ABSOLUTELY CHEBYSHEV SUBSPACES

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ABSTRACT: Let's say that a closed subspace M of a Banach space X is absolutely Chebyshev if it is Chebyshev and, for each $x \in X$, $\|x\|$ can be expressed as a function of only $d(x, M)$ and $\|P_M(x)\|$. A typical example is a closed subspace of a Hilbert space. Absolutely Chebyshev subspaces are, modulo renorming, the same as semi-L-summands. We show that any real Banach space can be absolutely Chebyshev in some larger space, with a nonlinear metric projection. Dually, it follows that if M has the 2-ball property but not the 3-ball property in X, no restriction exists on the quotient space X/M. It is not known whether such examples can be found in complex Banach spaces.

Everybody knows that a Чебышев (Chebyshev) subspace of a Banach space is one whose metric projection is single valued. This means that for each point x in the larger space, there is a unique point $Px = P_M x$ in the subspace which minimizes $\|x - Px\|$. In the case of a Hilbert space, the metric projection is just the orthogonal projection onto the (closed) subspace. It is linear and satisfies the identity $\|x\|^2 = \|Px\|^2 + \|x - Px\|^2$. This paper studies a natural generalization of the latter condition.

We will say that a closed subspace M of a Banach space X is *absolutely Chebyshev* if, in addition to being Чебышев, there is a function $\varphi : \mathbb{R}^2_+ \to \mathbb{R}$ satisfying the identity

$$\|x\| = \varphi(\|Px\|, \|x - Px\|)$$

for all $x \in X$. When φ needs to be emphasized, we will call M a φ-Chebyshev subspace of X. Choosing $x \in M$ and $y \in X$ with $\|x\| = 1 = \|y\| = d(y, M)$, we see easily that $\|\alpha x + \beta y\| = \varphi(|\alpha|, |\beta|)$ for all $(\alpha , \beta) \in \mathbb{R}^2$. Thus φ must be (the restriction to the positive quadrant of) an absolute norm on $\mathbb{R}^2$. Clearly $\varphi(1,0) = \varphi(0,1) = 1$, and $(0,1)$ must be an extreme point of the unit ball of $(\mathbb{R}^2, \varphi)$. Conversely, given any norm φ satisfying the preceding conditions, we can easily give examples of φ-Chebyshev subspaces.

The following characterization of Hilbert spaces shows that an arbitrary Banach space cannot have too many absolutely Chebyshev subspaces.

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Proposition 1. *If every (one dimensional) subspace of X is absolutely Chebyshev, then X is a Hilbert space.*

Proof. Consider $x, y \in X$ with $\|x\| \leq \|x + \alpha y\|$ for all scalars α. If $M = \text{span}(y)$, then $P_M(x) = 0$, whence $\|x + y\| = \varphi(\|y\|, \|x\|) = \|x - y\|$. This shows that Birkhoff orthogonality implies James orthogonality. A result of Ohira and Leduc [1, 4.4] then ensures that X is a Hilbert space. ♠

For spaces of dimension three or more, there are several alternative proofs of Proposition 1, using the Blaschke-Kakutani characterization of Hilbert spaces [1, Characterizations 12.7, 13.8 or 18.4].

Nevertheless, for $1 \leq p < \infty$, classical Banach spaces abound with examples of L_p-Chebyshev subspaces. There are no nondegenerate L_∞-Chebyshev subspaces, since $(0, 1)$ is not an extreme point for the L_∞ norm on $\mathbb{R}^2$. For $1 < p < \infty$, it is known [9, Corollary 1.9] that every L_p-Chebyshev subspace is already an L_p-summand. In this case, there is nothing new to be said.

The study of absolutely Chebyshev subspaces is really interesting only when their metric projections are nonlinear. The main purpose of this note is to show that examples of this phenomenon are far more common than previously thought.

In §1, we give some elementary results concerning absolutely Chebyshev subspaces. Every such subspace is equivalent, in a certain sense, to an L_1-Chebyshev subspace. Note that the L_1-Chebyshev subspaces are precisely the *semi-L-summands* first defined in [5].

