BANACH ALGEBRAS GENERATED BY ANALYTIC SEMIGROUPS HAVING COMPACTNESS PROPERTIES ON VERTICAL LINES *

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§0. INTRODUCTION

Let $(a^z)_{\mathrm{Re}\ z>0}$ be an analytic semigroup in a Banach algebra A such that $\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively weakly compact in A. Then the spectrum of a^1 is countable. This result was proved in [8], in order to investigate the relationships between the structure of a locally compact group G and the existence in the group algebra $L^1(G)$ of analytic semigroups with convenient behavior on vertical lines. The proof given in [8] was based on the fact that the function $y \mapsto a^{2+iy}$, $\mathbf{R} \to A$ is a vector version of the so-called weakly almost periodic functions defined by W.E. Eberlein in 1949, and it used elementary properties of this kind of function.

In this paper we give another proof of the same result, by considering a different approach. In fact, we are able to obtain a fairly complete description of the Banach algebras which are generated by semigroups with properties as above. We show that such an algebra A is semisimple if and only if $A^\perp = (0)$, where $A^\perp = \{b \in A : bA = (0)\}$, and then, that this class of algebras is exactly the class formed by the commutative Banach algebras A which are generated by their idempotents and for which the character space is discrete and countable. Moreover, this in turn is equivalent to the existence of a one-one, bounded algebra homomorphism $\psi : \ell^1(\mu) \to A$ such that $\psi(\ell^1(\mu))^- = A$, where $\mu = \{\mu_n\}_{n=1}^\infty$ is a sequence of numbers greater than or equal to one. (Here $\ell^1(\mu)$ is endowed with coordinatewise operations.) So the algebras $\ell^1(\mu)$ are canonical among this kind of algebras.

The strategy carried out in the proof of the main results of the paper is as follows. If an analytic semigroup $(a^z)_{\mathrm{Re}\ z>0}$ in a Banach algebra A satisfies $[a^1A]^- = A$ and

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$\sup_{y\in\mathbf{R}} \|a^{1+iy}\| < \infty$, then, for each y, we can define, formally, the operator a^{iy} by $a^{iy}ax = a^{1+iy}x$ $(x \in A)$. We then have $\sup_{n\in\mathbf{N}} \|(a^{iy})^n a\| = \sup_{n\in\mathbf{N}} \|a^{1+iny}a\| < \infty$. It turns out that this definition makes sense, a^{iy} becomes a densely defined and closed operator on A which satisfies the above boundedness conditions, and so we have indeed that a^{iy} $(y \in \mathbf{R})$ is a regular quasimultiplier, such as were defined by J. Esterle in [5]. This point of view was already adopted in [7], to get some results about weak amenability.

However, it is not necessary here to deal explicitly with regular quasimultipliers. What we really need is the notion of similar Banach algebras ([5, p.116]), by which, given a Banach algebra A and an analytic semigroup $(a^z)_{\text{Re } z>0}$ in A as before, we can associate to A a new Banach algebra $\mathcal{A}$ so that $(a^{iy})_{y\in\mathbf{R}}$ is a bounded group of multipliers of $\mathcal{A}$, i.e. $(a^{iy})_{y\in\mathbf{R}} \subseteq \text{Mul}(\mathcal{A})$ and $\mathcal{A}$ has the same character space as A. Then we use basic facts of the theory of topological semigroups, namely the Ellis Theorem ([4]), and results of Glicksberg and de Leeuw about semigroups of operators ([10], [12]), to obtain that the closure of $(a^{iy})_{y\in\mathbf{R}}$ for the weak operator topology on $\text{Mul}(\mathcal{A})$ is a metrizable, compact group, say G. Most of the remainder of the work consists in a detailed analysis of the integral

$$\int_G f(T)T\,dT \quad (f \in L^1(G)).$$

The arguments considered at this point are actually a reorganization of standard ideas, which can be seen in [14, p.41], or in [1, pp.86, 87] in another setting.

Let us also say that, on the other hand, operators T on a Banach space such that the weak-operator closure of $(T^n)_{n\in\mathbf{Z}}$ is a compact group were studied in [9]. These operators were called there G-operators. Following this terminology we say that an analytic semigroup $(a^z)_{\text{Re } z>0}$ in a Banach algebra A is *G-analytic* if $\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively weakly compact in A. If, besides, $\sup_{\text{Re } z>0} \|a^z\| < \infty$, then we say that $(a^z)_{\text{Re } z>0}$ is *bounded G-analytic*.

The paper is divided in three sections. Section 1 is devoted to an exposition of the basic facts on similar Banach algebras, topological semigroups, and analytic semigroups, which we shall need in Sections 2 and 3.

In Section 2 we prove the most difficult results, which involve Banach algebras generated by bounded G-analytic semigroups. In Section 3 we use the information obtained in §2 and the machinery supplied by the theory of similar Banach algebras to state and prove our main results.

