MOST OF THE TWO-PERSON ZERO-SUM GAMES HAVE UNIQUE SOLUTION

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Abstract : For a large class of compact topological spaces it is proved that the majority (in the Baire category sense) of the two-person zero-sum games have unique solution.

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1.Introduction.

Let f be a continuous function defined on the cartesian product XxY of the compact sets X and Y. Consider the two-person zero-sum game G_f generated by f.

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This means that the first player chooses a point x from X and the second player selects a point y from Y. They make their choices simultaneously and independently of each other. As a result of this game the second player pays to the first one the amount $f(x,y)$ (if $f(x,y)<0$, the first player pays to the second $-f(x,y)$ units of money).

If X and Y are finite sets, games like this are called matrix games (see [Ka]) because, in this case, f determines a matrix $A = (f(x_i,y_j))_{i,j}$.

If the game is to be repeated many times, it makes sense for every one of the players to determine his "strategy" showing the probability with which he chooses a given element from his set. For example, in the case of matrix games, where X is the set $\{1,2, \dots ,n\}$ and Y is equal to $\{1,2, \dots ,m\}$, every strategy of the first player is a nonnegative vector $(p_1,p_2, \dots ,p_n)$ for which $\sum_{i=1}^{n} p_i = 1$. Here p_i is the probability with which the first player chooses i. The strategies of the second player look similar . In the case of infinite compact spaces X and Y the strategies are Radon probability measures, i.e. nonnegative elements of $C(X)^*$ (resp. $C(Y)^*$) with norm one.

Let $\mathfrak{X} = \{ \mu \in C(X)^* : \mu \ge 0, \mu(1) = 1 \}$ be the set of strategies of the first player and $\mathfrak{Y} = \{ \nu \in C(Y)^* : \nu \ge 0, \nu(1) = 1 \}$ be the set of strategies of the second one. Throughout this paper we will consider $\mathfrak{X}$ and $\mathfrak{Y}$ endowed with the inherited weak-star topology from $C(X)^*$ and $C(Y)^*$ respectively. So $\mathfrak{X}$ and $\mathfrak{Y}$ are Hausdorff compact spaces.

If in the two-person zero-sum game G_f the first player chooses his strategy $\mu \in \mathfrak{X}$ and the second accepts $\nu \in \mathfrak{Y}$, the expected gain of the first player is $\Phi(\mu,\nu) = \int\int f(x,y)d\mu d\nu$. $\Phi : \mathfrak{X} \times \mathfrak{Y} \longrightarrow \mathbb{R}$ is a bilinear function defined on the cartesian product of $\mathfrak{X}$ and $\mathfrak{Y}$.

A couple $(\mu_0,\nu_0) \in \mathfrak{X} \times \mathfrak{Y}$ is said to be a solution to the game G_f if it is a saddle point of the function Φ, i.e.

(μ_0,ν_0) is a solution to the game G_f if the two inequalities

$$\iint f\, d\mu d\nu_0 \le \iint f\, d\mu_0 d\nu_0 \le \iint f\, d\mu_0 d\nu$$

hold for every $\mu \in \mathfrak{X}$ and for every $\nu \in \mathfrak{Y}$.

It is known (see for instance [A], p.223) that for each Hausdorff compact spaces X and Y every continuous function $f \in C(X \times Y)$ generates a game which has at least one solution. Of course, there are games with more than one solution. It turns out, however, that the majority of the games have unique solution. In 1950 Bohnenblust, Karlin and Shapley [BKS] observed that the set of all $m \times n$ matrix games with unique solution is open and dense in the (finite dimensional) space of all matrices of the same size. In 1969 Djubin [Dju] considered the case when X and Y are metrizable compacts and showed that the set of all continuous functions f which are defined on $X \times Y$ and generate two-person zero-sum game with unique solution contains a dense G_δ subset of $C(X \times Y)$. The main aim of this paper is to extend the result of Djubin to a more general class of compact spaces.

2. Definitions and notations.

First we recall some definitions.

