Lagrangian conditions for a minimax

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Abstract

A general approach is given to Lagrangian necessary conditions for a minimax problem. The necessary conditions become sufficient for a minimax under extra hypotheses, with either concave/convex or invex functions, and restrictions on the constraints. A minimax is shown to relate to a weak minimum of a vector function. The sensitivity of a minimax value to a perturbation is related to the gradient of a Lagrangian function with respect to the parameter.

Key words minimax, Lagrangian, concave/convex functions, invex, Robinson condition, weak vector minimization, sensitivity to perturbation.

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1. Introduction

Minimax problems are often associated with constrained minimization problems. Examples of functions $F(x,y)$ which are to be maximized with respect to y, then minimized with respect to x, include:

(i) $F(x,y) = f(y) - x^T g(y)$, (1)

a Lagrangian function from the problem

Maximize $f(y)$ subject to $g(y) \le 0$ (2)

[or to $-g(y) \in S$, where S is a closed convex cone];

(ii) $F(x,y) = f(y) - (\mu/2)\|[g(y)+\mu^{-1}x]_+\|^2$,

an augmented Lagrangian for (2); μ is a positive parameter, and $[t]_+ = t$ if $t \ge 0$, $[t]_+ = 0$ if $t<0$, for each component of $g(y)+\mu^{-1}x$.

[For a constraint $-g(x) \in S$, the expression $[g(y)+\mu^{-1}x]_+$ is replaced by $(I-P)[g(y)+\mu^{-1}x]$, where $Pv=v$ for $v \in S$, and, for $v \notin S$, Pv is the orthogonal projection of v onto S].

For (i) and (ii), the minimax problem is:

$$[\mathrm{MIN}_x \{\mathrm{MAX}_y F(x,y) : g(y) \le 0\} : x \ge 0],$$

with $x \ge 0$ replaced by $x \in S^*$, the dual cone of S, in case of a constraint $-g(y) \in S$. Another example of minimax occurs when an objective function is the pointwise maximum of several functions, namely

$$[\mathrm{MIN}_x \{\mathrm{MAX}_i f_i(x) \} : g(x) \le 0],$$

where $i = 1,2,\ldots r$.

A (global)minimax (x^*,y^*) for the problem:

$$\mathrm{MIN}_{x \in \Delta} \mathrm{MAX}_{y \in \Xi(x)} F(x,y), \qquad (3)$$

where Δ and $\Xi(x)$ are given sets, means that there exists a function $y\hat{}(x)$ such that $y^* = y\hat{}(x^*)$ and

$$(\forall x \in \Delta, \forall y \in \Xi(x)) \quad F(x,y\hat{}(x)) \ge F(x^*,y^*) \text{ and } F(x,y\hat{}(x)) \ge F(x,y). \qquad (4)$$

In contrast, (x^*,y^*) is a saddlepoint for (3) if, instead,

$$(\forall x \in \Delta, \forall y \in \Xi(x)) \quad F(x,y^*) \ge F(x^*,y^*) \ge F(x^*,y). \qquad (5)$$

It is well known - see, for example, Tanimoto [11], Craven and Mond [7], Bector and Chandra [1] - that a minimax problem is often associated with necessary conditions of Kuhn-Tucker type. It will now be shown that this holds under fairly general conditions, and also that such necessary conditions become also sufficient for a minimax, under suitable convexity hypotheses.

2. Necessary conditions for a minimax

Consider the problem:

$$[\mathrm{MIN}_x \{\mathrm{MAX}_y F(x,y) : -h(x,y) \in S\} : -g(x) \in T], \qquad (6)$$

in which X,Y,Z,U are normed spaces, $F: X \times Y \to \mathbb{R}$, $h: X \times Y \to U$, $g: X \to Z$ are continuously (Fréchet) differentiable functions, $S \subset U$ and $T \subset Z$ are closed convex cones, MIN denotes local minimum, and MAX denotes local maximum . (The spaces may, but need not, be finite-dimensional.) Assume that the inner (maximization) problem reaches a (local) maximum when $y = y\hat{}(x)$, with maximum value denoted by $m(x)$, and that a constraint qualification holds at this maximum. Then $m(x) = F(x,y\hat{}(x))$; and Kuhn-Tucker necessary conditions hold:

$$(\exists \lambda(x) \in S^*) \quad F_y(x,y\hat{}(x)) - \lambda(x)^T h_y(x,y\hat{}(x)) = 0, \; \lambda(x)^T h(x,y\hat{}(x)) = 0. \qquad (7)$$