In §2, we give some new examples of L_1-Chebyshev subspaces. Previously, every known example of an L_1-Chebyshev subspace either contained an L_1-summand, or was the range of a linear metric projection. We show that every real Banach space can be an L_1-Chebyshev subspace, with nonlinear metric projection, in some larger space.

It seems likely that, in a complex Banach space, the metric projection onto an absolutely Chebyshev subspace is automatically linear. In §3, we make some brief remarks about this, and an infamous problem which is dual to it.

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1. BASIC PROPERTIES OF φ-CHEBYSHEV SUBSPACES

The results in this section are special cases of results already established for *semisummands* in [9], and for *absolutely proximinal* subspaces in [10]. In fact, an absolutely Chebyshev subspace is nothing other than a Чебышев semisummand. The proofs are often simpler in the case of Чебышев subspaces, so we give a complete account.

The first interesting result is that if $(1,0)$ is a smooth point (of the unit ball of $(\mathbb{R}^2,\varphi)$), then every φ-Chebyshev subspace is a φ-summand. This is not too difficult to establish directly. Indeed, it is a special case of the main result of [15]. However, we will deduce this from a technical result which has several other applications.

Given any Banach space X, and a closed subspace M, let us put

$$D(M) = \{f \in X^* : \|f\| = 1 = \|f|M\|\}.$$

The *metric complement* of M is defined by $M^\perp = \{x \in X : \|x\| = d(x,M)\}$; it is closed under scalar multiplication but need not be convex. Note that $D(M)$ is just the intersection of $(M^\circ)^\perp$ with the unit sphere of X^*. If M is a Чебышев subspace of X, then every element of X can be written uniquely as the sum of an element of M and an element of $M^\perp$. This is false for nonЧебышев subspaces.

Given a normalized absolute norm φ on $\mathbb{R}^2$, we define an index $n = n_\varphi$ by

$$n_\varphi = \lim_{\varepsilon\downarrow 0}\frac{\varphi(1,\varepsilon)-1}{\varepsilon}.$$

It is clear from the triangle inequality and our assumptions on φ that $0 \le n_\varphi \le 1$. Geometrically, the ray $\{(1-n\beta,\beta) : \beta \ge 0\}$ is the one-sided tangent to the unit ball at $(1,0)$. The following lemma relates these definitions.

Lemma 2. *If M is a φ-Chebyshev subspace of X, and $f \in D(M)$, then $|f(y)| \le n\|y\|$ for all $y \in M^\perp$.*

Proof. Fix $y \in M^\perp$ with $\|y\| = 1$. For any $\varepsilon > 0$, we can find $m \in M$ with $\|m\| = 1$ and $f(m) > 1-\varepsilon^2$. If $x = m+\varepsilon y$ then $\varphi(1,\varepsilon) = \|x\| \ge f(x) > 1-\varepsilon^2+\varepsilon f(y)$. Thus $f(y) \le \varepsilon^{-1}(\varphi(1,\varepsilon)-1)+\varepsilon \to n$. ♠

Now let us define $\rho = \rho_M : X \to \mathbb{R}$ by $\rho_M(x) = \|Px\| + nd(x,M)$. The next result shows that ρ_M is a seminorm.

Proposition 3. *If M is a φ-Chebyshev subspace of X, then, for all $x \in X$,*

$$\rho_M(x) = \max\{f(x) : f \in D(M)\}.$$

Proof. Lemma 2 shows that $f(x) \le \rho(x)$ whenever $f \in D(M)$. Given $x \in X$, we must now find $f \in D(M)$ with $f(x) = \rho(x)$.

For $x \notin M^\perp$, we define a linear functional f on the span of Px and $x-Px$ by setting $f(Px) = \|Px\|$ and $f(x-Px) = n\|x-Px\|$. Clearly $f(x) = \rho(x)$. Since the unit ball of

$(\mathbb{R}^2, \varphi)$ is contained in $\{(\alpha, \beta) : |\alpha| + n|\beta| \leq 1\}$, we have $\|f\| \leq 1$. Extending f to all of X, and noting that $Px \neq 0$, we obtain $f \in D(M)$. The special case $x \in M^\perp$ follows by considering $x + m$, for any fixed $m \in M \setminus \{0\}$. ♠

The next result shows that the metric projection onto an absolutely Chebyshev subspace is always Lipschitz continuous.

