SPACES OF LARGE DIMENSION; SOME COUNTER-INTUITIVE RESULTS

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ABSTRACT

In recent years we have seen the important development in functional analysis of the study called "geometric analysis" which concerns asymptotic properties of sets in finite dimensional spaces where the dimension increases to infinity. The significance of this development is illustrated in four example areas where the author has been a contributor.

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1. INTRODUCTION

In this note we use the following standard notation. Let K be a complex symmetric compact body in $\mathbb{R}^n$ which we equip with the standard Euclidean inner product $(\cdot,\cdot)$, with norm $|x|^2 = (x,x)$ and Lebesgue volume. Let $X = X_K = (\mathbb{R}^n, \|\cdot\|_K)$ be a normed space such that the unit ball of X is K. Denote also by D the unit Euclidean ball (D_n, if we need to emphasize its dimension), the polar $K^\circ = \{x \in \mathbb{R}^n \mid (x,y) \leq 1 \text{ for all } y \in K\}$ and $X^* = (\mathbb{R}^n, \|\cdot\|_{K^\circ})$. In addition we often denote by sX a subspace of X, qY a quotient space of a space Y and $d_X \equiv d(X, l_2^{\dim X})$ the Banach-Mazur (multiplicative) distance between X and $l_2^{\dim X}$, that is, $d_X = \inf\{\|T\|.\|T^{-1}\| \mid T : X \to l_2^{\dim X} \text{ is a linear isomorphism}\}$. For convex bodies K and $T \subset \mathbb{R}^n$ we define $d(K,T) = \inf\{a, b \mid K \subset aT \subset abK\}$ and $d_K = d(K,D)$.

We study asymptotic properties of sets K and spaces X when n increases to infinity. This area of Functional Analysis is often known as the Local Theory of Normed Spaces and has been one of the most rapidly developing areas of Functional Analysis during the last decade. In this short note I would like to explain why this is no accidental development. Mathematicians in the past did not pay much attention to high dimensional spaces as such. At the beginning of this century, geometry (and our interest is in convex geometry) was mainly concerned with two and three dimensional study. Of course, some results were automatically extended to n-dimensional spaces but preserved (being isometrical) their low dimensional spirit. When it was realized that a study of high dimensional spaces was very important it was approached by infinite dimensional "approximation". And so infinite dimensional Functional Analysis flourished. After a number of decades of incredible success with Functional Analysis it had been completely overlooked that this development does not in fact give any knowledge about high dimensional normed spaces. An infinite dimensional normed space is usually a bad (and even wrong) approximation for a high dimensional space (which we could call an asymptotically infinite dimensional space). Later some other attempts at approximation were made, such as nuclear spaces and so on. They were good from some points of view but not at all satisfactory from others.

At the present time we have at least realized that asymptotic finite dimensional theory exists and is different from both its roots: low dimensional convex geometry and infinite dimensional Functional Analysis. We have begun to call it "geometric analysis" because it has both aspects as we will see.

This note is not intended to give any detailed account of the theory. What I will try to show is that "geometric analysis" develops a new intuition which we were lacking. Indeed, as we will see, generally in the past we did not have a good high dimensional intuition.

2. INTRODUCTION TO CONCENTRATION PHENOMENON AND THE CONCEPT OF A SPECTRUM OF A CONTINUOUS FUNCTION

Consider the classical isoperimetric problem on $S^n = \partial D_{n+1}$ equipped with the probability rotation-invariant measure μ and the geodesic distance ρ. Let $(*)_\epsilon \subset S^n$ be a cap of radius ϵ. Then for any Borel subset $A \subset S^n$, $\mu(A) = 1/2$, and $A_\epsilon = \{x \in S^n \mid \rho(x, A) \leq \epsilon\}$

$$\mu(A_\epsilon) \geq \mu((*)_{\frac{\pi}{2}+\epsilon}) = \int_0^{\frac{\pi}{2}+\epsilon} (\sin\theta)^{n-1}\, d\theta \Big/ \int_0^{\pi} (\sin\theta)^{n-1}\, d\theta$$
$$\geq 1 - \sqrt{\tfrac{\pi}{8}} e^{-\epsilon^2 n/2} \quad \to 1 \text{ if } n \to \infty \quad (\text{for any fixed } \epsilon > 0)\,.$$

This was an observation of P. Lévy dating from 1922 [13]. He used it to derive the following corollary.