(Here F_y means partial Fréchet derivative with respect to y, and superscript ^{T} denotes transpose in finite dimensions; in infinite dimensions, λ is a continuous linear functional, and $\lambda^T h_y$ means the composition $\lambda \circ h_y$.)

Since x is a parameter in the inner problem, it follows, under some regularity conditions, that the gradient m_x of $m(x)$ equals the Fréchet derivative, with respect to x, of the Lagrangian $F(x,y)-\lambda^T g(x,y)$, thus

$$m_x(x) = F_x(x,y\hat{}(x)) - \lambda(x)^T h_x(x,y\hat{}(x)) \,. \qquad (8)$$

Appropriate regularity conditions are [4] that $y\hat{}(.)$ is a Lipschitz function, $\lambda(.)$ is continuous, and a constraint qualification holds for the inner problem for each x, so that Kuhn-Tucker necessary conditions hold. Hypotheses sufficient for the first two requirements are discussed in [5].

Consider now the outer (minimization) problem. Assuming a constraint qualification (now relating to the constraint $-g(x)\in T$), necessary Kuhn-Tucker conditions for a minimum at $x=x^*$ are :

$$(\exists \mu \in T^*) \quad m_x(x^*) + \mu^T g_x(x^*) = 0,\ \mu^T g(x^*) = 0. \qquad (9)$$

Substituting from (8) for m_x gives

$$F_x(x^*,y\hat{}(x^*)) - \lambda(x^*)^T h_x(x^*,y\hat{}(x^*)) = 0. \qquad (10)$$

Define therefore a Lagrangian function for the minimax problem (6) as

$$L(x,y;\lambda,\mu) = F(x,y)-\lambda^T h(x,y)+\mu^T g(x). \qquad (11)$$

Denote $\nabla L:=[L_x,L_y]$. The following theorem has now been proved.

Theorem 1 In the minimax problem (6), assume that

(i) F,g and h are continuously Fréchet differentiable; the minimax is reached at $(x,y)=(x^*,y^*)$, with a constraint qualification holding there for the outer problem;

(ii) for $-g(x)\in T$ and $\|x-x^*\|$ sufficiently small, the inner problem reaches a local maximum at a point $y=y\hat{}(x)$, Kuhn-Tucker conditions hold there with Lagrange multiplier $\lambda\hat{}(x)$, $\lambda\hat{}(.)$ is continuous at x^*, and $y\hat{}(.)$ is a Lipschitz function, with $y\hat{}(x^*)=y^*$.

Then

$$(\exists \lambda^* \in S^*,\ \mu^* \in T^*) \quad \nabla L(x^*,y^*;\lambda^*,\mu^*)=0,\ \mu^{*T} g(x^*)=0;\ \lambda^{*T} h(x^*,y^*)=0, \qquad (12)$$

where $\lambda\hat{}(x^*)=\lambda^*$. Moreover, for $-g(x)\in T$ and $\|x-x^*\|$ sufficiently small,

$$L_y(x,y\hat{}(x);\lambda\hat{}(x),\mu^*)=0;\ \lambda\hat{}(x)^T h(x,y\hat{}(x))=0 \,. \qquad (13)$$

3. Sufficient conditions for a minimax

A converse result to Theorem 1 holds, under the serious restriction that h(x,y) does not depend on x. In this case, problem (6) takes the form:

$$\mathrm{MIN}_{x\in\Delta}\ \mathrm{MAX}_{y\in\Xi}\ F(x,y)\ , \tag{14}$$

in which $\Delta:=\{x\in X:-g(x)\in T\}$, and $\Xi:=\{y\in Y:-h(x,y)\in S$ is independent of x.

