A non-commutative joint spectral theory

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ABSTRACT

For certain m-tuples $a = (a_1,...,a_m)$ of elements a_j in a unital Banach algebra, we construct a joint spectrum $\gamma(a)$ and a functional calculus with a spectral mapping theorem. It is not assumed that the a_j commute but rather that they commute modulo the Jacobson radical of the algebra they generate. For matrices, this last condition is equivalent to their being simultaneously triangularizable. This work extends that of M.E. Taylor, R.F.V. Anderson, and A. McIntosh and A. Pryde.

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1. INTRODUCTION

Classical spectral theory relates to the spectrum of a single operator on a Banach space or, more generally, of a single element in a Banach algebra. There have been a number of successful attempts to extend this theory to a joint spectral theory for m-tuples $(a_1,...,a_m)$ of commuting operators or elements in a Banach algebra.

For example, the joint spectrum and analytic functional calculus of J.L. Taylor [8], [9] have become an essential part of spectral theory. For a description of the analytic functional calculus for m-tuples in a commutative Banach algebra, see Bonsall and Duncan [2] or Vasilescu [11].

By limiting the class of commuting m-tuples to those of type s (for the definition, see section 3 below), McIntosh and Pryde [4] developed a much richer functional calculus. Their results were stated for bounded linear operators on a Banach space but the proofs are equally valid for commuting m-tuples in a Banach algebra.

M.E. Taylor [10] and Anderson [1] constructed a functional calculus for (non-commuting) bounded self-adjoint operators on a Hilbert space or, more generally, on a Banach space. In contrast to the functional calculi mentioned previously, this one is not multiplicative and does not have a spectral mapping theorem (unless the a_j commute). This last fact will be demonstrated in example 4.3 below.

In this paper we describe a joint spectrum $\gamma(a)$ and a functional calculus with a spectral mapping theorem for certain m-tuples $a = (a_1,\dots,a_m)$ in a Banach algebra. Complete details will appear elsewhere.

Throughout, $\mathcal{B}$ is a Banach algebra with unit e and $a = (a_1,\dots,a_m) \in \mathcal{B}^m$. Also $\mathcal{A} = \mathrm{alg}(e,a_1,\dots,a_m)$ is the closed unital subalgebra of $\mathcal{B}$ generated by $a_1,\dots,a_m$.

Let $\mathcal{C}$ be any closed unital subalgebra of $\mathcal{B}$. The *spectrum* of an element x in $\mathcal{C}$ is denoted $\sigma_{\mathcal{C}}(x)$ or sometimes $\sigma(x)$. The group of invertible elements in $\mathcal{C}$ is denoted $G(\mathcal{C})$. An element q of $\mathcal{C}$ is called *quasinilpotent* if $\sigma_{\mathcal{C}}(q) = \{0\}$. The *Jacobson radical* of $\mathcal{C}$ is $\mathrm{rad}\,\mathcal{C} = \{q \in \mathcal{C} : qx$ is quasinilpotent for all $x \in \mathcal{C}\}$.

For a discussion of the Jacobson radical and other concepts, see Bonsall and Duncan [2]. In particular $\mathrm{rad}\,\mathcal{C}$ is a closed two-sided ideal of $\mathcal{C}$ and $\sigma_{\mathcal{C}}(x) = \sigma_{\mathcal{B}}(x)$ if $\sigma_{\mathcal{B}}(x) \subseteq \mathbb{R}$.

2. THE JOINT SPECTRUM

Following McIntosh and Pryde [4] we define a *spectral set* $\gamma(a)$ as follows. For any closed unital subalgebra $\mathcal{C}$ of $\mathcal{B}$ let $\gamma_{\mathcal{C}}(a) = \{\lambda \in \mathbb{R}^m : \sum_1^n (a_j-\lambda_j e)^2 \notin G(\mathcal{C})\}$. Then set $\gamma(a) = \gamma_{\mathcal{A}}(a)$.