Corollary 2.1 (P. Lévy). *Let $f(x) \in C(S^n)$ be a continuous function on S^n with the modulus of continuity $\omega_f(\epsilon)$. Let L_f define the median (Lévy mean) of $f(x)$, that is,*

$$\mu\{x \in S^n : f(x) \geq L_f\} \geq \tfrac{1}{2} \quad \text{and} \quad \mu\{x \in S^n : f(x) \leq L_f\} \geq \tfrac{1}{2}\,.$$

Then

$$\mu\{x \in S^n : |f(x) - L_f| > \omega_f(\epsilon)\} < \sqrt{\tfrac{\pi}{8}} \exp(-\epsilon^2 n/2)\,.$$

We consider now an abstract setting.

Let (X, ρ, μ) be a metric compact set with a metric ρ, $\operatorname{diam} X \geq 1$, and a probability measure μ. Define the concentration function $\alpha(X; \epsilon)$ of X by

$$\alpha(X;\epsilon) = 1 - \inf\{\mu(A_\epsilon) \mid A \text{ be a Borel subset of } X,\ \ \mu(A) \geq \tfrac{1}{2}\}$$

(here $A_\epsilon = \{x \in X \mid \rho(s, A) \leq \epsilon\}$). The above observation of P. Lévy implies

Example 2.2: Let S^n be the Euclidean sphere equipped with the geodesic distance ρ and the rotation-invariant probability measure μ_n. Then

$$\alpha(S^{n+1}; \epsilon) \leq \sqrt{\pi/8} \exp(-\epsilon^2 n/2) \to 0 \quad \text{for} \quad n \to \infty$$

for any fixed $\epsilon > 0$.

Following this example, we call a family (X_n, ρ_n, μ_n) of metric probability spaces a *Lévy family* ([9]) if, for any $\epsilon > 0$, $\alpha(X_n, \epsilon \cdot \operatorname{diam} X_n) \to 0$ for $n \to \infty$, and a *normal Lévy family* [2] with constant $(c_1; c_2)$ if,

$$\alpha(X_n; \epsilon) \leq c_1 \exp(-c_2 \epsilon^2 n) .$$

(When the factor $\operatorname{diam} X_n$ is omitted most of the examples below become normal Lévy families with their natural metric and natural enumeration.)

Let $f \in C(X)$ be a continuous function on a space X with the modulus of continuity $\omega_f(\epsilon)$. As in Corollary 1, define a median L_f (also called a *Lévy mean*) as being a number such that $\mu\{x \in X : f(x) \geq L_f\} \geq \frac{1}{2}$ and $\mu\{x \in X : f(x) \leq L_f\} \geq \frac{1}{2}$. Then $\mu(x : |f(x) - L_f| \leq \omega_f(\epsilon)\} \geq 1 - 2\alpha(X, \epsilon)$. This means that if $\alpha(X, \epsilon)$ is small, then "most" of the measure of X is concentrated "around" one value of $f(x)$.

In fact, the concept of a Lévy family (and especially a normal Lévy family) generalizes the concept behind the law of large numbers in two directions: a) the measures are not necessarily the product of measures (that is, no condition of "independence") and b) any Lipschitz function on the space is considered instead of linear functionals only.

During the last 10–15 years, many new examples of Lévy families have been discovered and different techniques of estimating the concentration function have been developed (see a recent survey [16]). We mention here only three more such examples.

Example 2.3: The family of orthogonal groups $\{S0(n)\}_{n \in N}$ equipped with the Riemannian metric ρ (which is equivalent up to $\pi/2$ to the Hilbert-Schmidt operator metric) and the normalized Haar measure μ_n:

$$\alpha(S0(n); \epsilon) \leq \sqrt{\tfrac{\pi}{8}} \exp(-\epsilon^2 n/8) .$$

(This follows from Gromov's isoperimetric inequality [8], see [9].)

Example 2.4: If $F_2^n = \{-1,1\}^n$ has the normalized Hamming metric

$$d(s,t) = \tfrac{1}{n}\big|\{i, s_i \neq t_i\}\big|$$

and the normalized counting measure μ, that is, $\mu(A) = |A|/2^n$, then

$$\alpha(F_2^n;\epsilon) \leq \tfrac{1}{2}\exp(-2\epsilon^2 n) \, .$$

(This follows from the Harper isoperimetric result [10]; see in such form [1].)

Example 2.5: The group $\prod_n$ of permutations of $\{1,\ldots,n\}$ with the normalized Hamming metric

$$d(\pi_1,\pi_2) = \tfrac{1}{n}\big|\{i : \pi_1(i) \neq \pi_2(i)\}\big|$$

and the normalized counting measure:

$$\alpha\Big(\prod_n;\epsilon\Big) \leq \exp(-\epsilon^2 n/64)$$

(B. Maurey [14]).

The concentration phenomenon is often used in a study of spaces of large dimension through a concept of the "spectrum" of a function with a small local oscillation. We outline this concept in the following example which was the original result in this direction (see [15]).