In order to apply an implicit function theorem, consider the system:

$$L_y(x,y;\lambda,\mu^*)=0;\ \lambda^T h(x,y)=0,\ -h(x,y)\in S,\ \lambda\in S^*, \tag{15}$$

written in the form $-K(y,\lambda;x)\in V$, where μ^* is fixed, x is a parameter, V is the convex cone $\{0\}\times\{0\}\times S\times S^*$, and solutions $(y,\lambda)=\psi(x)$ are sought, when $\|x-x^*\|$ is small. For this system, consider the Robinson condition

$$0\in \mathrm{int}[K(y^*,\lambda^*;x^*) + \mathrm{ran}\ P_{(y,\lambda)}K(y^*,\lambda^*;x^*) + V], \tag{16}$$

where int denotes interior, ran denotes range, and $P_{(y,\lambda)}$ denotes partial Fréchet derivative with respect to (y,λ). From (15) and (16), the condition requires that

$$0\in\mathrm{int}[L_y+L_{yy}(X\times Y)]; \tag{17}$$

$$0\in\mathrm{int}[\lambda^T h+\lambda^T h_y(X\times Y)+h^T(U^*)]; \tag{18}$$

$$0\in\mathrm{int}[h+h_y(X\times Y)+S]; \tag{19}$$

$$0\in\mathrm{int}[-\lambda-U^*+S^*]; \tag{20}$$

where all functions are evaluated at $(x,y,\lambda)=(x^*,y^*,\lambda^*)$. Note that (17) is equivalent to the surjectivity of L_{yy} ; and(20) holds trivially.

Theorem 2 For problem (6), assume that

(i) F,g and h are continuously Fréchet differentiable, and h(x,y) does not depend on x,

(ii) the Kuhn-Tucker necessary conditions (12) hold, with $-h(x^*,y^*)\in S$, $-g(x^*)\in T$;

(iii) F(x,.) is concave for each $x\in\Delta$, F(.,y) is convex for each $y\in Y$, and that g(.) is T-convex;

(iv) L_y is continuously Fréchet differentiable with respect to y, $L_{yy}(x^*,y^*)$ is surjective, the other Robinson conditions (18), (19) hold at (x^*,y^*,λ^*), and L_y is continuously differentiable with respect to x.

Then (x^*,y^*) is a local minimax point for (6).

Proof Robinson's theorem [9, Theorem 1] shows from (iv) that (13) has a continuous solution $(y,\lambda)=(\hat{y}(x),\hat{\lambda}(x))$, with $(\hat{y}(x^*),\hat{\lambda}(x^*))=(y^*,\lambda^*)$, valid when $\|x-x^*\|$ is sufficiently small. Implicit differentiation of (15) shows that $y\hat{}(.)$ is differentiable at x^*, hence Lipschitz. Hence there hold (15), the necessary Kuhn-Tucker conditions for a maximum of the inner problem in (14). Since $F(x,.)$ is concave, these necessary conditions are also sufficient for a maximum at $y=y\hat{}(x)$; thus

$$m(x) := F(x,y\hat{}(x))=\mathrm{MAX}_{y\in\Xi}\ F(x,y). \tag{21}$$

Since $m(.)$ is a maximum of a set of convex functions, $m(.)$ is convex. Since $\lambda\hat{}(.)$ is continuous and $y\hat{}(.)$ is Lipschitz, the gradient $m_x(x)$ is given by (8). If $-g(x)\in T$, then convexity of $F(.,y^*)$ and T-convexity of $g(.)$ show that, if $x\in\Delta$, then

$$\begin{aligned}F(x,y\hat{}(x))-F(x^*,y^*) &= m(x)-m(x^*) && \text{since } y^*=y\hat{}(x^*)\\ &\geq m_x(x^*)(x-x^*) && \text{since } m(.) \text{ is convex}\\ &= F_x(x^*,y^*))(x-x^*) && \text{by (8), since } h_x\equiv 0\\ &= -\mu^{*T}g_x(x^*,y^*)(x-x^*) && \text{by (12)}\\ &\geq -\mu^{*T}g(x)+\mu^{*T}g(x^*) && \text{since } \mu^{*T}g(.) \text{ is convex}\\ &\geq 0 + 0.\end{aligned}$$

Remark The proof does not need $\mathrm{MAX}_y\ F(.,y)$ differentiable; it may not be.