Corollary 4. *If M is a φ-Chebyshev subspace of X, then, for every $x, y \in X$, we have $\|Px - Py\| + n_\varphi|d(x, M) - d(y, M)| \leq \|x - y\|$.*

Proof.
$$\begin{aligned} \|Px - Py\| + nd(y, M) &= \rho(y - Px) \\ &\leq \rho(y - x) + \rho(x - Px) \\ &\leq \|y - x\| + nd(x, M). \end{aligned}$$
♠

For many absolutely Chebyshev subspaces, the metric projection is already linear.

Corollary 5. *Suppose that M is a φ-Chebyshev subspace of X, and that either $(1, 0)$ is a smooth point in $(\mathbb{R}^2, \varphi)$, or that M is the kernel of a norm one (linear) projection. Then P_M is linear.*

Proof. If $n_\varphi = 0$, then $M^\perp = \ker P_M = \{x : \rho(x) = 0\}$ must be a subspace.

If M is the kernel of a norm one projection, this projection must be a closest point mapping. ♠

Now we can show that every absolutely Chebyshev subspace is, in a suitable sense, equivalent to a semi-L-summand.

Theorem 6. *If M is an absolutely Chebyshev subspace of X, then there is an equivalent norm, $|||\cdot|||$, on X, under which M becomes an L_1-Chebyshev subspace.*

Conversely, if M is a semi-L-summand in X, and φ is a normalized absolute norm on $\mathbb{R}^2$, with $(0, 1)$ being an extreme point, and $(1, 0)$ a nonsmooth point, then X can be renormed so that M becomes a φ-Chebyshev subspace.

In both cases, $\|m\| = |||m|||$ for all $m \in M$, the metric projection is the same for both norms, and $\|x + M\| = k|||x + M|||$ for some constant k and all $x + M \in X/M$.

Proof. In the first case, define the new norm on X by $|||x||| = \|Px\| + \|x - Px\|$. Since $n \leq 1$, the identity $|||x||| = \rho(x) + (1 - n)d(x, M)$ shows that $|||\cdot|||$ is subadditive. It is easy to verify that P is also the metric projection for this new norm, with respect to which M is a semi-L-summand in X. In this case, $k = 1$.

Conversely, if M is an L_1-Chebyshev subspace in X, set $|||x||| = \varphi(\|Px\|, \frac{1}{n}\|x - Px\|)$. Let us also define ψ on $\mathbb{R}^2$ by $\psi(\alpha, \beta) = \varphi(|\alpha| - n|\beta|, \beta)$ if $|\alpha| > n|\beta|$, and $\psi(\alpha, \beta) = |\beta|$

when $|\alpha| \leq n|\beta|$. It is geometrically clear that $\{(\alpha,\beta) : \psi(\alpha,\beta) \leq 1\}$ is convex, and hence ψ is an absolute norm on $\mathbb{R}^2$. Since $|||x||| = \psi(\|x\|, \frac{1}{n}d(x,M))$, it is now easily verified that $|||\cdot|||$ is subadditive. The remaining details are easy to check, with $k = n$. Corollary 5 shows that the hypothesis $n_\varphi > 0$ is needed for this result. ♠

2. SOME NEW SEMI-L-SUMMANDS

It is ridiculously easy to construct examples of absolutely Chebyshev subspaces whose metric projections are linear. Finding examples with nonlinear metric projections is slightly more difficult. According to Theorem 6, this is tantamount to finding a semi-L-summand whose metric projection is not linear. A naturally occuring example is the subspace of constant functions, $\mathbb{R}1$, in a space of real valued continuous functions, $C_{\mathbb{R}}(K)$.

Further examples may be constructed by taking L-sums (or, with suitable care, projective tensor products). For example, if M_i are semi-L-summands in X_i $(i = 1,2)$ then $M_1 \oplus M_2$ is a semi-L-summand in $X_1 \oplus X_2$ – provided we use the L-norm on the direct sums. If P_{M_i} is nonlinear for at least one i, then so is $P_{M_1 \oplus M_2}$.