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§1. PRELIMINARIES

Let A be a commutative Banach algebra. We denote by $\mathcal{L}(A)$ the Banach algebra of bounded linear products from A into A. An element $T \in \mathcal{L}(A)$ is said to be a *multiplier* if $T(ab) = a(Tb)$ for every $a, b \in A$. The set $\mathrm{Mul}(A)$ of all multipliers of A is a Banach subalgebra of $\mathcal{L}(A)$ which is commutative if the span of all products in A form a dense subset of A. The mapping $a \mapsto T_a$, $A \to \mathrm{Mul}(A)$, where $T_a(b) = ab$ $(a, b \in A)$, is continuous. Furthermore, it is an isometry if A has an approximate identity bounded by one ([5, p.95]).

The *strong operator* (SO) topology in $\mathcal{L}(A)$ is the locally convex topology defined by the family of seminorms $(p_a)_{a \in A}$ given by $p_a(T) = \|T_a\|$ $(a \in A,\ T \in \mathcal{L}(A))$. If A has a bounded approximate identity, then A is dense in $\mathrm{Mul}(A)$ with respect to the SO topology. Moreover, if A has a sequential bounded approximate identity, the closed unit ball Ω of $\mathrm{Mul}(A)$ is complete and metrizable with respect to the SO topology. In fact, if $a \in A$ is such that $[aA]^- = A$, the distance d defined by $d(T, S) = \|Ta - Sa\|$ $(S, T \in \mathrm{Mul}(A))$, induces the SO topology ([5, p.95]) on Ω.

In this paper we are mainly concerned with the so-called *weak operator* (WO) topology on $\mathrm{Mul}(A)$, whose definition is as follows. Let A^* be the dual Banach space of A. The WO topology in $\mathcal{L}(A)$ is the one defined by the family of seminorms $(p_{a,\varphi})_{a \in A, \varphi \in A^*}$ where $p_{a,\varphi}(T) = |\varphi(Ta)|$ $(T \in \mathcal{L}(A))$.

PROPOSITION (1.1). $\mathrm{Mul}(A)$ *is* WO-*closed in* $\mathcal{L}(A)$.

Proof. Let $T_0 \in \mathcal{L}(A)$ and suppose that $(T_j)_{j\in J}$ is a net in $\mathrm{Mul}(A)$ such that $\lim_{j\in J} T_j = T_0$ in the WO topology. Take $a, b \in A$ and $\varphi \in A^*$. Then — the brackets below stand for duality —

$$\begin{aligned}\langle T_0(ab), \varphi\rangle &= \lim_{j\in J}\langle T_j(ab), \varphi\rangle \\ &= \lim_{j\in J}\langle aT_j(b), \varphi\rangle = \lim_{j\in J}\langle T_j(b), \varphi \cdot a\rangle \\ &= \langle T_0(b), \varphi\cdot a\rangle = \langle aT_0(b), \varphi\rangle\end{aligned}$$

where we have used the canonical definition of $\varphi \cdot a \in A^*$ ([1]). It follows that $T_0 \in \mathrm{Mul}(A)$.

The following notion was introduced by J. Esterle in [5], in the setting of the theory of regular quasimultipliers, in which such a notion plays a central role.

DEFINITION (1.2). Let A and $\mathcal{A}$ be two commutative Banach algebras. They are *similar* if there exist a commutative Banach algebra D with dense principal ideals, and two one-one continuous homomorphisms $\varphi : D \to A$, $\psi : D \to \mathcal{A}$ such that $\varphi(D)$, $\psi(D)$ are dense ideals of A and $\mathcal{A}$, respectively.

We shall use later the following elementary fact about similar Banach algebras. We denote by Φ_A the character space of A endowed with the Gelfand topology.

PROPOSITION (1.3). *Let A, $\mathcal{A}$ be two similar Banach algebras. Then Φ_A and $\Phi_{\mathcal{A}}$ are homeomorphic.*

Proof. Let D be as in Definition (1.2). We can suppose that D is an ideal of A and $\mathcal{A}$, respectively. Take $\varphi \in \Phi_D$ and $u \in D$ such that $[uD]^- = D$, and define $\tilde{\varphi}(a) = \varphi(au)\varphi(u)^{-1}$ $(a \in A)$. Then $\tilde{\varphi}$ is well defined and $\tilde{\varphi} \in \Phi_A$. Clearly, the mapping $\varphi \mapsto \tilde{\varphi}$, $\Phi_D \to \Phi_A$ is a bijection with continuous inverse. Let $(\varphi_j)_{j\in J}$ be a net in Φ_D which converges to $\varphi_0 \in \Phi_D$. Then

$$\lim_{j\in J} \varphi_j(au) = \varphi_0(au)$$

for all $a \in A$, and therefore

$$\lim_{j\in J} \tilde{\varphi}_j(a) = \tilde{\varphi}_0(a)$$

for all $a \in A$. Hence Φ_D and Φ_A are homeomorphic.

A *topological semigroup* is a non-empty set on which an associative product and a (Hausdorff) topology are defined, so that the product is separately continuous.

One of the most impressive results in the theory of topological semigroups is the Ellis Theorem ([4]) which we state now.

THEOREM (1.4). *Let G be a locally compact topological semigroup which has an identity element and for which each element has an inverse. Then G is a locally compact group, i.e. the product is jointly continuous and inverse mapping is continuous.*

In this paper we are interested in two kinds of topological semigroups.