Remark If the hypothesis (iv) is omitted, then (21) only holds for $x=x^*$, and only a saddlepoint can be deduced, by

$$\begin{aligned}F(x,y^*)-F(x^*,y^*) &\geq F_x(x^*,y^*)(x-x^*)\\ &= -\mu^{*T}g_x(x^*)(x-x^*)\\ &\geq -\mu^{*T}g(x)+\mu^{*T}g(x)\\ &\geq 0+0.\end{aligned}$$

Remark If $h(x,y)$ depends on x, $\Phi:\Delta\times Y\to R^-:=R\cup\{+\infty\}$ may be defined (in the manner of Rockafellar [10]) as $\Phi(x,y)=F(x,y)$ when $-h(x,y)\in S$, otherwise $\Phi(x,y)=+\infty$. Then (6) is equivalent to the problem

$$\mathrm{MIN}_{x\in\Delta}\ \{\mathrm{MAX}_{y\in Y}\ \Phi(x,y)\}\ . \tag{22}$$

Then Theorem 2 may be applied with Φ replacing F and Y replacing Ξ. But the necessary conditions so obtained are not useful, because the concave/convex properties assumed for Φ only hold when Φ takes only finite values for $x\in\Delta$, thus when h does not depend on x.

A less restrictive sufficiency theorem can be given, when the dependence of $h(x,y)$ on x takes a certain form. Consider the form:

$$h(x,y)=q(r(x)+y)\ , \tag{23}$$

where q and r are differentiable functions.

Theorem 3 For the minimax problem (6), assume that F,g,h are continuously Fréchet differentiable, and satisfy hypotheses (ii) and (iv) of Theorem 2, where h(x,y) has the special form (23) with q and r differentiable. Define $\Psi(x,w):=F(x,w-r(x))$ and $\Omega:=\{w:-q(w)\in S\}$. Assume also that $\Psi(x,.)$ is concave on Ω for each $x\in\Delta$, $\Psi(.,w)$ is convex for each $w\in\Omega$, and that g(.) is T-convex. Then (x^*,y^*) is a local minimax point for (6).

Proof From (ii) and (iv) there follow, as in the proof of Theorem 2, the necessary Kuhn-Tucker conditions for a maximum of the inner problem of (6) at $(x,y^\frown(x)$. The (invertible) change of variable from (x,y) to (x,w), where $w:=r(x)+y$ converts the problem (6) to:

$$MIN_{x\in\Delta}\ MAX_{w\in\Omega}\ \Psi(x,w). \qquad (24)$$

If $z=(x,y)$, $\rho=(\lambda^\frown(x),\mu^*)$, and $k(z):=[-h(x,y),g(x)]$, then the Lagrangian L in (11) becomes $F(z)+\rho^T k(z)$, and the Lagrangian conditions (12) and (13) become $\nabla_z L(z;\rho)=0$, $\rho^T k(z)=0$, where $z=(x,y^\frown(x))$. The invertible transformation given by $w:=r(x)+y$ may be expressed as $z=\varphi(\zeta)$, where $\zeta=(x,w)$. It follows that $\nabla_\zeta L(\varphi(\zeta);\rho)=0$ and $\rho^T k(\varphi(\zeta))=0$, where $z=\varphi(\zeta)$. Thus the Lagrangian necessary conditions hold also for problem (24). Since $\Psi(x,.)$ is concave on Ω for each $x\in\Delta$, and $\Psi(.,w)$ is convex on Δ for each $w\in\Omega$, the last part of the proof of Theorem 2 shows that ζ^* is a minimax point for (24), and hence (x^*,y^*) is a minimax point for (6).