This last construction is rather elementary. It would be more satisfying to find examples which are not formed from L-sums of previous examples. This is possible. Taking the dual of any of the examples in [11] will give us a Banach space, which admits no L-projections, but which does contain a 2-dimensional semi-L-summand. However, in each of these examples, the semi-L-summand is the 2-dimensional ℓ_1 space.

This suggests that whenever M is a semi-L-summand in X, then either P_M is linear, or M contains an L-summand. We now show that nothing could be further from the truth: even with P nonlinear, the subspace M can be any real Banach space whatsoever. Furthermore, the quotient X/M can be almost any Banach space.

Let us say that Z is a semi-L-sum of X and Y if X is (isometric to a subspace which is) a semi-L-summand in Z, and Z/X is isometric to Y. When P_X is nonlinear, we will say that Z is a proper semi-L-sum of X and Y. The formation of semi-L-sums is not a commutative operation.

A map $\Omega : Y \to X$ between two Banach spaces will be called pseudolinear if it is homogeneous, and satisfies the inequality $\|\Omega(x) + \Omega(y) - \Omega(x+y)\| \leq \|x\| + \|y\| - \|x+y\|$. A proper pseudolinear map is one which is not linear.

Theorem 7. *Given real Banach spaces X and Y, the following are equivalent.*
(i) There is a proper semi-L-sum of X and Y.
(ii) The unit ball of Y^* is weak reducible, i.e. there is an asymmetric, weak* compact, convex set $S \subset Y^*$ such that $S - S$ is a closed ball.*
(iii) There is a proper semi-L-sum of $\mathbb{R}$ and Y.

(iv) There is a proper pseudolinear map $\Omega : Y \to \mathbb{R}$.
(v) There is a proper pseudolinear map $\Omega : Y \to X$.

Proof. (i)⇒(ii). We will use some duality theory, although we prefer not to give all the necessary definitions until the next section. If X is a proper semi-L-summand in Z, then X° has the 2-ball property, but not the 3-ball property, in Z^*, by [5, §6] or [16]. Then $S = X^\circ \cap B(f, d(f, X^\circ))$ satisfies $S - S = B(0, 2d(f, X^\circ))$ for every $f \in Z^*$; and S will not be a ball for some $f \in Z^*$ [7, §1]. Since X° can be identified with Y^*, (ii) is satisfied.
(ii)⇒(iii). Let $Z = A(S)$ be the space of affine real valued functions on S, continuous in the relative weak* topology. It is routine to check that $\mathbb{R}1$ is a semi-L-summand in Z, with Pf being the constant function $\frac{1}{2}(\max f(S) + \min f(S))$. Since S is not symmetric, P cannot be linear. If we identify S with the evaluation functionals in Z^*, then $co(-S \cup S)$ is the unit ball of Z^*. Then $\frac{1}{2}(S - S)$ is the unit ball of $(\mathbb{R}1)^\circ$, which is thus isometric to Y^*. This shows that Y^* is isometric and weak* isomorphic to $(Z/\mathbb{R}1)^*$. See also [6, §§2-3].
(iii)⇒(iv). Let $\mathbb{R} = \mathbb{R}e$ be a proper semi-L-summand in Z, with $Z/\mathbb{R} = Y$. Choose $f \in Z^*$ with $f(e) = 1$, and define $\Omega : Y \to \mathbb{R}$ by $\Omega(z + \mathbb{R}) = P(z) - f(z)$. A short calculation shows that Ω is well defined. For any $x + \mathbb{R}$, $y + \mathbb{R} \in Y$, we have

$$\begin{aligned}
|\Omega(x + y + \mathbb{R}) - \Omega(x + \mathbb{R}) - \Omega(y + \mathbb{R})| &= \|P(x + y) - Px - Py\| \\
&= \|x + y - Px - Py\| - \|x + y - P(x + y)\| \\
&\leq \|x - Px\| + \|y - Py\| - \|x + y - P(x + y)\| \\
&= \|x + \mathbb{R}\| + \|y + \mathbb{R}\| - \|x + y + \mathbb{R}\|.
\end{aligned}$$