Theorem 2.6. *There exists a universal constant $c > 0$ such that, for every integer n and $k = [c\epsilon^2 n/\log 1/\epsilon]$ and any continuous function $f \in C(S^n)$, there exists a k-dimensional subspace E such that, for any x and $y \in S^{k-1} = E_k \cap S^n$*

$$|f(x) - f(y)| < \omega_f(2\epsilon) \, ,$$

where $\omega_f(\epsilon)$ is the modulus of continuity of $f(x)$.

Remarks. 1. Recently, Y. Gordon [7] removed the $\log 1/\epsilon$ factor in the above formula for k.

2. Using this theorem, we choose a function f in such a way that $f =$ Const. means a given geometric property. Then, by the theorem, we find subspaces of large dimension where this property is "almost" satisfied. (See [18] for a number of such applications.)

The estimate on dimension k in the above theorem is important and leads to numerous geometric results. As an example, we mention the famous Dvoretsky theorem [5] about almost Euclidean sections of a convex symmetric body in $\mathbb{R}^n$ (see [15]).

3. APPROXIMATION BY MINKOWSKI SUMS

Let $A + B = \{x + y \mid x \in A, y \in B\}$ be the Minkowski sum of two sets A and B in $\mathbb{R}^n$. Let $I_i = [-x_i, x_i] \subset \mathbb{R}^n$ be intervals of length, say, 1. Consider $T = \sum_{i=1}^{N} I_i$. We want to approximate a Euclidean ball by such sums, that is, for a given $\epsilon > 0$ we would like to have $d(T, D) \leq 1 + \epsilon$. Obviously, if $N = n$ then $d(T, D) \geq \sqrt{n}$ and, by an entropy consideration, it looks as if we need at least an exponential by n of a number of intervals to achieve a good approximation to D. However, as was observed in [17], an easy geometric interpretation of the same old result from [6] shows that there exist intervals $I_i \subset \mathbb{R}^n$, $i = 1, \ldots, N_0$, for $N_0 \leq c\frac{n}{\epsilon^2} \log \frac{1}{\epsilon}$ (c is a numerical constant) such that $d(\sum_1^{N_0} I_i, D) \leq 1 + \epsilon$.

This direction was recently intensively treated in [3] and [4]. It is shown there that the above situation is essentially preserved when we substitute intervals by other convex bodies or if we consider approximation by sums of other convex bodies instead of D. For example

Theorem [4]. *Let a convex compact body $K \subset \mathbb{R}^n$ be given. There exist orthogonal operators $A_i \in S0_n$, $i = 1, \ldots, N_0$, for $N_0 \leq c\frac{n}{\epsilon^2} \log \frac{1}{\epsilon}$ such that $T = \frac{1}{N_0} \sum_1^{N_0} A_i K$ satisfies $d(T, D) \leq 1 + \epsilon$ (as usual, c is a numerical constant).*

Moreover, if D is the ellipsoid of maximal volume of symmetric K then we need in fact only $N_0 \sim \frac{n}{\log n} \cdot \frac{\log \frac{1}{\epsilon}}{\epsilon^2}$ rotations to achieve an ϵ-approximation of the Euclidean ball.

The above result may be strengthened further for some special convex bodies. For example, the dual form of a result of Kašin [11] gives us that for the cube $C^n = [-1,1]^n \subset \mathbb{R}^n$ there exists a rotation $A \in SO_n$ such that

$$d(C^n + AC^n, D) \le C$$

(that is, uniformly bounded; C is a universal constant).

In fact we show in [4] that for every $K \subset \mathbb{R}^n$ such that the polar K° is the unit ball of cotype 2 space X, for every $\epsilon > 0$ we may find a finite number $p(\epsilon)$ of rotations $A_i \in SO_n$ (where $p(\epsilon)$ depends on $\epsilon > 0$ and cotype 2 constant $C_2(X)$ but not on n) such that $d(\sum_{i=1}^{p(\epsilon)} A_i K, D) \le 1 + \epsilon$.

4. GEOMETRIC FORM OF THE QUOTIENT OF A SUBSPACE THEOREM

Using the language of Minkowski sums we may give an easily visualized version of the so called QS-Theorem from [19] which in its analytic form states

Theorem 4.1. *([19], see [17] in this form). For any space $X = (\mathbb{R}^n \| \cdot \|)$ and any $\frac{1}{2} < \lambda < 1$, there exists qsX — a quotient of a subspace of X — such that $\dim qsX \ge \lambda n$ and $d_{qsX} \le C(1-\lambda)^{-1} \log(1-\lambda)^{-1}$. If $0 < \lambda < \frac{1}{2}$ then there exists $Y = qsX$ such that $\dim Y \ge \lambda n$ and $d_Y \le 1 + C\sqrt{\lambda}$.*

(In fact, I knew of an estimate $d_Y \le 1 + c\sqrt{\lambda \log \frac{1}{\lambda}}$ and the log factor was removed in [7] as I have already noted after Theorem 2.6.)

