ON THE CONNECTEDNESS PROPERTIES OF SUNS IN FINITE DIMENSIONAL SPACES

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ABSTRACT

The author introduced in [3] the notion of an M-connected closed subset of a normed linear space and defined the class of (BM)-spaces. An M-connected closed subset of a finite dimensional normed linear space is a sun and a sun in a space which is either of dimension two or is a finite dimensional (BM)-space is M-connected. Theorem 1 asserts that an M-connected closed subset of a finite dimensional space is n-connected for all n = 0,1,2.... Theorem 2 relates Theorem 1 to the results of [3]. Theorem 3 is an improvement of a result of Koshcheev and asserts that a sun in a finite dimensional space is path-connected.

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1. INTRODUCTION

The concept of an M-connected closed subject of a normed linear space (or even of a metric space) was introduced in [3]. The definition is in terms of the metric structure of the space but it has topological consequences. The principal result of this paper is the terminologically satisfying

THEOREM 1 *An M-connected closed subset of a finite dimensional real normed linear space is* n*-connected for all* n = 0, 1, 2, ...

In the class of (BM)-spaces introduced in [3] M-connectedness provides a characterisation of those subsets of the space which are suns. The second of our theorems is obtained by combining Theorem 1 with results of [3]. Let G be an arbitrary non-trivial abelian group. A space (all our spaces are metric) will be said to be acyclic if its Cech (or Alexander) cohomology with coefficients in G is trivial.

THEOREM 2. *Let* K *be a closed subset of a finite dimensional real normed linear space* X. *Each of the first four of the five conditions*

(1) K *is M-connected,*

(2) *For each closed ball* B *of* X *the set* $K \cap B$ *is n-connected for all* $n = 0,1,2,\ldots,$

(3) *For each closed ball* B *of* X *the set* $K \cap B$ *either is empty or is acyclic,*

(4) K *is P-acyclic in the sense of Vlasov (see* [10] *or* [3]),

(5) K *is a sun,*

implies the succeeding one. If X *either is of dimension two or is a (BM)-space then the conditions are equivalent.*

The implication (1) $\Rightarrow$ (2) is an immediate consequence of Theorem 1 and a property of M-connectedness : if K is M-connected and B is a closed ball in X then $K \cap B$ is also M-connected. The implication (2) $\Rightarrow$ (3) is purely topological being a direct consequence of [9, Theorems 7.5.5, 5.5.3 (applied to singular homology and cohomology) and 6.9.5].
The other implications of the theorem (and the direct implication (1) $\Rightarrow$ (3)) are discussed in [3].

Conditions under which the results of [3] can be extended to an infinite dimensional space and a boundedly compact subset K are discussed in [4].

Koshcheev [6] has proved that a sun in a finite dimensional space is connected. He has also obtained results for suns in infinite dimensional spaces [7,8]. Here we obtain an improvement (mentioned in [3]) of Koshcheev's result.

THEOREM 3. *Let* X *be a real normed linear space of finite dimension. If* K *is a closed subset of* X *and* K *is a sun, then* K *is path-connected and locally path-connected. More precisely, there exist positive constants* M *and* α *depending only upon* X *such that if* x_0, y_0 *are distinct points of* K *then there exists a path* $s : [0,1] \longrightarrow K$ *from* x_0 *to* y_0 *which satisfies the condition*

$$(*) \qquad \|s(\xi) - s(\eta)\| \leq M \|x_0 - y_0\| \, |\xi-\eta|^{\alpha}$$

for all $\xi,\eta \in [0,1]$.

Theorems 2 and 3 together add point to the question "What connectedness properties are shared by all suns in finite dimensional spaces?" Theorems 1 and 3 are proved in Sections 2 and 3 respectively. The definitions and properties of suns and M-connected sets are discussed to the extent necessary to make the account readable.

The author's interest in the subject of suns was much increased by the work of Berens and Hetzelt [1,2]. Theorem 2 of this paper is a direct response to a remark of Prof. Berens who suggested that 'P-acyclicity is not the right condition'.

2. M-CONNECTED IMPLIES n-CONNECTED

Throughout the paper X will be a fixed real normed linear space of finite dimension m. The norm on X will be denoted by $\|\cdot\|$.