Thus Ω is pseudolinear, but clearly is not linear.
(iv)⇒(v) is obvious.
(v)⇒(i) Let us take $Z = X \oplus Y$ to be the algebraic direct sum. We shall equip Z with the norm $\|(x, y)\| = \|x + \Omega(y)\| + \|y\|$. It is easy to check that this is a norm, and that $X \oplus \{0\}$ is isometric to X. However $\{0\} \oplus Y$ need not be closed in Z, if Ω is not continuous. We do not claim that X is topologically complemented in Z.

Routine calculations show that X is a **Чебышев** subspace of Z, whose metric projection is given by $P(x, y) = x + \Omega(y)$. It follows that Z/X is isometric to Y, that Z is complete, and that X is a semi-L-summand in Z. If Ω is nonlinear, so also is P. ♠

The implication (i)⇒(v) can be proved directly, without using any duality theory. If X is a proper semi-L-summand in Z, let $Q : Z \to X$ be any linear projection, not necessarily continuous. We define $\Omega : Y = Z/X \to X$ by $\Omega(z + X) = Pz - Qz$. The argument from (iii)⇒(iv) can easily be modified to show that Ω is a proper pseudolinear map.

As a special case of (v)⇒(i), we get the implication (iv)⇒(iii). Now (v)⇒(iv) follows immediately from the Hahn-Banach Theorem. Thus the equivalence of (i), (iii), (iv) and

(v) can be proved without any duality theory, using an argument that is also valid when the scalars are complex.

It is instructive to exhibit some proper pseudolinear maps on 2-dimensional Banach spaces. It is well known [2] that a 2-dimensional space satisfies (ii) of Theorem 7 if and only if its unit ball has more than 4 extreme points. The simplest such example is $Y = \mathbb{R}^2$, equipped with the hexagonal norm $\|(\alpha, \beta)\| = \max\{|\alpha|, |\beta|, |\alpha - \beta|\}$. Given any real Banach space X, choose $e, f \in X$ with $0 < \|e + f\| \leq 1$. We may then define a map $\Omega : Y \to X$ by

$$\Omega(\alpha, \beta) = \begin{cases} (\alpha - \beta)e, & \text{if } 0 < \beta \leq \alpha \text{ or } \alpha < \beta < 0 \\ (\beta - \alpha)f, & \text{if } 0 < \alpha < \beta \text{ or } \beta \leq \alpha < 0 \\ \alpha e + \beta f, & \text{if } \alpha\beta \leq 0. \end{cases}$$

It is straightforward but tedious to verify that Ω is pseudolinear. The proper semi-L-sum resulting from this map was, in fact, our first example.

Another interesting example is $Y = \mathbb{R}^2$, with the euclidean norm. The existence of sets of constant width in $Y = Y^*$ shows that its unit ball is reducible in the sense of (ii) in Theorem 7. Working backwards from one such set, the Reuleaux triangle, gives the following construction. First define a norm on $\mathbb{R}^3$ by

$$\|(\alpha, \beta, \gamma)\| = \max(\{|2\cos\theta - 1| \cdot |\alpha| + |2\sin\theta \cdot \beta + \gamma| : 0 \leq \theta \leq \pi/3\}$$
$$\bigcup\{|2\sin\theta \cdot \alpha + (\sqrt{3} - 2\cos\theta)\beta + \gamma| : -\pi/6 \leq \theta \leq \pi/6\}).$$

Then define $\Omega(\alpha, \beta)$ to be the unique real number γ which minimizes $\|(\alpha, \beta, \gamma)\|$. It can be shown that $\Omega : Y \to \mathbb{R}$ is a proper pseudolinear map.

It is well known [14] that n-dimensional cubes and octahedrons are irreducible, and so any pseudolinear map on $\ell_1(n)$ or $\ell_\infty(n)$ is automatically linear. Let us now deduce this directly from a more general result.