Previously, this set $\gamma(a)$ was only defined for commuting m-tuples $a = (a_1,\dots,a_m)$ with $a_j \in \mathcal{B}(X)$ the space of bounded linear operators on the Banach space X. In that case, if also $\sigma_{\mathcal{B}}(a_j) \subseteq \mathbb{R}$ for each j, where $\mathcal{B} = \mathcal{B}(X)$, then $\gamma_{\mathcal{A}}(a) = \gamma_{\mathcal{B}}(a) = Sp(a)$. The second equality was proved in McIntosh, Pryde and Ricker [5], along with equality with various other joint spectra. The first equality follows from Taylor's spectral mapping theorem [9]. Ricker and Schep [7] have proved that for commuting operators with arbitrary spectrum, $\gamma_{\mathcal{B}(X)}(a)$ is non-empty whenever $m \geq 2$.

We will see that for certain m-tuples $a = (a_1,\dots,a_m) \in \mathcal{B}^m$ with $\sigma_{\mathcal{B}}(a_j) \subseteq \mathbb{R}$, the set $\gamma(a)$ gives an adequate notion of *joint spectrum*. In that case $r(a) = \sup\{|\lambda| : \lambda \in \gamma(a)\}$ will be called the *joint spectral radius* of a.

3. THE FUNCTIONAL CALCULUS

Following McIntosh and Pryde [4], we say the m-tuple $a = (a_1,\dots,a_m) \in \mathcal{B}^m$ is of *type s*, where $s \geq 0$, if there is a constant M such that

(3.1) $\|e^{i\langle a,\xi\rangle}\| \leq M(1+|\xi|)^s$ for all $\xi \in \mathbb{R}^m$.

Of course $\langle a,\xi\rangle = \sum_1^m a_j\xi_j \in \mathcal{A}$ and $e^{i\langle a,\xi\rangle} \in \mathcal{A}$. It follows that

(3.2) $\sigma_{\mathcal{A}}(\langle a,\xi\rangle) = \sigma_{\mathcal{B}}(\langle a,\xi\rangle) \subseteq \mathbb{R}$ for all $\xi \in \mathbb{R}^m$.

Further, we shall say that $a = (a_1,\dots,a_m)$ is of *type* (s,r), where $s \geq 0$ and $r > 0$, if there is a constant M such that

(3.3) $\qquad \|e^{i\langle a,\xi+i\eta\rangle}\| \leq M(1+|\xi|)^s e^{r|\eta|}$ for all $\xi,\eta \in \mathbb{R}^m$.

The space $\mathcal{F} = \check{L}_1^s(\mathbb{R}^m)$ is discussed in [4]. It consists of the set of inverse Fourier transforms $f = \check{g}$ of the functions $g : \mathbb{R}^m \to \mathbb{C}$ for which $(1+|\xi|)^s g \in L_1(\mathbb{R}^m)$. We shall write $\hat{f}$ for g. The Fourier inversion formula we are using is

$$f(\lambda) = (2\pi)^{-m}\int e^{i\langle\lambda,\xi\rangle} g(\xi)d\xi,\ \lambda \in \mathbb{R}^m.$$

It follows that $\mathcal{F}$ is a Banach algebra with respect to pointwise addition and multiplication and with norm

$$\|f\| = (2\pi)^{-m}\|(1+|\xi|)^s\hat{f}\|_{L_1}.$$

Moreover, the space $C_c^\infty(\mathbb{R}^m)$ of infinitely differentiable functions on $\mathbb{R}^m$ with compact support is dense in $\mathcal{F}$.

For m-tuples $a = (a_1,\dots,a_m)$ of type s we define a linear map $\Phi_a : \mathcal{F} \to \mathcal{A}$ by

$$\Phi_a(f) = (2\pi)^{-m}\int e^{i\langle a,\xi\rangle}\hat{f}(\xi)d\xi.$$

This integral exists as a Bochner integral because the integrand is a strongly measurable $\mathcal{A}$-valued function of ξ and the integral is absolutely convergent by (3.1). Moreover, $\|\Phi_a(f)\| \leq M\|f\|$.