First we introduce some notation. We choose, once for all, $f_1,\ldots,f_m$ in $\operatorname{ext} S(X^*)$, the set of extreme points of the unit sphere of X^*, such that

$f_1, \ldots, f_m$ is a basis of X^*. It will be convenient to work with the norm $|\cdot|$ on X, equivalent to $\|\cdot\|$, defined by

$$|x| = \max_{1 \leq i \leq m} |f_i(x)|$$

For any bounded subset A of X and for $i = 1, \ldots, m$ let

$$l_i(A) = \inf f_i(A), \quad m_i(A) = \sup f_i(A), \quad d_i(A) = m_i(A) - l_i(A).$$

and let

$$d(A) = \max_{1 \leq i \leq m} d_i(A)$$

Thus $d(A)$ is the diameter of A with respect to the norm $|\cdot|$.

If $\alpha_i \leq \beta_i$ for $i = 1, \ldots, m$ then the set

$$b = \bigcap_{i=1}^{m} f_i^{-1}([\alpha_i, \beta_i])$$

will be called a box. If A is a bounded subset of X and $\alpha_i = l_i(A)$, $\beta_i = m_i(A)$ then b is the smallest box which contains A and will be denoted $b(A)$.

If A is a bounded subset of X then the Banach-Mazur Hull of A is the set

$$M(A) = \bigcap \{B : B \text{ is a closed ball of } X,\ A \subseteq B\}.$$

A closed subset K of X is said to be M-connected [3] if

$$K \cap (M(\{x,y\}) \setminus \{x,y\}) \neq \emptyset$$

whenever x,y are distinct points of K. The properties of M-connected sets which are required, and which are established in [3] are the following.

(M1) If K is M-connected and B is a closed ball of X then $K \cap B$ is M-connected (one can read this as including the statement that M-connectedness implies local M-connectedness).

(M2) If K is M-connected, $f \in \text{ext } S(X^*)$ and $\alpha \in \mathbb{R}$ then $K \cap f^{-1}((-\infty, \alpha])$ is also M-connected; consequently for each box b the set $K \cap b$ is M-connected.

(M3) If K is M-connected then K is path-connected.

In the proof of Theorem 1 only the properties (M2) and (M3) are needed.

Given a non-negative integer n let

$$E_0 = \{x = (\xi_1,\dots,\xi_{n+1}) \in \mathbb{R}^{n+1} : 0 \le \xi_i \le 1 \quad \text{for} \quad i = 1,2,\dots,n+1\}.$$

The boundary of a subset A of $\mathbb{R}^{n+1}$ relative to $\mathbb{R}^{n+1}$ will be denoted ∂A. Thus the pair $(E_0,\partial E_0)$ is homeomorphic to the pair (E^{n+1},S^n), the standard euclidean $(n+1)$-ball and its boundary sphere. Theorem 1 is equivalent to the statement that if K is an M-connected subset of X then for each non-negative integer n any continuous mapping

$$\varphi : \partial E_0 \longrightarrow K$$

has a continuous extension over E_0. The proof is by induction on n. Property (M3) is the case $n = 0$.

Now let n be a positive integer and assume that every M-connected subset of X is $(n-1)$-connected. Let

$$\varphi : \partial E_0 \longrightarrow K$$

be a continuous mapping. It must be shown that a continuous extension of φ over E_0 can be constructed. The construction will be in five steps. Step 1 is the basic one. Step 2 involves a finite repetition of Step 1. Step 3 consists of an infinite repetition of Step 2. Throughout the discussion the same symbol φ will be used for the initial mapping and for extensions of it.

The first step of the extension process depends upon an

EXTENSION LEMMA *If* $n \ge 1$, Y *is an* $(n-1)$*-connected space and* (X,A) *is a finite relative CW-complex with* $\dim(X\setminus A) \le n$ *then any continuous mapping* $f : A \longrightarrow Y$ *can be extended over* X.

The proof of the lemma is an elementary exercise. The lemma is a very simple case (in which there are no obstructions) of results in obstruction theory, e.g. [5, Proposition VI.6.6] or [9, Theorem 8.1.17].