2.1. Definition. A multivalued mapping $F : U_1 \longrightarrow U_2$ from the topological space U_1 into the topological space U_2 is said to be upper semi-continuous at the point $u_0 \in U_1$ if for every open set V in U_2 which contains $F(u_0)$ there exists a neighbourhood W of u_0 such that $F(u) \subset V$ for every $u \in W$. F is said to be upper semicontinuous if it is upper semi-continuous at every point $u \in U_1$. The correspondence F is called *usco*, if it is upper semi-continuous and F(u) is non-empty and compact for every $u \in U_1$.

<u>2.2. Definition.</u> A multivalued mapping $F : U_1 \longrightarrow U_2$ from a topological space U_1 into a topological space U_2 is said to have the property (*) at $u \in U_1$ if there exists a point w in $F(u)$ such that for each neighbourhood W of w there exists a neighbourhood V of u with $F(v) \cap W \neq \emptyset$ whenever $v \in V$.

In an implicit form this notion was used in [DK]. Explicitly it was involved in [Ch] and [ChK].

Our method allows us to prove a "Djubin-like" result for generic uniqueness of the solution of two-person zero-sum games whenever the topological space $\mathfrak{X} \times \mathfrak{Y}$ is such that for every complete metric space B and every *usco* correspondence $F : B \longrightarrow \mathfrak{X} \times \mathfrak{Y}$ the subset of B consisting of all points at which F doesn't have the property (*) is of first Baire category.

In [Ch] it is shown that if X and Y are Eberlein compacts (i.e. each of these spaces is homeomorphic to a weakly compact subset of some Banach space), $\mathfrak{X} \times \mathfrak{Y}$ has the desired property. Debs in [D] proved the same (among other things) for X, Y Talagrand compacta and Kenderov in [K3] generalized his result for Gul'ko compact spaces. Combining some results from [ChK] and [N] we can obtain the desired conclusion for X, Y which are Radon-Nikodym compacts (see [N], Theorem 5.6). The widest class (to our knowlege) of compact spaces such that for every X, Y in it $\mathfrak{X} \times \mathfrak{Y}$ has the former property, is the class of the so called fragmentable compacta :

<u>2.3. Definition.</u> (see [JR]) Let X be a topological space and ρ be a metric defined on $X \times X$. X is said to be fragmented by the metric ρ if, for every $\varepsilon > 0$ and each nonempty subset Z of X there is a nonempty relatively open subset U of Z such that $\rho\text{-diam}(U) \leq \varepsilon$. The space X is said to be fragmentable, if there exits a metric ρ on it which fragments X.

The class of fragmentable spaces contains the Radon-Nikodym compacts and (in increasing generality) all Eberlein compacta, all Talagrand compacta and all Gul'ko compact spaces (see [R]). It is stable under various topological operations, for instance under countable products, countable unions (of closed subsets), continuous images (see [R], Proposition 2.8). Moreover, if X is fragmentable, then $C(X)^*$ with the weak star topology is fragmentable as well (see [R], Theorem 3.1). We will prove our "Djubin-like" result for X, Y fragmentable compacts.

3. Main result.

3.1. Theorem. Let X and Y be fragmentable Hausdorff compact spaces. Then the set of all continuous functions $f \in C(X \times Y)$, for which the corresponding game G_f has unique solution, contains a dense G_δ subset of the space $C(X \times Y)$ equipped with the usual uniform convergence norm.

Proof.

Let us consider the multivalued mapping

$S : C(X \times Y) \longrightarrow \mathfrak{X} \times \mathfrak{Y}$

assigning to each continuous function f the set of solutions to the two-person zero-sum game G_f. Here the domain space is endowed the uniform convergence topology and the range space is endowed the product topology inherited from $(C(X)^*, w^*) \times (C(Y)^*, w^*)$.

We need the following

3.2. Lemma. The above defined mapping S is *usco*.

Proof of the lemma.