(i) *Semigroups of operators.* Let S be a topological semigroup and let X be a Banach space. A WO-*continuous representation* of S as a semigroup of operators on X is a linear mapping $s \mapsto T_s$, $S \to \mathcal{L}(X)$ such that $T_{s+r} = T_s \circ T_r$ $(s, r \in S)$, and $\lim_{s \to r} T_s = T_r$ in the WO-topology on A. Then $(T_s)_{s \in S}$ is a topological semigroup with respect to the WO-topology in $\mathcal{L}(X)$. It is said to be *weakly almost periodic* (WAP) if it is WO-relatively compact in $\mathcal{L}(X)$.

Suppose that S is WAP and consider the WO closure of S in $\mathcal{L}(X)$, say $\bar{S}$. Then $\bar{S}$ is a compact topological semigroup in $\mathcal{L}(X)$. Moreover, if $\bar{S}$ is amenable, it gives rise to a canonical decomposition of X: let $X_r = \{x \in X : \text{ for each } U \in \bar{S} \text{ there exists some } V \in \bar{S} \text{ such that } VUx = x\}$, and $X_0 = \{x \in X : 0 \text{ is weakly adherent to } (T_j x)_{j \in S}\}$. Then X_r and X_0 are closed subspaces of X and $X = X_r \oplus X_0$.

The reader is referred to [12, pp.80, 81, 85, 86] for these results. See also [11].

We now apply this to the situation where $X = A$ and S is a semigroup of multipliers of A. If A is supposed to have a b.a.i., then $\text{Mul}(A)$ is commutative and so $\bar{S} \subseteq \text{Mul}(A)$ is amenable. We have the following.

PROPOSITION (1.5). *Let $S \to \text{Mul}(A) : s \mapsto T_s$ be a WO-continuous representation of S as a WAP semigroup of operators on A. Then A_r and A_0 are closed ideals of A.*

Proof. Take $a \in A_r$ and $x \in A$. For every $T \in \bar{S}$ there exists $U \in \bar{S}$ such that $UTa = a$ then $UTax = ax$ since $U, T \in \mathrm{Mul}(A)$ and so $ax \in A_r$. Similarly, take $a \in A_0$ and $x \in A$. There exists a subnet $(T_j)_{j \in J}$ of $(T_s)_{s \in S}$ such that, for each $\varphi \in A^*$, $\langle T_j a, \varphi \rangle \to 0$. Then $\langle T_j ax, \varphi \rangle = \langle T_j a, x \cdot \varphi \rangle \to 0$ for every $\varphi \in A^*$; therefore $ax \in A_0$.

(ii) *Analytic semigroups.* Let H be the open right-hand half plane in $\mathbf{C}$. An *analytic semigroup* (AS) in A is an analytic function $z \mapsto a^z$, $H \to A$ such that $a^{z+w} = a^z \cdot a^w$ $(z, w \in H)$. We identify an AS with its semigroup range, denoted by $(a^z)_{\mathrm{Re}\ z>0}$. Throughout, whenever we consider AS, we shall write a instead of a^1. It is well known that $[a^z A]^- = [aA]^-$ $(z \in H)$. We shall consider AS for which $\sup_{y \in \mathbf{R}} \|a^{1+iy}\| < \infty$. In this case, if B is the closed subalgebra of A generated by $(a^z)_{\mathrm{Re}\ z>0}$ then B equals the closed subalgebra of A polynomially generated by a, say $P(a)$ ([6, p.379]). Also, $B = [aB]^-$ ([3, Theorem]). If $\varphi \in \Phi_B$ there is $\alpha \in \mathbf{R}$ such that $\varphi(a^z) = e^{\alpha z}$ $(z \in H)$ ([18, p.78]).

Finally, let us note that we shall use on several occasions the following fact. If

$$(*) \qquad \int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2}\, dy < \infty,$$

and a is quasinilpotent, then $a = 0$ ([18, p.82]).

Next, we state the result which connects the study of AS with "good" growth properties on vertical lines, with the theory of similar Banach algebras. Put $A^\perp = \{b \in A : bc = 0\ (c \in A)\}$.

PROPOSITION (1.6). *Let A be a commutative Banach algebra such that $A^\perp = (0)$. Suppose that A has an* AS $(a^z)_{\mathrm{Re} z>0}$ *such that* $[aA]^- = A$ *and* $\sup_{y \in \mathbf{R}} \|a^{1+iy}\| < \infty$. *Then there exists a Banach algebra $\mathcal{A}$ such that*

(i) *A is densely and continuously contained in $\mathcal{A}$, and A, $\mathcal{A}$ are similar.*

(ii) *$(a^{1/n})_{n \geq 1}$ is an approximate identity in $\mathcal{A}$, bounded by one.*

(iii) *The norm limit $a^{iy} b = \lim_{z \to iy} a^z b$ exists in $\mathcal{A}$ for every $y \in \mathbf{R}$ and every $b \in \mathcal{A}$.*

(iv) $(a^{iy})_{y\in\mathbf{R}}$ *is a group in* $\mathrm{Mul}(A)$ *such that* $a^0 = I$ *(identity),* $a^{iy}a^{-iy} = I$, *and* $\|a^{iy}\| = 1$ *(*$y \in \mathbf{R}$*).*

A detailed proof of Proposition (1.6) is in [7] where a more general statement is given.