We shall work with subsets of $\mathbb{R}^{n+1}$ which are unions of cubical cells of certain lattices. For each positive integer p the set $2^{-p}\,\mathbb{Z}^{n+1}$ is a lattice in $\mathbb{R}^{n+1}$; the sets of the form

$$\{x = (\xi_1,\ldots,\xi_{n+1}) : 2^{-p}a_i \le \xi_i \le 2^{-p}b_i \quad \text{for } i = 1,\ldots,n+1\}$$

where $a_i, b_i \in \mathbb{Z}$ and $a_i \le b_i \le a_i+1$ for $i = 1,\ldots,n+1$ will be referred to as the cubical cells of the 2^{-p}-lattice. The dimension of such a cell is $\operatorname{card}\{i : b_i \ne a_i\}$. Any union of cubical cells of the 2^{-p}-lattice is a CW-complex in a natural and obvious way.

Step 1 of the extension process Let E be a subset of E_0 such that E is a union of $(n+1)$-cells of some 2^{-p}-lattice and let

$$\varphi : \partial E \longrightarrow K$$

be a continuous mapping. Let $j \in \{1,\ldots,n\}$.

Then there exists a finite family of subsets $A_1,\ldots,A_M$ of E and a continuous extension of φ over $E \setminus \bigcup_{k=1}^{M} \operatorname{int} A_k$ such that

(1.1) For each $k = 1,\ldots,M$ the set A_k is a union of $(n+1)$-cells of some 2^{-q}-lattice,

(1.2) $\bigcup_{k=1}^{M} A_k = E$,

(1.3) $\operatorname{int} A_k \cap \operatorname{int} A_l = \emptyset$ for $k \ne l$,

(1.4) $\varphi(E \setminus \bigcup_{k=1}^{M} \operatorname{int} A_k) \subseteq b(\varphi(\partial E))$,

(1.5) $d_j(\varphi(\partial A_k)) \le \frac{2}{3}\, d_j(\varphi(\partial E))$ for $k = 1,\ldots,M$.

(The first digit of the numeration refers to Step 1.)

Proof First note that ∂E is a union of n-cells of the 2^{-p}-lattice.

We can suppose that $d_j(\varphi(\partial E)) > 0$ for otherwise we can take $M = 1$, $A_1 = E$. Let

$$\alpha_i = l_j(\varphi(\partial E)) + \frac{i}{3}\,(m_j(\varphi(\partial E)) - l_j(\varphi(\partial E)))$$

for $i = 0,1,2,3$. Let $I_i = [\alpha_i,\alpha_{i+1}]$ and $F_i = \varphi^{-1}(f_j^{-1}(I_i))$ for $i = 0,1,2$. Thus F_0, F_2 are disjoint closed subsets of ∂E.

It will be shown that there exist an integer q and a subset D of E such that

(i) D is a union of cells of the 2^{-q}-lattice and $D \subseteq E \setminus (F_0 \cup F_2)$,

(ii) D separates F_0 and F_2 in E, and

(iii) D is minimal subject to the conditions (i) and (ii).

Let d here denote distance with respect to the max norm in $\mathbb{R}^{n+1}$. In order to find q and D let

$$\Lambda = \{x \in E : d(x,F_0) = d(x,F_2)\},$$
$$\delta = d(\Lambda,F_0) = d(\Lambda,F_2)$$

and choose $q \geq p$ such that $2^{-q} < \delta$. Let Λ' be the union of all those (n+1)-cells of the 2^{-q}-lattice which are contained in E and meet Λ. Thus Λ' satisfies conditions (i) and (ii). If a subset Λ' of E satisfies (i) and (ii) then so does the union of all those cells of the 2^{-q}-lattice which are contained in Λ' and are of dimension $\leq n$. Consequently there exists a set D which satisfies conditions (i), (ii) and (iii) and it is a union of cells of dimension $\leq n$. It is easily seen that, by (iii), for such a set D the set $D \cap \partial E$ is a union of (n-1)-cells of the lattice.

Now $D \cap (F_0 \cup F_2) = \emptyset$ so $\varphi(D \cap \partial E) \subseteq f_j^{-1}((\alpha_1,\alpha_2))$.
Therefore

$$\varphi(D \cap \partial E) \subseteq K \cap b(\varphi(\partial E)) \cap f_j^{-1}([\alpha_1,\alpha_2]).$$

The set on the right is M-connected by (M2) and so (n-1)-connected by the inductive hypothesis. Therefore, by the Extension Lemma, the mapping

$$\varphi|D \cap \partial E : D \cap \partial E \longrightarrow K \cap b(\varphi(\partial E)) \cap f_j^{-1}([\alpha_1,\alpha_2])$$

has a continuous extension over D. An extension gives a continuous extension of

$$\varphi : \partial E \longrightarrow K \cap b(\varphi(\partial E))$$

over $D \cup \partial E$.