Because of the compactness of $\mathfrak{X} \times \mathfrak{Y}$, it suffices to show that the graph of S is closed. Let $\{ f_\alpha \}_{\alpha \in A}$ and

$\{(\mu_\alpha,\nu_\alpha)\}_{\alpha\in A}$ be two convergent nets with limits f_o and (μ_o,ν_o) respectively and such that $(\mu_\alpha,\nu_\alpha) \in S(f_\alpha)$ for every $\alpha \in A$. Since

$$| \iint f_\alpha d\mu_\alpha d\nu_\alpha - \iint f_o d\mu_o d\nu_o | \leq$$

$$\| f_\alpha - f_o \| + | \iint f_o d\mu_\alpha d\nu_\alpha - \iint f_o d\mu_o d\nu_o |$$

tends to zero, we get

$$\iint f_\alpha d\mu_\alpha d\nu_\alpha \longrightarrow \iint f_o d\mu_o d\nu_o .$$

Similarly

$$\iint f_\alpha d\mu_\alpha d\nu \longrightarrow \iint f_o d\mu_o d\nu \quad \text{and}$$

$$\iint f_\alpha d\mu d\nu_\alpha \longrightarrow \iint f_o d\mu d\nu_o$$

for every fixed $\mu \in \mathfrak{X}$, $\nu \in \mathfrak{Y}$. Now from the inequalities

$$\iint f_\alpha d\mu d\nu_\alpha \leq \iint f_\alpha d\mu_\alpha d\nu_\alpha \leq \iint f_\alpha d\mu_\alpha d\nu$$

which are fulfilled for every α and for every fixed $\mu \in \mathfrak{X}$, $\nu \in \mathfrak{Y}$ we conclude that

$$\iint f_o d\mu d\nu_o \leq \iint f_o d\mu_o d\nu_o \leq \iint f_o d\mu_o d\nu$$

for every $\mu \in \mathfrak{X}$, $\nu \in \mathfrak{Y}$, i..e. (μ_o,ν_o) is a solution to the game generated by f_o. This means that the graph of S is a closed subset of $(C(X \times Y), \|.\|) \times (\mathfrak{X} \times \mathfrak{Y}, w^*)$.■

Since X is a fragmentable compact space, Theorem 3.1 of [R] yields that $C(X)^*$ endowed with the weak star topology, is a fragmentable space as well. This means that $\mathfrak{X}$ is also fragmentable. Similar arguments show that $\mathfrak{Y}$ is fragmentable. Using the stability properties of the class of fragmentable spaces (see Proposition 2.8, (d) from [R]), we conclude that the product $\mathfrak{X} \times \mathfrak{Y}$ is fragmentable. Now the *usco* correspondence S is defined on the complete metric space $(C(X \times Y),\|.\|)$ and its range space is the fragmentable compact $\mathfrak{X} \times \mathfrak{Y}$. Therefore there exists a dense and G_δ subset A of $(C(X \times Y),\|.\|)$ such that S has the property (*) at every point of A (see [R], Proposition 2.5). The following lemma completes the proof of the theorem :

<u>3.3.Lemma.</u> Let S have the property (*) at f_o and (μ_o,ν_o)

be the point, mentioned in (*), i.e. for every neighbourhood U of (μ_0,ν_0) there exists $\varepsilon > 0$ such that the intersection $S(f) \cap U$ is nonempty whenever $f \in C(X \times Y)$ and $\| f - f_0 \| < \varepsilon$. Then S is single-valued at f_0, i.e. $S(f_0)=\{(\mu_0,\nu_0)\}$.

<u>Proof of the lemma.</u>

To prove this we recall an elementary fact from the game theory, namely that for every $f \in C(X \times Y)$ $S(f)$ is the cartesian product of the set

$$\{ \mu_0 \in \mathfrak{X} : \min \{ \int\int f \, d\mu_0 d\nu : \nu \in \mathfrak{Y} \} = \max \{ \min \{ \int\int f \, d\mu d\nu : \nu \in \mathfrak{Y} \} : \mu \in \mathfrak{X} \} \}$$

(which is called the set of optimal strategies of the first player) and the set

$$\{ \nu_0 \in \mathfrak{Y} : \max \{ \int\int f \, d\mu d\nu_0 : \mu \in \mathfrak{X} \} = \min \{ \max \{ \int\int f \, d\mu d\nu : \mu \in \mathfrak{X} \} : \nu \in \mathfrak{Y} \} \}$$

Let us suppose that $S(f_0)$ contains more than one element. Since $(\mu_1,\nu_1) \in S(f_0)$ and $(\mu_2,\nu_2) \in S(f_0)$ imply $(\mu_1,\nu_2) \in S(f_0)$, we can assume without loss of generality that there exists a point $(\mu_1,\nu_0) \in S(f_0)$ such that $\mu_1 \neq \mu_0$. Hence there exists a continuous function $a \in C(X)$ with $\| a \|_{C(X)} = 1$, $\mu_1(a) \le 0$ and $\mu_0(a) > 0$.