Proposition 8. *Suppose that the unit ball of Y contains a closed face F for which the convex hull of $-F \cup F$ is the closed unit ball. Then every pseudolinear map on Y is linear.*

Proof. Let C be the cone generated by the face F. Then for any $x \in Y$, we can find $a, b \in C$ with $x = a - b$ and $\|x\| = \|a\| + \|b\|$. Now let Ω be a pseudolinear map on Y, and fix $x, y \in Y$. Note that Ω is additive on any facial cone of Y. We can find $a_i, b_i \in C$ with $x = a_1 - b_1$, $y = a_2 - b_2$, $x + y = a_3 - b_3$ and $\|a_i - b_i\| = \|a_i\| + \|b_i\|$ for $i = 1, 2, 3$. The last condition implies that $\Omega(a_i - b_i) = \Omega(a_i) - \Omega(b_i)$ for each i. Since $a_1 + a_2 + b_3 = a_3 + b_1 + b_2$, pseudolinearity also implies that $\Omega(a_1) + \Omega(a_2) + \Omega(b_3) = \Omega(a_3) + \Omega(b_1) + \Omega(b_2)$. Thus $\Omega(x + y) = \Omega(x) + \Omega(y)$, as required. ♠

We now see that any pseudolinear map on $C(K)$, $L_\infty(\mu)$ or $L_1(\mu)$ is automatically linear.

Corollary 9. *Let Y be any real Banach space, e an extreme point of its unit ball with the property that every maximal face contains either e or $-e$. Then balls in Y^{**} are weak* irreducible.*

Proof. Let $F = \{f \in Y^* : \|f\| = f(e) = 1\}$. Standard duality arguments (e.g. [6] or [11]) then show that $co(-F \cup F) = B(0,1)$ in Y^*. The conclusion now follows from Proposition 8 and Theorem 7. ♠

Even in the finite dimensional case, Corollary 9 gives us examples of irreducible convex bodies which are not covered by the results of [14]. For instance, the convex hull of the twelve points $(\pm5, \pm5, 0)$, $\pm(\pm4, 8, -1)$, $\pm(0, 10, 0)$ and $\pm(0, 0, 5)$ is an irreducible set in $\mathbb{R}^3$, even though none of its 2-faces are triangles. (Here $(0, 0, 5)$ plays the role of e.)

Recall that $L_1(0,1)$ is isometric to a quotient, but not isomorphic to any subspace, of ℓ_1. The kernel of the quotient map is thus an uncomplemented subspace which cannot be made into a semi-L-summand, even if we renorm ℓ_1 in the manner of Theorem 6.

Given a pseudolinear map $\Omega : Y \to X$, we have seen how to construct a semi-L-sum, Z, of X and Y. If Ω is not continuous, then $\{0\} \oplus Y$ will not be closed in Z, and will not be isomorphic to Y. This suggests that X might not be complemented in Z, but discontinuity of Ω is not sufficient to guarantee that. Examples with Ω discontinuous can easily be constructed by replacing Ω with $\Omega - T$, where $T : Y \to X$ is any discontinuous linear map. But if X is complemented in X^{**} then, by Corollary 4 and [8, Corollary 2] or [12, p62], X will be complemented in Z. This all seems a bit weird at first, but the next result clarifies things.

Proposition 10. *Let $\Omega : Y \to X$ be pseudolinear, and Z the corresponding semi-L-sum. Then X is complemented in Z if, and only if, $\Omega = T + A$ where $T : Y \to X$ is a linear map and $A : Y \to X$ is continuous.*

Proof. ($\Rightarrow$) Given a continuous linear projection $Q : Z \to X$, define $A, T : Y \to X$ by $Ty = Q(0, y)$ and $Ay = \Omega y - Ty$. Clearly T is linear, and $\Omega = A + T$. For any $y \in Y$,

$$\begin{aligned} \|Ay\| = \|\Omega y - Ty\| &= \|(-Ty, y)\| - \|y\| \\ &= \|(I - Q)(-\Omega y, y)\| - \|y\| \\ &\leq \|I - Q\| \cdot \|(-\Omega y, y)\| - \|y\| \\ &= (\|I - Q\| - 1)\|y\|. \end{aligned}$$

Thus A is continuous at 0 and so, by pseudolinearity, continuous everywhere.