The open subset $E \setminus (D \cup \partial E)$ of $\mathbb{R}^{n+1}$ is dense in E and has a finite number of components; let the closures of these components be $A_1,\ldots,A_M$. Then conditions (1.1)-(1.4) are satisfied. If U is a component of $E \setminus (D \cup \partial E)$ then U^- cannot meet both F_1 and F_2, consequently $\varphi(\partial U)$ is a subset either of $f_j^{-1}([\alpha_0,\alpha_2])$ or of $f_j^{-1}([\alpha_1,\alpha_3])$. This proves that (1.5) is satisfied.

Step 2 of the extension process Let E be a subset of E_0 such that E is a union of (n+1)-cells of some 2^{-p}-lattice and let

$$\varphi : \partial E \longrightarrow K$$

be a continuous mapping. Then there exists a finite family of closed subsets $E_1',\dots,E_N'$ of E and a continuous extension of φ over $E \setminus \bigcup_{k=1}^{N} \operatorname{int} E_k'$ such that

(2.1) For each $k = 1,\dots,N$ the set E_k' is a union of (n+1)-cells of some 2^{-q}-lattice,

(2.2) $\bigcup_{k+1}^{M} E_k' = E$,

(2.3) $\operatorname{int} E_k' \cap \operatorname{int} E_l' = \emptyset$ for $k \neq l$,

(2.4) $\varphi(E \setminus \bigcup_{k=1}^{N} E_k') \subseteq b(\varphi(\partial E))$,

(2.5) $d(\varphi(\partial E_k)) \leq \frac{2}{3} d(\varphi(\partial E))$ for $k = 1,\dots,N$.

Proof The sets $E_1',\dots,E_N'$ and the extension φ are obtained by applying Stage 1 to $j = 1$ and E to obtain sets $A_1,\dots,A_M$, then applying Stage 1 to $j = 2$ and each of $A_1,\dots,A_M$ and so on for $j = 3,\dots,m$.

Step 3 of the extension process There exist positive integers $N(i)$ for $i = 0,1,2,\dots,$ sets $E_j^{(i)}$ for $j = 1,\dots,N(i)$ and $i = 0,1,2,\dots,$ and an extension of $\varphi : \partial E_0 \longrightarrow K$ over the set

$$\bigcup_{i=0}^{\infty} (E_0 \setminus \bigcup_{i=1}^{N(i)} \operatorname{int} E_j^{(i)})$$

such that

(3.1) $N(0) = 1$, $E_1^{(0)} = E_0$;

(3.2) $\bigcup_{j=1}^{N(i)} E_j^{(i)} = E_0$ for each $i = 0,1,2,\dots$;

(3.3) $\operatorname{int} E_j^{(i)} \cap \operatorname{int} E_k^{(i)} = \emptyset$ for $j \neq k$ and $i = 1,2,\dots$;

(3.4) For each $i+1, j$ such that $1 \leq j \leq N(i+1)$ there exists j' such that $E_j^{(i+1)} \subseteq E_{j'}^{(i)}$ and

(3.5) $\varphi(\partial E_j^{(i+1)}) \subseteq b(\varphi(\partial E_{j'}^{(i)}))$; and

(3.6) $d(\varphi(\partial E_j^{(i)})) \leq (\frac{2}{3})^i \, d(\varphi(\partial E))$ for all $j = 1,\dots,N(i)$ and $i = 0,1,2,\dots$.

Proof $N(0)$ and $E_1^{(0)}$ are defined by (3.1). If $N(r)$ for $r = 0,\dots,i-1$, the sets $E_j^{(r)}$ for $j = 1,\dots,N(r)$ and $r = 0,\dots,i-1$ and a continuous extension

$$\varphi : (E_0 \setminus \bigcup_{j=1}^{N(i-1)} \text{int } E_j^{(i-1)}) \longrightarrow K$$

have been defined then $N(i)$, the sets $E_j^{(i)}$ for $j = 1,\dots,N(i)$ and a continuous extension

$$\varphi : (E_0 \setminus \bigcup_{j=1}^{N(i)} \text{int } E_j^{(i)}) \longrightarrow K$$

are obtained by applying Step 2 to each of the sets $E_1^{(i-1)},\dots,E_{N(i-1)}^{(i-1)}$.