§2. ON THE STRUCTURE OF BANACH ALGEBRAS WITH BOUNDED G-ANALYTIC GENERATORS

In all this section $\mathcal{A}$ will denote a Banach algebra generated by an AS $(a^z)_{\mathrm{Re}\ z>0}$ such that $\sup_{\mathrm{Re}\ z>0}\|a^z\| < \infty$. For such an algebra we have that $(a^{iy})_{y\in\mathbf{R}} \subseteq \mathrm{Mul}(A)$, with $\|a^{iy}\| = 1$ $(y \in \mathbf{R})$ ([18, p.84]).

LEMMA (2.1). *Suppose that* $\{a^{1+iy} : y \in \mathbf{R}\}$ *is relatively weakly compact in* $\mathcal{A}$. *Then* $(a^{iy})_{y\in\mathbf{R}}$ *is a* WAP *group in* $\mathrm{Mul}(A)$.

Proof. Let $(a^{iy})_{y\in J}$ be a net in $(a^{iy})_{y\in\mathbf{R}}$. By hypothesis, there is a subnet $(a^{1+iy})_{y\in J'}$ of $(a^{1+iy})_{y\in J}$ which converges weakly to an element of $\mathcal{A}$, say α. Take $b \in \mathcal{A}$, $\epsilon > 0$, and choose $x \in \mathcal{A}$ such that $\|b - ax\| < \epsilon$. Then, for every $\varphi \in \mathcal{A}^*$, there exists $y \in J'$ such that

$$\begin{aligned}
|\langle a^{iu}b - a^{iv}b, \varphi\rangle| &\le |\langle a^{iu}b - a^{iu}(ax), \varphi\rangle| \\
&\quad + |\langle a^{1+iu}x - a^{1+iv}x, \varphi\rangle| + |\langle a^{iv}(ax) - a^{iv}b, \varphi\rangle| \\
&\le 2\|\varphi\|\|b - ax\| + |\langle a^{1+iu} - a^{1+iv}, x\cdot\varphi\rangle| \\
&\le (2\|\varphi\| + 1)\epsilon
\end{aligned}$$

for every $u, v \ge y$ in J'. It follows that the subnet $(a^{iy}b)_{y\in J'}$ converges weakly to a certain element Tb of $\mathcal{A}^{**}$, for each $b \in \mathcal{A}$. Clearly, T is a linear operator. Furthermore,

$$\begin{aligned}
\|Tb\| &= \sup\{|\langle Tb, \varphi\rangle| : \|\varphi\| \le 1\} \\
&\le \limsup\{|\langle a^{iy}b, \varphi\rangle| : \|\varphi\| \le 1, y \in J'\} \le \|b\| \quad (b \in \mathcal{A})
\end{aligned}$$

whence $T(b) = \lim_n T(ax_n)$ in the norm of $\mathcal{A}^{**}$ if $b = \lim_n ax_n$ in the norm of $\mathcal{A}$ for some sequence $(x_n)_{n=1}^{\infty} \subseteq \mathcal{A}$. Also,

$$\begin{aligned}\langle T(ax), \varphi\rangle &= \lim_{y\in J'} \langle a^{1+iy}x, \varphi\rangle \\ &= \lim_{y\in J'} \langle a^{1+iy}, x\cdot\varphi\rangle \\ &= \langle Ta, x\cdot\varphi\rangle \\ &= \langle (Ta)x, \varphi\rangle\end{aligned}$$

for all $x \in \mathcal{A}$ and all $\varphi \in \mathcal{A}^*$; hence $T(ax) = (Ta)x = ax \in \mathcal{A}$. So $Tb \in \mathcal{A}$ for all $b \in \mathcal{A}$, and we have proved that $(a^{iy})_{y\in\mathbf{R}}$ is WAP.

Note that the above lemma remains true if we replace the condition $\mathcal{A} = P(a)$ by the weaker one $\mathcal{A} = [a\mathcal{A}]^-$.

LEMMA (2.2). *Let $\mathcal{A}$ be as above and let G be the* WO*-closure in* Mul(A) *of the group $(a^{iy})_{y\in\mathbf{R}}$. Then G is a compact, metrizable, abelian group.*

Proof. Since we are supposing that $\mathcal{A}$ has a b.a.i. we may apply Proposition (1.5) to show that $\mathcal{A} = \mathcal{A}_r \oplus \mathcal{A}_0$. Take $b \in \mathcal{A}_0$. Then there is a net $(a^{iy})_{y\in J}$ such that $\lim_{y\in J} \varphi(a^{iy}b) = 0$ for every $\varphi \in \mathcal{A}^*$. So, if $\varphi \in \Phi_{\mathcal{A}}$,

$$\begin{aligned}\varphi(a^{iy}b) &= \varphi(a^{iy}b)\varphi(a)\varphi(a)^{-1} = \varphi(a^{iy}ba)\varphi(a)^{-1} \\ &= \varphi(a^{1+iy})\varphi(b)\varphi(a)^{-1} = e^{\alpha(1+iy)}\varphi(b)e^{-\alpha} = e^{i\alpha y}\varphi(b),\end{aligned}$$

for some $\alpha \in \mathbf{R}$, and $|\varphi(b)| = |e^{i\alpha y}\varphi(b)| \to 0$. This means that b is in the radical of $\mathcal{A}$. Now, because $\mathcal{A}_r$ is a closed ideal of $\mathcal{A}$ the projection $\pi_0 : \mathcal{A} \to \mathcal{A}_0$ is a bounded algebra homomorphism and so $(\pi_0(a^z))_{\mathrm{Re}\ z>0}$ is an AS satisfying the condition (*) before Proposition (1.6), with $\pi_0(a)$ quasinilpotent. Therefore $\pi_0(a) = 0$ and so $a \in \mathcal{A}_r$.

