a risk-free security whose return function is a constant) which is adjusted by trading so that it replicates the return of the contingent security. The option must have the same present value as the portfolio, otherwise buying the portfolio and selling the security, or vice versa, would give certain profit now, while the future returns of the two assets would cancel. Under the basic assumptions that the underlying stock price evolves through a geometric Brownian motion and that one can trade continuously in the market, a simple p.d.e. can be derived that describes the evolution of the portfolio, and this can then be integrated backward to determine the present option price. This approach is of great practical use, however its validity is restricted by the strong assumptions made about the market. Our aim is to find alternative characterizations of V.

In a perfect market where all contingent securities could be traded, the prices of securities contingent on a particular underlying security must be consistent if the market is in equilibrium. If not, there would be the possibility of riskless profit taking, which would in turn move prices. These consistency conditions essentially require that the functional V be linear and positive. Thus if L is a subset of any reasonable function space, e.g. L is the positive cone of $L^p([0,\infty],\mu)$ for some p and finite measure μ, then V must be a bounded linear functional [6, p. 84].

The problem brought to the workshop arose from an attempt to further characterize the pricing functional through the general equilibrium theory of Arrow and Debreu [1]. Briefly, given a collection of N individuals with utility functions $U_i(f)$ and initial endowments f_i of securities who then trade among themselves, the theory relates the final allocations f_i^*, the U_i and the valuation functional V^* when equilibrium is reached (i.e. when no one can make himself better off by trading at the prices V^* without someone else becoming worse off). The theory in [1], however, only covers the finite dimensional case where $f(s)$ is replaced by a finite vector with components f_m. Moreover, the extension to the present infinite dimensional setting is not straightforward, because the sets L of feasible allocations are subsets of the positive cone of $L^p([0,\infty] : \mu)$; if $p < \infty$, L will not have a nonempty interior and the standard separation theorems cannot be applied.

The particular problem was to show that at equilibrium V^* must be proportional to the gradient $\partial U_i(f_i^*)$. The following special case of this result was quickly proved at the workshop using the duality theory for constraint sets with nonempty quasi-relative interiors recently developed by Borwein and Lewis [2,3].

Theorem: If $v \in L^q([o,\infty),\mu)$ and $\alpha \geq 0$ are given, $f^*(s)$ solves the optimization problem:

$$\sup_{f \in L^p,\ f \geq 0} \quad E[U(f)] \;\equiv\; \int_0^\infty U(f(s))\, d\mu \qquad \text{subject to:} \quad \int_0^\infty f(s)\, v(s)\, d\mu \;\leq\; \alpha$$

with a finite value, and there exists a feasible $g \in qri(dom(E[U(.)])$, then $\beta v(s) \in \partial U(f^*(s))$ for some $\beta \geq 0$. Conversly, if $v \in L^q([o,\infty),d)$ and $v(s) \in \partial U(f^*(s))$ for some $\beta \geq 0$, then $f^*(s)$ solves:

$$\sup_{f \in L^p,\ f \geq 0} \quad E[U(f)] \qquad \text{subject to:} \quad \int_0^\infty f(s)\, v(s)\, d\mu \;\leq\; \int_0^\infty f^*(s)\, v(s)\, d\mu$$

We hope to extend these results to establish an infinite dimensional version of the full general equilibrium theory for contingent securities, and to relate the results to those presented recently by Mas-Collel [5].

References:

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