For every positive real ε we consider the function $f_\varepsilon : X \times Y \longrightarrow \mathbb{R}$, defined by

$$f_\varepsilon(x,y) = f(x,y) - \varepsilon . a(x).$$

It is clear that f_ε is continuous and $\| f_\varepsilon - f_0 \| \le \varepsilon$. Let us denote by v_0 and v_ε the quantities

$$\max \{ \min \{ \int\int f_0 d\mu d\nu : \nu \in \mathfrak{Y} \} : \mu \in \mathfrak{X} \} \text{ and}$$

$$\max \{ \min \{ \int\int f_\varepsilon d\mu d\nu : \nu \in \mathfrak{Y} \} : \mu \in \mathfrak{X} \}$$

respectively. Then for every $\mu \in \mathfrak{X}$ with $\mu(a) > 0$ we have

$$\min \{ \int\int f_\varepsilon d\mu d\nu : \nu \in \mathfrak{Y} \} =$$

$$\min \{ \int\int f_0 d\mu d\nu : \nu \in \mathfrak{Y} \} - \varepsilon . \mu(a) \le$$

$$v_0 - \varepsilon . \mu(a) < v_0 .$$

On the other hand, since μ_1 is optimal for the first player in the game generated by f_0 and $\mu_1(a) \le 0$, we get

$v_0 = \min \{ \int\int f_0 d\mu_1 d\nu : \nu \in \mathfrak{Y} \} \le$

$$\min \{ \int\int f_\varepsilon d\mu_1 d\nu : \nu \in \mathfrak{Y} \} \le v_\varepsilon .$$

Therefore

$\min \{ \int\int f_\varepsilon d\mu d\nu : \nu \in \mathfrak{Y} \} < v_0 \le v_\varepsilon$

whenever $\mu \in \mathfrak{X}$, $\mu(a) > 0$. This means that μ isn't optimal for the first player in the game generated by f_ε.

Let us denote by V the open subset $\{ \mu \in \mathfrak{X} : \mu(a) > 0 \} \times \mathfrak{Y}$ of $\mathfrak{X} \times \mathfrak{Y}$. We proved that for every positive ε a function $f_\varepsilon \in C(X \times Y)$ can be found for which $\| f - f_\varepsilon \| \le \varepsilon$ and $S(f_\varepsilon) \cap V = \emptyset$. As $(\mu_0, \nu_0) \in V$, this contradicts the fact that S has the property (*) at the point f_0. ■

Let us consider the particular case when the space Y is a singleton, i.e. $Y = \{y_0\}$. Then $C(Y)$ and $C(Y)^*$ coincide with the real line $\mathbb{R}$ and $\mathfrak{Y}$ is a single point set $\{1\}$. Each continuous function $f \in C(X \times Y) = C(X \times \{y_0\})$ is identified with the function $g(x) \equiv f(x,y_0) \in C(X)$. Moreover, for $\mu \in C(X)^*$ the pair $(\mu,1)$ provides a solution of the game generated by $f(x,y_0)$ if and only if

$\text{supp}\, \mu \subset \{ x_0 \in X : \max \{ g(x) : x \in X \} \}$.

Therefore, such a game will have unique solution just in the case when the set $\{ x_0 \in X : g(x_0) = \max \{ g(x) : x \in X \} \}$ is a singleton. In this way we get the following corollary (see [K1], [K2] and [KR2] for more general results).

<u>3.4. Corollary.</u> If X is a fragmentable compact, then the set of all functions $g \in C(X)$ which attain their maximum at just one point of X contains dense and G_δ subset of $(C(X), \|.\|)$.

<u>3.5. Remarks.</u> Let us note that if X is an uncountable product of segments $[0,1]$ or $X = \beta\mathbb{N} \setminus \mathbb{N}$, where $\mathbb{N}$ is the set of all integers and $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$, then no function $g \in C(X)$ attains its maximum at just one point. For these spaces theorem 3.1 is not valid and therefore they are not fragmentable.

In [KR1] we announced theorem 3.1 for the particular case when X and Y are Eberlein compacts.

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