A notable special case is that of a linear constraint $-h(x,y)\in S$. Consider a constraint $Ax + By \le c$, where $x\in R^n$, $y\in R^p$, $c\in R^m$, A is an m×n matrix, B is an m×p matrix, and $p\le m<n$. The matrix B has full rank if it has rank MIN{m,p}. Assume that B has full rank. If p<m, additional columns may then be adjoined to B, to make an invertible m×m matrix $B^\sim$; let the additional components of y form a vector $y_{(a)}$; let $y^{\sim T}:=[y^T,y_{(a)}{}^T]$. Define the m×n matrix $K^\sim\equiv K^\sim(A,B):=B^{\sim-1}A$. Then $Ax+By=B(K^\sim x+y)$. Denote by $K\equiv K(A,B)$ the matrix obtained from $K^\sim$ by deleting rows corresponding to elements of $y_{(a)}$.

Theorem 4 In the minimax problem (6), let the inner constraint $-h(x,y)\in S$ take the linear form $Ax+By\le c$, where the matrix B has full rank, and $p\le m<n$. Let $\Psi(x,w):=F(x,w-K(A,B)x)$, and $\Pi:=\{w:Bw\le c\}$. Assume that F and g are continuously differentiable, g is T-convex, $\Psi(x,.)$ is concave on Π for each $x\in\Delta:=\{x:-g(x)\in T\}$, $\Psi(.,w)$ is convex on Δ for each $w\in\Pi$, hypothesis (iv) of Theorem 2 holds, and the necessary conditions (12) hold at a point (x^*,y^*) satisfying the constraints of (6). Then (x^*,y^*) is a minimax point for (6).

Proof Construct the matrices $K^{\sim}\equiv K^{\sim}(A,B)$ and $K\equiv K(A,B)$ as above. Let e be a vector of ones, and let M be a sufficiently large positive number. The modified inner problem,

$$MAX_y \{F(x,y)-Me^T y_{(a)} : B(K^{\sim}x+y^{\sim})\leq c\}, \qquad (25)$$

reaches the same maximum as the given inner problem in (6), since maximization eliminates the artificial variables $y_{(a)}$. Let $w:=K^{\sim}x+y^{\sim}$. Since the transformation $(x,y^{\sim})\rightarrow(x,w)$ is invertible, the minimax problem is equivalent to the problem

$$MIN_{x\in\Delta}\, MAX_{w\in\Pi}\, \Psi(x,w)-Me^T y_{(a)}\, . \qquad (26)$$

The concave/convex hypotheses on Ψ imply similar properties for the objective function of (26), since linear terms are concave and convex, and Ψ does not involve $y_{(a)}$. As in the proof of Theorem 3, the necessary Lagrangian conditions for (6) imply necessary Lagrangian conditions for (26). Hence, by Theorem 2, these conditions are also sufficient for a minimax of (26), and so of (6) in this case.

Corollary For a linear minimax problem, thus when $F(x,y)$ is bilinear in x and y, and g and h are affine functions, with h satisfying the rank requirement of Theorem 4 and (iv) holding, the necessary Lagrangian conditions at a feasible point are also sufficient for a minimax.

Some relaxation of the concave/convex hypothesis of Theorem 2 is discussed below, in Section 5.