The following recent geometric form of this theorem better corresponds to the purpose of this note.

Theorem 4.2. *([20]). Let K be any convex symmetric compact body in $\mathbb{R}^n$. Then there exist a linear operator $u : \mathbb{R}^n \to \mathbb{R}^n$, $\det u = 1$, and two orthogonal operators*

$A_1, A_2 \in O(n)$ such that if

$$T = uK + A_1 uK$$

then the distance

$$d(T^0 + A_2 T^0, D) \leq C$$

where C is a universal constant (independent of K and n).

Also the following ϵ-version is true: for every $\epsilon > 0$ there exists an integer $p(\epsilon)$, independent of n or K, such that for some operators $A_i \in O(n)$, $i = 2, \ldots, p(\epsilon)$,

$$d\left(\sum_{i=2}^{p(\epsilon)} A_i T^0, D\right) \leq 1 + \epsilon \, .$$

5. ENTROPY AND COVERING NUMBERS

Recall that the covering number

$$N(K,T) = \inf\left\{N \mid \text{there exists an } x_i \quad \text{and} \quad K \subset \bigcup_1^N (x_i + T)\right\} ,$$

that is, the minimal number of shifts of $T \subset \mathbb{R}^n$ needed to cover $K \subset \mathbb{R}^n$. If $d = d(K,T)$ then it is expected that either $N(K,T)$ or $N(T,K)$ would be at least D^{cn} (for some universal $c > 0$).

However, for every $K \subset \mathbb{R}^n$ there exists a linear transform $u_K \in SL_n$ such that for $\hat{K} = u_K K$ the situation is different.

We call *a position* of K any affine image uK for $u \in GL_n$. Clearly, every position of K produces the unit ball of isometrically the same normed space as X_K. It is an interesting feature of the (asymptotic) high dimensional theory of convex sets that we are, in fact, forced to consider the family of all positions of a given K (that is, all affine images uK, $u \in GL_n$) even when we are aiming at some volume inequalities or other properties of an individual K.

Theorem 5.1. *([20]). For every convex compact body $K \in \mathbb{R}^n$ there corresponds a volume preserving position (called "canonical") $\hat{K} = u_K K$ ($u_K \in SL_n$) such that for any two bodies K and T,* vol. $K =$ vol. T,

$$N(\hat{K}, \hat{T}) \le C^n \quad \text{and} \quad N(\hat{T}, \hat{K}) \le C^n ,$$

where, as usual, C is a numerical constant.

Note that it is easy to give an example of such K and T with $d(X_K, X_T) \ge \sqrt{n}$. Therefore, $d(\hat{K}, \hat{T}) \ge \sqrt{n}$ for such K and T.

The next fact is the volume interpretation of the above theorem. It shows that, from the point of view of *volume ratio*, any K behaves as a suitable ellipsoid. Or precisely,

Theorem 5.2. *(See [21], [24].) For any convex compact body K there exists an ellipsoid $\mathcal{M}_K$ such that* vol. $K =$ vol. $\mathcal{M}_K$ *and for any other convex body T*

$$\tfrac{1}{C^n}\text{vol.}\,(\mathcal{M}_K + T) \le \text{vol.}\,(K + T) \le C^n\text{vol.}\,(\mathcal{M}_K + T)$$

(C is a universal constant).

Remark 1. In the above theorem the family of ellipsoids does not play a special role. We could take any fixed body P and replace the ellipsoid $\mathcal{M}_K$ by some affine image P_K(-position) of P.

Remark 2. I like to emphasize that G. Pisier has simplified the original proofs of the results of this section which originated in [22]. His approach is a mixture of *volume* and *entropy* points of view which unexpectedly becomes much simpler than steadily pursuing the fixed line of volume (or entropy) inequalities (see [23], [24]).

The last result which we will mention in this section does not require any position.

Theorem 5.3. *([12]). There exist universal constants $c > 0$ and C such that for any $n \in \mathbb{N}$ and any convex symmetric compact bodies K and T in $\mathbb{R}^n$*

$$c \leq \left(\frac{N(K,T)}{N(T^0, K^0)} \right)^{1/n} \leq C \, .$$

So we conclude this list of results (or examples of results) which, from our point of view, do not fit exactly with our standard intuition. I feel this to be a very preliminary note on the subject, and it has not included other important examples such as a study of Khinchine type inequalities, symmetrizations and many others. However, I hope that the results presented will bring some understanding of those new areas where the dimension of the space is a parameter of study.

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