We define the *support* $\operatorname{supp}\Phi_a$ of Φ_a to be the smallest closed set K in $\mathbb{R}^m$ such that $\Phi_a(f) = 0$ for all $f \in \mathcal{F}$ with compact support disjoint from K. Using a partition of unity argument, it is easy to see that the intersection of all closed sets K with the given property also has that property. Hence there is a smallest K.

The argument in [4, lemma 8.4] based on the Paley-Wiener theorem can be repeated to prove

THEOREM 3.4. *If* $a = (a_1,\dots,a_m)$ *is an m-tuple in* $\mathcal{B}$ *of type* (s,r) *then* Φ_a *has compact support. In fact* $\operatorname{supp} \Phi_a \subseteq \{\lambda \in \mathbb{R}^m : |\lambda| \le r\}$.

As mentioned above, Taylor [10] and Anderson [1] considered the case of m-tuples $a = (a_1,\dots,a_m)$ of bounded self-adjoint operators. That is to say, $e^{i\langle a,\xi\rangle}$ is an isometry for each $\xi \in \mathbb{R}^m$. So a is of type 0. In fact, using the Trotter product formula, Taylor proved that such an m-tuple is of type $(0,\|a\|)$ where $\|a\| = (\sum_1^m \|a_j\|^2)^{1/2}$. The arguments used in [10] and [1] may be used to prove

THEOREM 3.5. *Let* $a = (a_1,\dots,a_m)$ *be an m-tuple in* $\mathcal{B}$ *of type* s *such that* Φ_a *has compact support. Let* $\theta \in C_c^\infty(\mathbb{R}^m)$ *be identically* 1 *on a neighbourhood of* $\operatorname{supp} \Phi_a$.

(a) *If* $p(\lambda_j)$ *is a polynomial in* λ_j, *then* $\Phi_a(\theta p) = p(a_j)$.

(b) *If* $p(\lambda) = \lambda_1^{k_1}\dots\lambda_m^{k_m}$, *then* $\Phi_a(\theta p) = \dfrac{k_1!\dots k_m!}{k!} \sum_\tau a_{\tau(1)} \dots a_{\tau(k)}$,

where $k = k_1 + \dots + k_m$ *and the sum is over every map* $\tau : \{1,\dots,k\} \to \{1,\dots,m\}$ *which assumes the value* j *exactly* k_j *times for* $1 \le j \le m$.

4. THE SPECTRAL MAPPING THEOREM

As before, let $a = (a_1,\dots,a_m) \in \mathcal{B}^m$ and set $\mathcal{A} = \operatorname{alg}(e,a_1,\dots,a_m)$. Let $\mathcal{A}' = \operatorname{span}\{xy-yx : x,y \in \mathcal{A}\}$ be the *commutator subspace* of $\mathcal{A}$ and $\operatorname{rad} \mathcal{A}$ its Jacobson radical. So $\mathcal{A}/\operatorname{rad} \mathcal{A}$ is a commutative Banach algebra and we let $\pi : \mathcal{A} \to \mathcal{A}/\operatorname{rad} \mathcal{A}$ be the canonical homomorphism. It follows that $\sigma(\pi(x)) = \sigma(x)$ for all $x \in \mathcal{A}$.

We will impose the condition that $\mathcal{A}' \subseteq \text{rad}\,\mathcal{A}$, which is equivalent to the a_j commuting modulo $\text{rad}\,\mathcal{A}$. Under this condition, $\pi(a) = (\pi(a_1),\ldots,\pi(a_m))$ is a commuting m-tuple in $\mathcal{A}/\text{rad}\,\mathcal{A}$. If in addition $\sigma_{\mathcal{B}}(a_j) \subseteq \mathbb{R}$ for each j then $\gamma(a) = \gamma(\pi(a))$.