($\Leftarrow$) Given T and A as above, let Y_0 be the subspace $\{(-Ty, y) : y \in Y\}$ of Z. Since $\|(-Ty, y)\| = \|Ay\| + \|y\|$, we see that Y_0 is naturally isomorphic to Y and hence closed in Z. The identity $(x, y) = (-Ty, y) + (x + Ty, 0)$ shows that Y_0 is complementary to X. ♠

Thus any pseudolinear map on a finite dimensional space is automatically continuous.

Ribe [13] and Kalton [3] studied quasilinear maps between various spaces. These are maps $\Omega : Y \to X$ satisfying an inequality $\|\Omega(x) + \Omega(y) - \Omega(x+y)\| \le K(\|x\| + \|y\|)$, for some constant K. These were originally introduced to produce twisted sums of certain quasinormed spaces, and later [4] of Banach spaces. Recall that a quasinorm on a vector space is a real valued function, $\|\cdot\|$, satisfying the usual positivity and homogeneity axioms, but only a weak triangle inequality $\|x+y\| \le K(\|x\|+\|y\|)$, where K is a constant. To say that Z is a nontrivial twisted sum of X and Y means that Z contains an uncomplemented subspace isomorphic to X, the quotient by which is isomorphic to Y. The proof of (v)$\Rightarrow$(i) in Theorem 7 is rather similar to the construction in [13, Lemma 1] and [4, p5]. That construction gives only a quasinorm, not necessarily a norm, on Z, even when X and Y are both Banach spaces. Under additional hypotheses [3, Theorem 2.6], it sometimes happens that Z is isomorphic to a Banach space.

It is natural to ask what strengthening of quasilinearity will ensure that the triangle inequality holds in Z, without any need for renorming. Pseudolinearity will certainly achieve this, but it does much more besides. Moving from quasilinear to pseudolinear maps changes the problem from that of finding twisted sums (an isomorphic problem) to that of finding proper semi-L-sums (an isometric problem). Finding a proper semi-L-sum of two Banach spaces is a nontrivial exercise even in finite dimensions, whereas there are obviously no nontrivial finite dimensional twisted sums. There are also, as we have seen, uncomplemented subspaces which cannot be made into semi-L-summands.

Proposition 10 is the analogue of [3, Proposition 3.3], but here the difference between the two theories becomes more apparent. Much of [3, §4] is devoted to showing there is a quasilinear map from ℓ_1 to $\mathbb{R}$ which is not "close" to a linear map. Similar maps on ℓ_1 are then used as the building blocks for other examples of quasilinear maps. On the other hand, Proposition 8 shows that every pseudolinear map on ℓ_1 is linear. We have used the terminology proper semi-L-sum rather than twisted semi-L-sum, because a proper semi-L-summand may, as we have seen, be complemented. Despite the expenditure of much elbow grease, we have not yet been able to find an uncomplemented semi-L-summand.

Combining Theorems 7 and 6 returns us to absolutely Chebyshev subspaces.

Proposition 11. *Let X, Y be real Banach spaces, such that the unit ball of Y^* is weak* reducible. Let φ be a normalized absolute norm on $\mathbb{R}^2$, with respect to which $(0,1)$ is an extreme point of the unit ball, and $(1,0)$ is not a smooth point. Then there is a Banach space Z containing X in such a way that X is a φ-Chebyshev subspace, P_X is not linear, and Z/X is isometric to Y.*

3. DUALITY AND COMPLEX BANACH SPACES

It is curious that no example is known of a proper semi-L-summand in a complex Banach space. Before discussing this problem, it is pertinent to introduce some duality theory.

Recall that M is said to have the n-ball property in X (where $n \in \mathbb{N}$) if, whenever $B_1, ..., B_n$ are open balls in X, with $M \cap B_i$ nonempty for each i, and $\bigcap_{i=1}^n B_i$ nonempty, then $M \cap \bigcap_{i=1}^n B_i$ is nonempty. It is known that the 3-ball property implies the n-ball property for all n, and that any L_∞-summand has these properties. It turns out that the 2-ball property is dual to being a semi-L-summand : that is, a subspace of a Banach space has one of these properties if and only if its polar has the other property in the dual space. Likewise, the 3-ball property and being an L-summand are dual properties. Proofs of these results may be found in [5, §§5-6] or [16].