Step 4 of the extension process The extension

$$\varphi : A = \bigcup_{i=1}^{\infty} (E_0 \setminus \bigcup_{j=1}^{N(i)} E_j^{(i)}) \longrightarrow K$$

is uniformly continuous and so extends over A^-.

Proof Let i be a positive integer. Let

$$\delta_i = \min\{d(E_j^{(i)},E_k^{(i)}) : 1 \leq j,k \leq N(i),\ E_j^{(i)} \cap E_k^{(i)} = \emptyset\}.$$

Then $\delta_i > 0$. If $x,y \in E$ and $d(x,y) < \delta_i$ then there exist j,k such that $x \in E_j^{(i)}$, $y \in E_k^{(i)}$ and $E_j^{(i)} \cap E_k^{(i)} \neq \emptyset$. If also $x,y \in (E_0 \setminus \bigcup_{j=1}^{N(i)} \text{int } E_j^{(i)})$ then

$$d(\varphi(x),\varphi(y)) \leq d(\varphi(\partial E_j^{(i)}) + d(\varphi(\partial E_k^{(i)})) \leq 2(\tfrac{2}{3})^i \, d(\varphi(\partial E)).$$

This proves the assertion.

Step 5 of the extension process Let U be a component of the open subset $E \setminus A^-$ of $\mathbb{R}^{n+1}$. For each $i = 0,1,2,\ldots$ the set U is a subset of int $E_j^{(i)}$ for some j and $\partial U \subseteq E_j^{(i)}$. Then, by 3.5,

$$\varphi(\partial U) \subseteq b(\varphi(\partial E_j^{(i)})$$

and so, by 3.6,

$$d(\varphi(\partial U)) \leq d(\varphi(E_j^{(i)})) \leq (\tfrac{2}{3})^i \, d(\varphi(\partial E_0)).$$

This inequality holds for all $i = 0,1,2,\ldots$ and therefore $\varphi(\partial U)$ is a single point. It follows that

$$\varphi : A^- \longrightarrow K$$

can be extended over E_0 by defining it to be constant on the closure of each component of $E_0 \setminus A^-$. It is easy to see that the resulting extension is continuous. Thus we have obtained a continuous extension of the original mapping $\varphi : \partial E_0 \longrightarrow K$ and the proof of the theorem by induction is complete.

3. SUNS ARE PATH-CONNECTED.

A closed subset K of the normed linear space X (of finite dimension m) is a sun if for each $x \in X \setminus K$ there exists a point $s \in K$ (a *solar point for* x *in* K) such that

$$\|(s+\lambda(x-s)) - s\| = d(s+\lambda(x-s),K)$$

for all $\lambda \geq 0$. Let

$$B(x,r) = \{y \in X : \|x-y\| < r\}$$

$$B'(x,r) = \{y \in X : \|x-y\| \leq r\}$$

$$S(x,r) = \{y \in X : \|x-y\| = r\}$$

denote the open ball, closed ball and sphere in X of centre x and radius r respectively. The proof of Theorem 3 depends on the following well-known property of solar points.

Proposition (v.[3],Prop 2.1). *If* K *is a sun in* X, $x \in X \setminus K$ *and* $s \in K$ *is a solar point for* x *in* K *then* $[s,k] \cap B(x,\|x-s\|) = \emptyset$ *for all* $k \in K$.

Let K be a sun in X and let x_0,x_1 be distinct points of K. First we describe the construction of a path in K from x_0 to x_1. The idea arose from consideration of the case in which X is a space of dimension two (see[2,3]). Let

$$D = \{2^{-k}i : i = 0,1,\ldots,2^k,\ k = 0,1,\ldots\}$$

be the set of dyadic rationals in $[0,1]$. A mapping $s : D \longrightarrow K$ can be chosen inductively in the following way:

(i) Let $s(0) = x_0$, $s(1) = x_1$.