If $a = (a_1,\ldots,a_m)$ is an m-tuple of type s then $\pi(a)$ is a commuting m-tuple of type s. So $\pi(a)$ has a functional calculus $\Phi_{\pi(a)} : \mathcal{F} \to \mathcal{A}/\text{rad}\,\mathcal{A}$ and the results of McIntosh and Pryde [4] may be applied. Moreover, $\Phi_{\pi(a)} = \pi \circ \Phi_a$ and we have the following theorem

THEOREM 4.1. *If* $a = (a_1,\ldots,a_m)$ *is an m-tuple in* $\mathcal{B}$ *of type* s *and* $\mathcal{A}' \subseteq \text{rad}\,\mathcal{A}$ *then*

(a) $\pi \circ \Phi_a : \mathcal{F} \to \mathcal{A}/\text{rad}\,\mathcal{A}$ *is a homomorphism,*

(b) $\text{supp}(\pi \circ \Phi_a)$ *is compact,*

(c) $\text{supp}(\pi \circ \Phi_a) = \gamma(a)$,

(d) $\sigma(\Phi_a(f)) = f(\gamma(a))$ *for all* $f \in \mathcal{F}$.

Part (d) of Theorem 4.1 is the spectral mapping theorem. Applying it to the functions θp_j, where $p_j(\lambda) = \lambda_j$ for $\lambda \in \mathbb{R}^m$ and $\theta \in C_c^\infty(\mathbb{R}^m)$ is identically 1 on a neighbourhood of $\gamma(a)$, and to $\theta(p_j+p_k)$ we obtain

COROLLARY 4.2. *If* $a = (a_1,\ldots,a_m)$ *is an m-tuple in* $\mathcal{B}$ *of type* s *and* $\mathcal{A}' \subseteq \text{rad}\,\mathcal{A}$ *then*

(a) $\gamma(a) \subseteq \sigma(a_1) \times \ldots \times \sigma(a_n)$,

(b) $\sigma(a_j+a_k) \subseteq \sigma(a_j) + \sigma(a_k)$.

EXAMPLE 4.3. Let $a = (a_1,a_2)$ where $a_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $a_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then a_1,a_2 are non-commuting self-adjoint matrices with $\sigma(a_1) = \sigma(a_2) = \{1,-1\}$. Also $\sigma(a_1+a_2) = \{\sqrt{2},-\sqrt{2}\}$ so there is no spectral mapping theorem for Φ_a. Here (a_1,a_2) is of type $(0,\sqrt{2})$ but of course $\mathcal{A}' \not\subseteq \text{rad}\,\mathcal{A}$.

5. ALGEBRAS OF MATRICES

In this section we provide examples of m-tuples of matrices satisfying the conditions of Theorem 4.1.

It is well known that if $a_1,\dots,a_m$ are commuting matrices then they are simultaneously triangularizable. That is, there is an invertible matrix b such that $b^{-1}a_j b$ is (upper) triangular for $1 \le j \le m$. Then, $b^{-1}xb$ is triangular for all $x \in \mathcal{A} = \mathrm{alg}(e,a_1,\dots,a_m)$ and we say that $\mathcal{A}$ is *triangularizable*.

It is clear that condition 4.2(b) is satisfied if $\mathcal{A}$ is triangularizable. Moreover, if in addition $\sigma(a_j) \subseteq \mathbb{R}$ for each $j = 1,\dots,m$ then $\gamma(a) = \{\lambda \in \mathbb{R}^m :$ there exists k, $1 \le k \le m$, such that $\lambda_j = (b^{-1}a_j b)_{kk}$ for each j, $1 \le j \le n\}$. So 4.2(a) is then also satisfied.