4. The relation of a minimax to a weak vector minimization

Consider the minimax problem (6) when $Y=\{1,2,...,r\}$, and write $f_i(x):=F(x,i)$ $(i=1,2,...,r)$. This may be related to the weak vector minimization problem:

$$WEAKMIN_x\ f(x):=\{f_1(x),f_2(x),...,f_r(x)\}\ \text{subject to}\ -g(x)\in T. \qquad (27)$$

The weak minimization [2] is with respect to a convex cone Q; thus x^* is a weak minimum of (27) if $f(x)-f(x^*)\notin -\text{int}\, Q$ for all feasible points x, sufficiently close to x^*. Assume initially that $Q=R_+^r$. Let $e^T:=(1,1,...,1)$; and note that $e\in\text{int}\, Q$. Let $m(x):=MAX\{f_1(x),f_2(x),...,f_r(x)\}$; then $m(x^*)e-f(x^*)$ has all components ≥ 0, and at least one zero component. This may be expressed by $m(x^*)e-f(x^*)\in\partial Q$, where ∂ denotes boundary.

Thus, for the minimax problem considered, there hold:
(i) $m(x^*)e-f(x^*)\in\partial Q$; (ii) $m(x)e-f(x)\in\partial Q$; (iii) $m(x)e-m(x^*)e\in Q$; where (ii) holds for all x satisfying $-g(x)\in T$, and expresses the inner maximization, and (iii) holds for all x satisfying $-g(x)\in T$, sufficiently close to x^*, and expresses the outer minimization. Suppose, if possible, that x^* is not a weak minimum of (15). Then, for some such x, $f(x)-f(x^*)\in -\text{int } Q$. From (i) and (iii), $m(x)e-f(x^*)\in Q+\partial Q\subset Q$. From the supposition, $f(x^*)-f(x)\in\text{int } Q$. Adding these inclusions,

$$m(x)e-f(x)\in Q+\text{int } Q \in \text{int } Q,$$

contradicting (ii). Hence x^* is a weak minimum of (27).

This relation generalizes to weak minimization with respect to some other cones Q than R_+^r, provided that minimization is suitably defined. Let $Q\subset R^r$ be a convex cone with interior; and fix $e \in \text{int } Q$. Now define, for a vector $f(x)$, the **maximum of $f(x)$ with respect to Q** (denoted by $MAX_Q\, f(x)$) as $m(x)$, satisfying

$$m(x)e - f(x) \in\partial Q. \qquad (28)$$

Then the proof of the previous paragraph shows that an optimum of

$$MIN_x \{MAX_Q\, f(x)\} \quad \text{subject to } -g(x)\in T \qquad (29)$$

must be a weak minimum of $f(x)$ subject to $-g(x)\in T$.

It follows that Kuhn-Tucker necessary conditions hold for a considerable class of optimization problems, that imply weak vector minimization. Some other examples arise in generalized fractional programming (see [6, Chapter 6]).

5. Using invex hypotheses

In problem (14), the hypothesis that $F(x,.)$ is concave on Ξ may be weakened as follows. Assume that $-F(x,.)$ is invex on Ξ, defined [8,3] by

$$(\forall x\in\Delta,\ \forall y,y'\in\Xi) \qquad -F(x,y') + F(x,y) \geq F_y(x,y)\theta(x,y,y'), \qquad (30)$$

and that $h(y)\equiv h(x,y)$ is also invex, thus

$$(\forall y,y'\in\Xi) \qquad h(y')-h(y)\geq_S h'(y)\theta(x,y,y'), \qquad (31)$$

with the same function θ, with $a\geq_S b \Leftrightarrow a-b\in S$. It is known then [8] that the Kuhn-Tucker necessary conditions (12),(13) for the inner problem in (14) are also sufficient; thus when $\|x-x^*\|$ is sufficiently small,

$$q(x):= F(x,y\hat{}(x))=MAX_{y\in\Xi}\ F(x,y). \qquad (32)$$

Assume also that each function $F(.,y)$, $y\in\Xi$, is invex, thus

$$(\forall x,x'\in\Delta,\ \forall y\in\Xi) \qquad F(x',y)-F(x,y)\geq F_x(x,y)\sigma(x,x'), \qquad (33)$$

thus with the function σ independent of $y\in\Xi$; and assume that g is invex, thus

$$g(x')-g(x)\geq_T g_x(x)\sigma(x,x'), \qquad (34)$$

with the same function σ.