(ii) If the points $s(\frac{i}{2^j})$ have been chosen for $i = 0,\ldots,2^j$ and $j = 0,\ldots,k$ then for $i = 0,\ldots,2^{k-1}$ let

$$z(i,k) = \frac{1}{2}\left(s\left(\frac{i}{2^k}\right) + s\left(\frac{i+1}{2^k}\right)\right)$$

and choose

$$s\left(\frac{2i+1}{2^{k+1}}\right) = \begin{cases} z(i,k) & \text{if } z(i,k) \in K, \\ \text{a solar point for } z(i,k) \text{ in } K & \text{if } z(i,k) \notin K. \end{cases}$$

It will be shown that a mapping $s : D \longrightarrow K$ chosen in this way satisfies condition (*) of Theorem 3 (for some M and α). It will then follow that $s : D \longrightarrow K$ is uniformly continuous and so has a continuous extension $s : [0,1] \longrightarrow K$. The extension will also satisfy (*). The proof will be complete.

The proof depends upon the principal Lemma 3 of [6] which we reformulate as follows.

LEMMA *There exists* $\mu' \in (0,1)$, *a constant depending only upon the space* X, *such that if* $((x_r,y_r,z_r,s_r) : r = 0,\ldots,m-1)$ *is an m-tuple of points in* X^4 *(that is a point of* $(X^4)^m$*) which satisfies the conditions*

(1) $x_r \neq y_r$, $z_r = \frac{1}{2}(x_r+y_r)$, $s_r \notin [x_r,y_r]$ *for* $r = 0,\ldots,m-1$,

(2) *Either* $(x_{r+1},y_{r+1}) = (x_r,s_r)$ *or* $(x_{r+1},y_{r+1}) = (s_r,y_r)$ *for* $r = 0,\dots,m-2$,

(3) $([x_r,s_r] \cup [y_r,s_r] \cap B(z_r,\|s_r-z_r\|) = \emptyset$ *for* $r = 0,\dots,m-1$,

then

$$\max\{\|s_{m-1} - x_{m-1}\|,\ \|y_{m-1} - s_{m-1}\|\} \le \mu'\|y_0-x_0\|.$$

Let $\mu = \max\{\frac{1}{2},\mu'\}$. Let $s : D \longrightarrow K$ be a mapping chosen to satisfy conditions (i) and (ii). It will be shown that if

$$M = \frac{4m}{\mu(1-\mu)}, \qquad \alpha = -\frac{1}{m}\log_2\mu$$

then $s : D \longrightarrow K$ satisfies condition (*) of Theorem 3.

In order to describe the proof we introduce the set

$$T = \{(i,j) : i = 0,\dots,2^j-1;\ j = 0,1,2,\dots\}$$

which will be regarded as a directed tree in which (i,j) has the two immediate successors $(2i,j+1)$ and $(2i+1,j+1)$. Then $(0,0)$ is the root of the tree and (i,j) is a point at height j in the tree. Let $\le$ denote the partial order in T. To each point $(i,j) \in T$ we associate the line segment $[s(\frac{i}{2^j}),\ s(\frac{i+1}{2^j})]$, its mid-point $z(i,j)$ and its length $l(i,j) = \|s(\frac{i+1}{2^j}) - s(\frac{i}{2^j})\|$. The line segments corresponding to a level of the tree form a polygonal path which is an approximation to the path $s : [0,1] \longrightarrow K$.

Consider a point $(i,j) \in T$. For the moment write $x = s(\frac{i}{2^j})$, $y = s(\frac{i+1}{2^j})$ and $s = s(\frac{2i+1}{2^j})$. If $z = z(i,j) \notin K$ then s is a solar point for z in K and by the Proposition

$$([x,s] \cup [y,s] \cap B(z,\|s-z\|) = \emptyset.$$

Thus to any sequence $(i_0,j) \le (i_1,j+1) \le \dots$ of successive points in the tree there corresponds a sequence of points in X^4. If none of the points $z(i_r,j+r)$ is in K then the sequence of points satisfy conditions (1), (2) and (3) (for all relevant r).