Now let $a_1,\dots,a_m$ be N by N matrices. Set $\mathcal{A}^{(0)} = \mathcal{A} = \mathrm{alg}(e,a_1,\dots,a_m)$ and $\mathcal{A}^{(n)} = \mathrm{span}\{xy-yx : x,y \in \mathcal{A}^{(n-1)}\}$ for $n \ge 1$. Recall that $\mathcal{A}$ is called *solvable* if $\mathcal{A}^{(n)} = \{0\}$ for some $n \in \mathbb{N}$.

It is clear that if $\mathcal{A}$ is triangularizable then $\mathcal{A}$ is solvable. The converse is known as Lie's theorem. A proof may be found in Jacobson [3].

If $\mathcal{A}$ is triangularizable then $\mathrm{rad}\,\mathcal{A} = \{q \in \mathcal{A} : q \text{ is nilpotent}\}$ and $\mathcal{A}' = \mathcal{A}^{(1)} \subseteq \mathrm{rad}\,\mathcal{A}$. Conversely, if $\mathcal{A}' \subseteq \mathrm{rad}\,\mathcal{A}$ then $(xy-yx)z$ is nilpotent, and hence has zero trace, for all $x,y,z \in \mathcal{A}$. By a recent result of Radjavi [6] this trace condition is equivalent to $\mathcal{A}$ being triangularizable. So we have the following

THEOREM 5.1. *For an algebra* $\mathcal{A}$ *of (complex) matrices the following are equivalent*

(a) $\mathcal{A}$ *is triangularizable,*

(b) $\mathcal{A}$ *is solvable,*

(c) $\mathcal{A}' \subseteq \operatorname{rad} \mathcal{A}$,

(d) $\operatorname{trace}(xyz-yxz) = 0$ *for all* $x,y,z \in \mathcal{A}$.

Finally we have the following

THEOREM 5.2. *If* $a_1,\dots,a_m$ *are simultaneously triangularizable 2 by 2 matrices with real eigenvalues then* $\mathrm{a} = (a_1,\dots,a_m)$ *is of type* $(1,r)$ *for any* $r > r(\mathrm{a})$ *the joint spectral radius of* a.

PROOF. It suffices to suppose that the a_j are triangular matrices. Let $a^{kh} = (a_1^{kh},\dots,a_m^{kh})$ where a_j^{kh} is the kh component of a_j for $1 \le k,h \le 2$. For $\zeta = \xi + i\eta$ where $\xi,\eta \in \mathbb{R}^m$ let $z = e^{i\langle \mathrm{a},\xi\rangle}$. Then $z_{11} = e^{i\langle a^{11},\zeta\rangle}$, $z_{21} = 0$, $z_{22} = e^{i\langle a^{22},\zeta\rangle}$, $z_{12} = \langle a^{12},\zeta\rangle \dfrac{e^{i\langle a^{22},\zeta\rangle}-e^{i\langle a^{11},\zeta\rangle}}{\langle a^{22},\zeta\rangle-\langle a^{11},\zeta\rangle}$ if $\langle a^{22},\zeta\rangle \neq \langle a^{11},\zeta\rangle$, and $z_{12} = i\langle a^{12},\zeta\rangle e^{i\langle a^{11},\zeta\rangle}$ if $\langle a^{22},\zeta\rangle = \langle a^{11},\zeta\rangle$. So $|z_1| = e^{-\langle a^{11},\eta\rangle} \le e^{|a^{11}||\eta|}$, $|z_{22}| = e^{-\langle a^{22},\eta\rangle} \le e^{|a^{22}||\eta|}$, and $|z_{12}| \le 4|a^{12}||\zeta|(e^{|a^{22}||\eta|} + e^{|a^{11}||\eta|})$, giving $\|e^{i\langle \mathrm{a},\zeta\rangle}\| \le c_\epsilon(1+|\xi|)e^{r_\epsilon|\eta|}$ where $\epsilon > 0$ is arbitrary and $r_\epsilon = \epsilon + \max_k|a^{kk}| = \epsilon + r(\mathrm{a})$.

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