This implies that for every $T \in G$ there is $S \in G$ such that $STa = a$, and so, since $\mathcal{A} = P(a)$, $STb = b$ for every $b \in \mathcal{A}$. Thus G is an algebraic group. By Theorem (1.4) G is in fact a compact group. Furthermore, the SO and WO topologies on $\mathcal{L}(A)$ are equal on G ([10, p.94]), whence it turns out that G is also metrizable.

For G as in Lemma (2.2), let $M(G)$ be the measure algebra on G. As it is well known, $M(G) = \mathrm{Mul}(L^1(G))$. The fact that G is compact allows us to consider the mapping $\Lambda : M(G) \to \mathrm{Mul}(A)$ defined by $\Lambda\mu = \int_G T d\mu(T)$ $(\mu \in M(G))$. This mapping Λ is in fact a bounded algebra homomorphism, and it was used by F.P. Greenleaf in order to characterize quotients of measure algebras ([14, p.36]). In the same way, we shall exploit this kind of idea to describe the structure of $\mathcal{A}$. For this, the idempotents of $\mathcal{A}$ are very important.

Let $\hat{G}$ be the dual group of G, say $\hat{G} \cong \{\sigma_n\}_{n=1}^{\infty}$. For $n \geq 1$, define $E_n = \int_G \sigma_n(T) T dT$, where dT denotes the normalized Haar measure on G.

Let $\mathbf{E} = \{n \in \mathbf{N} : E_n \neq 0\}$.

THEOREM (2.3). *The set $\{E_n\}_{n\in\mathbf{E}}$ is an orthogonal set of idempotents in $\mathcal{A}$ which generates $\mathcal{A}$. Furthermore, $\mathcal{A}$ is semisimple, $\Phi_{\mathcal{A}} \cong \{\varphi_n\}_{n\in\mathbf{E}}$ is discrete (and countable), and $E_n a = \varphi_n(a) E_n$ $(n \in \mathbf{E})$.*

Proof. First of all, observe that, if $\sigma \in \hat{G}$, the continuous mapping $h(y) = \sigma(a^{iy})$ $(y \in \mathbf{R})$ satisfies $h(y+v) = \sigma(a^{i(y+v)}) = \sigma(a^{iy}a^{iv}) = \sigma(a^{iy})\sigma(a^{iv}) = h(y)h(v)$ for every $y, v \in \mathbf{R}$, and $|h(y)| = 1$ $(y \in \mathbf{R})$. Therefore there is $\alpha \in \mathbf{R}$ such that $\sigma(a^{iy}) = e^{i\alpha y}$ $(y \in \mathbf{R})$.

Now, E_n $(n \geq 1)$ is an idempotent, since

$$\begin{aligned} E_nE_n &= \left(\int_G \sigma_n(T)TdT\right)\left(\int_G \sigma_n(S)SdS\right) \\ &= \int_G \left[\int_G TS\sigma_n(TS)dT\right]dS = \int_G \left[\int_G U\sigma_n(U)dU\right]dS \\ &= \int_G E_ndS = E_n. \end{aligned}$$

Also, if $S \in G$,

$$\begin{aligned} SE_n &= \int_G ST\sigma_n(T)dT = \int_G U\sigma_n(US^{-1})dU \\ &= \sigma_n(S^{-1})\int_G U\sigma_n(U)dU = \sigma_n(S^{-1})E_n \quad (n \geq 1). \end{aligned}$$

In particular, $E_na^{iy} = \sigma_n(a^{-iy})E_n$, that is, $E_na^{iy} = e^{-i\alpha_n y}E_n$ ($y \in \mathbf{R}$, $n \geq 1$) for some $\alpha_n \in \mathbf{R}$.

If $\varphi \in \mathcal{A}^*$ then the function

$$z \mapsto g(z) = \varphi(E_na^za - e^{-\alpha_n z}E_na), \quad H^- \to \mathbf{C},$$

is analytic in H, continuous in H^-, and null on $H^-\backslash H$. So, by the uniqueness theorem $g(z) \equiv 0$, whence $\varphi(E_na^2 - e^{-\alpha_n}E_na) = 0$ for every $\varphi \in \mathcal{A}^*$. Hence $(E_na - e^{-\alpha_n}E_n)a = 0$, i.e. $E_na = e^{-\alpha_n}E_n$, and so $E_n \in \mathcal{A}$ $(n \geq 1)$.