For each $(i,j) \in T$ the point $s(\frac{2i+1}{2^j})$ is a point of K closest to $z(i,j)$. It follows that if $(i_o,j) \leq (i_1,j+1)$ then

$$(4) \qquad l(i_1,j+1) \leq l(i_0,j)$$

and

$$(5) \qquad \|z(i_1,j+1) - z(i_0,j)\| \leq l(i_0,j).$$

Thus l is a monotonically decreasing function on T. If $s(\frac{2i+1}{2^j}) = z(i_0,j) \in K$ then

$$(6) \qquad l(i_1,j+1) = \frac{1}{2}\, l(i_0,j).$$

Suppose that $(i_0,j) \leq (i_1,j+1) \leq \ldots \leq (i_{m-1},j+m-1) \leq (i_m,j+m)$ are $m+1$ successive points in the tree. If $z(i_r,j+r) \in K$ for some $r \in \{0,\ldots,m-1\}$ then by (6) and the monotonicity of l

$$l(i_m,j+m) \leq \frac{1}{2}\, l(i_0,j)$$

and otherwise, by the lemma,

$$l(i_m,j+m) \leq \mu'\, l(i_0,j).$$

Consequently, in either case

$$(7) \qquad l(i_m,j+m) \leq \mu\, l(i_0,j).$$

Also, by (5) and the monotonicity of l,

$$(8) \qquad \|z(i_r,j+r) - z(i_0,j)\| \leq r\, l(i_0,j) \leq m\, l(i_0,j).$$

Now we climb up the tree m levels at a time. Suppose that $(i,j+n) \geq (i_0,j)$ and that $km \leq n < (k+1)m$. Then there is a sequence

$$(i_0,j) \leq (i_1,j+m) \leq \ldots \leq (i_k,j+km) \leq (i,j+n)$$

in the tree. By (5), the monotonicity of l, (8) and (7),

$$\begin{aligned}
\|s(\tfrac{2i+1}{2^{j+n+1}}) - z(i_0,j)\| &\leq \|s(\tfrac{2i+1}{2^{j+n+1}}) - z(i,j+n)\| + \|z(i,j+n) - z(i_k,j+km)\| \\
&\quad + \sum_{r=1}^{k} \|z(i_r,j+rn) - z(i_{r-1},j+(r-1)n)\| \\
&\leq \tfrac{1}{2} l(i,j+n) + m\, l(i_k,j+km) + \sum_{r=1}^{k} m\, l(i_{r-1},j+(r-1)m) \\
&\leq (\tfrac{1}{2}\mu^k + \sum_{r=1}^{k+1} m\,\mu^{r-1})\, l(i_0,j) \\
(9) \qquad &< \frac{m}{1-\mu}\, l(i_0,j).
\end{aligned}$$

It follows that if $\xi \in D \cap [s(2^{-j}i_0),\ s(2^{-j}(i_0+1))]$ then

(10) $$\|s(\xi) - z(i_0,j)\| < \frac{2m}{1-\mu}\, l(i_0,j).$$

(if ξ is in the open interval then (10) is given by (9); (10) clearly holds for the end points of the interval.)

Now suppose that $\xi,\eta \in D$. Let $N = [-\frac{1}{m}\log_2|\xi-\eta|]$.

Then

$$|\xi-\eta| \le \frac{1}{2^{Nm}}, \qquad \mu^N < \frac{1}{\mu}\,|\xi-\eta|^{\alpha}$$

and there exists an i such that $\{\xi,\eta\} \subseteq [\frac{i-1}{2^{Nm}}, \frac{i+1}{2^{Nm}}]$.

It follows from (10) and (7) that

$$\begin{aligned}\|s(\xi) - s(\eta)\| &\le \|s(\xi) - s(\tfrac{i}{2^{Nm}})\| + \|s(\tfrac{i}{2^{Nm}}) - s(\eta)\| \\ &\le \frac{2m}{1-\mu}\, 2 \max\{l(i-1,Nm),\ l(i,Nm)\} \\ &\le \frac{4m}{1-\mu}\, \mu^N\, l(0,0) \\ &< M\|y_0-x_0\|\ |\xi-\eta|^{\alpha}.\end{aligned}$$

The proof is complete.

We conclude by remarking that Theorem 3 is true with $\alpha = 1$ if the space X either is two-dimensional or is a BM-space, and, more generally, that the conclusion holds with $\alpha=1$ if K is an M-connected set, whatever the finite dimensional space X. In the case $\dim X = 2$ the result follows from those of [2], for $X = \ell^{\infty}(m)$ it follows from those of [1] and for M-connected sets in general it is proved in [3].

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