From (31) and (32), if $y^*=y\hat{}(x^*)$ and $-g(x)\in T$,

$$\begin{aligned} Q(x) - Q(x^*) &= F(x,y\hat{}(x))-F(x^*,y\hat{}(x^*)) \\ &\geq F_x(x^*,y^*)\sigma(x^*,x) && \text{from (33) and } y\hat{}(x^*)=y^* \\ &= -\mu g_x(x^*,y^*)\sigma(x^*,x) && \text{from (12)} \\ &\geq -\mu g(x) + \mu g(x^*) \\ &\geq 0 + 0. \end{aligned}$$

This has proved

Theorem 5 For the minimax problem (6), assume that F, g,h satisfy hypotheses (i), (ii) and (iv) of Theorem 2, and also the invex hypotheses (30), (31), (33),(34). Then (x^*,y^*) is a minimum point for (6).

6. Sensitivity of minimax value to perturbations

Consider now problem (6), with a perturbation parameter $p\in R^s$ included in each function, thus:

$$J(p):= [MIN_x\{MAX_y F(x,y;p):-h(x,y;p)\in S\}:-g(x;p)\in T] . \qquad (35)$$

The Lagrangian for (35) is

$$L(x,y:\lambda,\mu;p):=F(x,y;p)-\lambda^T h(x,y;p)+\mu^T g(x;p). \qquad (36)$$

Let ∇_p denote gradient with respect to p. Assume the hypotheses of Theorem 1, for each fixed p in a neighbourhood N of 0. Then

$$m(x;p):=\{MAX_y F(x,y;p):-h(x,y;p)\in S\}=F(x,y\hat{}(x;p);p), \qquad (37)$$

for a suitable function $y\hat{}(x;p)$; and, having assumed suitable regularity conditions for the inner problem, $m_p(x;p) =\nabla_p[F(x,y;p) -\lambda^T g(x,y;p)]$ at $y=y\hat{}(x;p),\lambda=\lambda\hat{}(x;p)$, from [4,Theorem 1]. For the outer problem,

$$J(p):= MIN_x \{m(x;p) : -g(x;p)\in T\}; \qquad (38)$$

the Lagrangian is $F(x,y\hat{}(x);p)+\mu^T g(x;p)$; the optimal point x and multiplier μ are functions $x^{\#}(p),\mu^{\#}(p)$. Assuming suitable regularity, $\Phi'(p)$ equals the gradient, with respect to p, of the Lagrangian $m(x;p) +\mu^T g(x;p)$. Hence, substituting for $m_p(x;p)$,

$$\begin{aligned} J'(0) &= m_p(x^*;0) + \mu^T g_p(x^*;0) \\ &= F_p(x^*,y^*;0) - \lambda^{*T} g_p(x^*,y^*;0) + \mu^{*T} g_p(x^*;0), \end{aligned} \qquad (39)$$

where $(x^*,y^*)=(x^{\#}(0),y\hat{}(x^*;0)$ is the optimum at $p=0$, with Lagrange multipliers $\lambda^*=\lambda\hat{}(x^*;0)$, $\mu^*=\mu^{\#}(0)$. Hence, citing appropriate regularity conditions from [4], the following Theorem is proved.

Theorem 6 For the parametrized minimax problem (35), assume the hypotheses of Theorem 1, for each p in a neighbourhood N of 0; also that $y\hat{}(x;.)$ is Lipschitz , and $\lambda\hat{}(x;.)$ and $\mu^{\#}(.)$ are continuous at 0. Then the optimum value function J(p) of (35) is Fréchet differentiable at 0, with

$$J'(0) = L_p(x^*,y^*:\lambda^*,\mu^*;0). \qquad (40)$$

Remark For conditions sufficient for such Lipschitz conditions, with continuity of Lagrange multipliers as functions of p, see [5]. Conditions for the multipliers relate to a dual problem. In particular, if problem (35) is linear in all variables, then $\lambda\hat{}(x;.)$ and $\mu^{\#}(.)$ are locally constant functions, with jumps when a basis changes in a dual linear program.

7. References

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