Suppose that $\varphi \in \mathcal{A}^*$ is such that $\varphi(E_n) = 0$ $(n \geq 1)$. Then, since $E_na = e^{-\alpha_n}E_n$, $\varphi(E_np(a)) = 0$ for every polynomial p. It follows that

$$\int_G \sigma_n(T)\varphi(Tp(a))dT = 0$$

for every $n \geq 1$, whence $\varphi(Tp(a)) = 0$ $(T \in G)$. In particular $\varphi(p(a)) = 0$ for every polynomial p and so $\varphi \equiv 0$. It follows that $\{E_n\}_{n=1}^{\infty}$ spans a dense subset of $\mathcal{A}$. As a result, we obtain that if $\varphi \in \Phi_{\mathcal{A}}$ then there is $n \in \mathbf{E}$ such that $\varphi(E_n) \neq 0$ (in fact, $\varphi(E_n) = 1$). Hence, from $E_na = e^{-\alpha_n}E_n$, we get $\varphi(a) = e^{-\alpha_n}$.

Conversely, if n is given in $\mathbb{E}$, E_n cannot be in the radical of $\mathcal{A}$, so there exists $\varphi \in \Phi_{\mathcal{A}}$ such that $\varphi(E_n) = 1$. Using again that $E_n a = e^{-\alpha_n} E_n$ we deduce that $\varphi(a) = e^{-\alpha_n}$ (so φ must be unique). Put φ_n for the unique character on $\mathcal{A}$ such that $\varphi_n(a) = e^{-\alpha_n}$. Then $\Phi_{\mathcal{A}} \cong \{\varphi_n\}_{n\in\mathbb{E}}$ and it is clear that $\Phi_{\mathcal{A}}$ is discrete. It is also clear that $E_n E_m = 0$ if $n \neq m$ since $E_n E_m$ is in the radical of $\mathcal{A}$ and it is also idempotent.

Finally, take r in the radical of $\mathcal{A}$, and let $n \geq 1$. Since $E_n a \in \mathbb{C}E_n$ and $\mathcal{A} = P(a)$ we have that $E_n\mathcal{A} \subseteq \mathbb{C}E_n$ and so $E_n r = \lambda(r) E_n$, for some $\lambda(r) \in \mathbb{C}$. Since $E_n r$ is quasinilpotent, $\lambda(r) = 0$. Now $\mathcal{A} = \overline{\text{span}}\{E_n\}_{n=1}^{\infty}$, whence $\mathcal{A}r = 0$. As $(a^{1/n})_{n\geq 1}$ is an approximate identity in $\mathcal{A}$ we obtain that $r = 0$.

We finish the section proving that $\mathcal{A}$ is similar to a quotient of $L^1(G)$.

THEOREM (2.4). *If $f \in L^1(G)$, then*

$$\int_G f(T)T\,dT \in \mathcal{A}.$$

Let

$$N = \{f \in L^1(G) : \int_G f(T)T\,dT = 0\}.$$

Then N is a closed ideal of $L^1(G)$, and the mapping

$$\Psi : f + L^1(G) \mapsto \int_G f(T)T\,dT,$$

$L^1(G)/N \rightarrow \mathcal{A}$, is an injective and bounded algebra homomorphism such that $\Psi(L^1(G)/N)^- = \mathcal{A}$. Indeed, $L^1(G)/N$ and $\mathcal{A}$ are similar.

Proof. The mapping

$$\Lambda : f \mapsto \int_G f(T)T\,dT, \quad L^1(G) \rightarrow \text{Mul}(A)$$

is a bounded homomorphism. Now, $\Lambda(\sigma_n) = E_n \in \mathcal{A}$ $(n \geq 1)$ and therefore $\Lambda(p) \in \mathcal{A}$ for every trigonometric polynomial in $L^1(G)$. So it is apparent that $\Lambda(L^1(G)) \subset \mathcal{A}$. Moreover, $\Lambda(L^1(G))^- = \mathcal{A}$ because $\{E_n\}_{n=1}^{\infty}$ spans $\mathcal{A}$, and the norm in Mul(A) is the same as in $\mathcal{A}$.

Consider now the Banach algebra $\ell^1(\mathbf{E}) = D$ with coordinatewise product, and the algebra homomorphisms

$$\varphi : \{c_n\}_{n\in\mathbf{E}} \mapsto \sum_{n\in\mathbf{E}} c_n(\sigma_n + N), \quad D \to L^1(G)/N,$$

and

$$\psi : (c_n)_{n\in\mathbf{E}} \mapsto \sum_{n\in\mathbf{E}} c_n E_n, \quad D \to \mathcal{A}.$$

These homomorphisms are injective because if $0 = \psi((c_n)_{n=1}^{\infty}) = \Psi(\varphi((c_n)_{n=1}^{\infty}))$ then $\sum_{n\in\mathbf{E}} c_n E_n = 0$ in $\mathcal{A}$ and so $c_n = 0$ $(n \in \mathbf{E})$. It is also straightforward that D has dense principal ideals, that φ, ψ are continuous (observe that

$$1 \leq \|E_n\| \leq \int_G |\sigma_n(T)| \|T\| dT \leq 1 \quad (n \geq 1)),$$

and that $\varphi(D)$, $\psi(D)$ are dense in $L^1(G)/N$ and $\mathcal{A}$, respectively.