Examples in real Banach spaces of subspaces with the 2-ball property but not the 3-ball property are well known. (By Corollary 5, no such subspace can be the range of a norm one projection.) Whether the 2-ball property implies the 3-ball property in complex Banach spaces is a long standing open question. One possible approach to this problem might be to show that whenever M has the 2-ball property, but not the 3-ball property, in a real Banach space X, then X/M is not isometric to any complex Banach space. This is the case in every hitherto known example. The following result shows that this approach does not work.

Theorem 12. *Given real Banach spaces M and Y, with M being finite dimensional, the following are equivalent.*
(i) There is a Banach space X containing M, such that M has the 2-ball property but not the 3-ball property in X, and X/M is isometric to Y.
(ii) There is a closed convex asymmetric set S in M, for which $S - S$ is a closed ball.

Proof. (i)$\Rightarrow$(ii) is clear from the results of [7, §1]. This part of the proof does not require finite dimensionality of M.
(ii)$\Rightarrow$(i). Theorem 7 shows that there is a proper semi-L-sum, say Z, of Y^* and M^*. Finite dimensionality of M ensures that $\Omega : M^* \to \mathbb{R} \subset Y^*$ is continuous for both the norm and weak* topologies, and that $Z = Y^* \oplus M^*$ is a topological direct sum. Giving Z the weak* topology induced from Y^* and M^*, we can check that its norm is weak* lower semicontinuous, and that $Y^* \oplus \{0\}$ is weak* closed. Let X be the predual of Z (for this weak* topology) and let M_1 be the polar of $Y^* \oplus \{0\}$ in X. Routine arguments now show that M_1^* is isometric and weak* isomorphic to $X^*/(Y^* \oplus \{0\})$ and hence to M^*; that M_1 has the 2-ball property but not the 3-ball property in X; and that $(X/M_1)^*$ is isometric and weak* isomorphic to Y^*. ♠

Theorem 12, together with the existence of sets of constant width in euclidean $\mathbb{R}^n$, shows that M having the 2-ball property but not the 3-ball property in X gives us no information about X/M. It is cumbersome to state a version of Theorem 12 which holds when M is not finite dimensional. The existence of a weak* continuous proper pseudolinear map $\Omega : M^* \to \mathbb{R}$ is a sufficient condition for (i), but it is not necessarily implied by (ii). To see this, let $S = \{x \in \ell_2 : \|x\| \leq 1 \text{ and } x_n \geq 0 \text{ for all } n\}$, and let M be ℓ_2 equipped with the norm whose unit ball is $S - S$. Clearly M is reflexive, S is weakly compact, and (ii) is satisfied. Let (e_n) be the usual basis for ℓ_2, and (f_n) the corresponding coefficient functionals, considered as elements of $A(S)$. Then $P(f_n) = \frac{1}{2}$ for all n, even though $f_n \to 0$ weak*. Thus $P : A(S) \to \mathbb{R}1$ is not weak* continuous.

It has long been known that the problem "does the 2-ball property imply the 3-ball property in complex Banach spaces?" is equivalent to an easily stated problem concerning 2-dimensional convex sets. One proof of this can be found in the final section of [17]. Combining this with some of the results above gives our last result.

Proposition 13. *The following conjectures are equivalent.*

C1. *In complex Banach spaces, every subspace with the 2-ball property already has the 3-ball property.*

C2. *Whenever $S \subset \mathbb{C}^2$ is a compact convex set, such that $f(S)$ is a disc for all linear $f : \mathbb{C}^2 \to \mathbb{C}$, then S is symmetric.*

C3. *Every pseudolinear map between two complex Banach spaces is already linear.*

C4. *Whenever M is an absolutely Chebyshev subspace of a complex Banach space, then P_M is linear.*

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