Remark. Since $L^1(G)/N$ and $\mathcal{A}$ have bounded approximate identities, the structures of their closed ideals are exactly the same ([5, p.74]).

§3. CHARACTERIZATIONS OF BANACH ALGEBRAS WITH G-ANALYTIC GENERATORS

We say that an AS $(a^z)_{\mathrm{Re}\ z>0}$ in a Banach algebra A is *cyclic* on $\{\mathrm{Re}\ z = 1\}$ if there is $y_0 \in \mathbf{R}\backslash\{0\}$ such that $a^{1+iy} = a^{1+i(y+y_0)}$ $(y \in \mathbf{R})$. In this case, $\{a^{1+iy} : y \in \mathbf{R}\}$ is norm compact in A, for the mapping $y \mapsto a^{1+iy}$, $[0, y_0] \to A$ is continuous.

THEOREM (3.1). *Let A be a commutative Banach algebra. The following properties are equivalent.*

(i) Φ_A *is discrete and countable, and A is generated by its idempotents.*

(ii) *There is an AS $(a^z)_{\mathrm{Re}\ z>0}$ in A such that $A = P(a)$ and $\{a^{1+iy} : y \in \mathbf{R}\}$ is*

(a) *cyclic, or*

(b) *norm compact in A, or*

(c) *relatively compact in A, or*

(d) *relatively weakly compact in A.*

Proof. (i) $\Rightarrow$ (ii.a). Put $\Phi_A = \{\varphi_n\}_{n=1}^{\infty}$ and let E_n be the idempotent supported by φ_n $(n \geq 1)$. Then $\{E_n\}_{n=1}^{\infty}$ is an orthogonal set of idempotents. As Φ_A is discrete, every idempotent of A is supported by a finite number of characters and so it is a linear combination of E_n's. So $\{E_n\}_{n=1}^{\infty}$ generates A. For each n choose an integer $\lambda_n \geq \|E_n\|$, then define

$$a^z = \sum_{n=1}^{\infty} e^{-(n+\lambda_n)z} E_n \quad (\mathrm{Re}\ z > 0).$$

Clearly, $(a^z)_{\mathrm{Re}\ z>0}$ is an AS in A such that $a^{1+i(y+2\pi)} = a^{1+iy}$ $(y \in \mathbf{R})$. Moreover, if $\varphi \in A^*$ satisfies $\varphi(a^z) = 0$ $(z \in H)$, then

$$\sum_{n=1}^{\infty} e^{-(n+\lambda_n)z} \varphi(E_n) = 0,$$

and therefore $\varphi(E_n) = 0$ $(n \geq 1)$.

(ii.d) $\Rightarrow$ (i) We can suppose that A is semisimple so that $A^{\perp} = (0)$. Let $\mathcal{A}$ be the Banach algebra similar to A, given by Proposition (1.6). By Theorem (2.3), $\Phi_{\mathcal{A}}$ (and so Φ_A) is discrete and countable. Take idempotents $\{E_n\}$ as above and put

$$J = \{\lambda_1 E_1 + \ldots + \lambda_n E_n : \lambda_j \in \mathbf{C} \quad (j = 1, \ldots, n), \quad n \geq 1\}^{-}.$$

Then J is a closed ideal of A, and $(a^z + J)_{\mathrm{Re}\ z>0}$ satisfies condition (*) before Proposition (1.6). Therefore $a^z + J \equiv 0$, $a \in J$. It follows that $A = J$.

We find particularly interesting the case where A is semisimple for the following reasons.

If A is a commutative Banach algebra which contains an AS $(a^z)_{\mathrm{Re}\ z>0}$ such that $[aA]^- = A$ and

$$(*) \qquad \int_{-\infty}^{+\infty} \frac{\log^+ \|a^{1+iy}\|}{1+y^2} dy < \infty,$$

then A satisfies the *Wiener tauberian property*, i.e. every proper closed ideal of A is contained in some modular maximal ideal (see [18, p.78]). This property also holds for Banach algebras which are regular, semisimple, and tauberian (see [15]). So, it seems natural to ask for relationships between these last conditions, and the above condition (*), in Banach algebras. By using tensor products, it is quite elementary to find examples of Banach algebras with $(a^z)_{\mathrm{Re}\ z>0}$ satisfying (*) but such that they are neither regular, nor semisimple. Nevertheless, it was shown in [6] that if $A = P(a)$ with $(a^z)_{\mathrm{Re}\ z>0}$ satisfying (*) then A is regular. Such an algebra is also tauberian (the proof is similar to the end of the proof of Theorem (3.1)).

On the other hand, it is not easy, in general, to determine whether a polynomially generated Banach algebra is semisimple or not. A. Sinclair proved in [17] that a Banach algebra generated by a hermitian element with countable spectrum must be semisimple, and he gave also an example of a non- semisimple Banach algebra generated by a certain hermitian element (of course, with noncountable spectrum).

Here, thanks to Theorem (2.3), we have the following characterization.

PROPOSITION (3.2). *Let $(a^z)_{\mathrm{Re}\ z>0}$ be an AS in A such that $A = P(a)$ and $\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively weakly compact in A. Then A is semisimple if and only if $A^\perp = (0)$.*

Proof. Assume $A^{\perp} = (0)$, and consider the Banach algebra $\mathcal{A}$, similar to A, given by Proposition (1.6). Then $\mathcal{A}$ is semisimple (Theorem (2.3)) and, since A is continuously contained in $\mathcal{A}$, A is semisimple too. The converse is obvious.

Certainly, there are non-semisimple Banach algebras satisfying the AS hypothesis of the above proposition. We are going to show an example.

Recall that *spectral synthesis* holds for a *commutative* Banach algebra A if each closed ideal is the intersection of modular maximal ideals. This property is equivalent to the fact that each quotient Banach algebra of A is semisimple (see [16], [17]). In [16], H. Mirkil give an example of regular, semisimple, tauberian Banach algebra M generated by its idempotents and such that Φ_M is discrete and countable — of course, there are $(a^z)_{\mathrm{Re}\ z>0}$ in M satisfying the hypothesis of Proposition (3.2) — for which spectral synthesis does not hold. Thus it is clear that some quotient A of M has an AS as in the first part of Proposition (3.2) but it is not semisimple (i.e. $A^{\perp} \neq (0)$).

We are now going to see that, for semisimple Banach algebras, the only possible way to produce an AS compact on $\{\mathrm{Re}\ z = 1\}$ is to consider AS cyclic on $\{\mathrm{Re}\ z = 1\}$.

PROPOSITION (3.3). *Let A be a Banach algebra such that $A^{\perp} = (0)$ and suppose that there is an* AS $(a^z)_{\mathrm{Re}\ z>0}$ *such that $[aA]^- = A$ and $\{a^{1+iy} : y \in \mathbf{R}\}$ is norm compact. Then $(a^z)_{\mathrm{Re}\ z>0}$ is cyclic on $\{\mathrm{Re}\ z = 1\}$.*

Proof. Let $\mathcal{A}$ be the Banach algebra associated to A as above. Then $G = (a^{iy})_{y\in\mathbf{R}}$ is a compact group in $\mathrm{Mul}(A)$ for the SO topology. Consider then the mapping $\rho : y \mapsto a^{iy}$, $\mathbf{R} \to G$. This mapping ρ is a continuous, surjective, group homomorphism. Since $\mathbf{R}$ is $\sigma-$compact and G is compact, ρ is an open mapping ([13, Theorem 5.29]). Now suppose, if possible, that ρ is one-one. In this case ρ is a homeomorphism and so $\mathbf{R}$ is compact, which is not possible. So, there exists $y_0 \in \mathbf{R}\backslash\{0\}$ such that a^{iy_0} is the identity of G; hence we have that $a^{1+iy+iy_0} = aa^{iy}a^{iy_0} = aa^{iy} = a^{1+iy}$ $(y \in \mathbf{R})$, as we wanted to prove.

The following result is the main result of the paper. Let $\{\mu_n\}_{n=1}^{\infty}$ satisfy $\mu_n \geq 1$ for all n. Then we denote by $\ell^1(\mu)$ the Banach algebra of all sequences $\{c_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |c_n|\mu_n < \infty$$

with coordinatewise multiplication.

THEOREM (3.4). *Let A be a Banach algebra such that $A^{\perp} = (0)$. Then the following properties are equivalent.*

(i) *Φ_A is discrete and countable, and A is generated by its idempotents.*

(ii) *There exists an AS $(a^z)_{\mathrm{Re}\ z>0}$ in A such that $A = P(a)$ and*

(a) *$\{a^{1+iy} : y \in \mathbf{R}\}$ is norm compact (i.e. cyclic) in A, or*

(b) *$\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively compact in A, or*

(c) *$\{a^{1+iy} : y \in \mathbf{R}\}$ is relatively weakly compact in A.*

(iii) *There is a sequence $\{\mu_n\}_{n=1}^{\infty}$ with $\mu_n \geq 1$, and a one-one continuous homomorphism $\eta : \ell^1(\mu) \rightarrow A$ with dense image.*

Proof. By Theorem 3.1 it suffices to prove (iii) is equivalent to any of the others. (ii.c) $\Rightarrow$ (iii). A is similar to $\mathcal{A}$, where $\mathcal{A}$ is as in Proposition (1.6). Let $\{E_n\}_{n\in\mathbf{E}}$ be the set of idempotents defined before Theorem (2.3). Then $\Phi_A \cong \Phi_{\mathcal{A}} \cong \{\varphi_n\}_{n\in\mathbf{E}}$ where φ_n is the unique character of $\mathcal{A}$ such that $\varphi_n(E_n) = 1$ $(n \in \mathbf{E})$. Choose for every $n \in \mathbf{E}$ an idempotent e_n in A such that $\varphi_n(e_n) = 1$ and $\varphi_m(e_n) = 0$, if $m \in \mathbf{E}$ and $m \neq n$. Then $\varphi_m(E_n - e_n) = 0$ $(m \in \mathbf{E})$ and so $E_n = e_n \in A$ $(n \in \mathbf{E})$.

Let $\mu = \|E_n\|_A$ and define $\eta : \ell^1(\mu) \to A$ by $\eta(\{c_n\}_{n=1}^{\infty}) = \sum_{n\in\mathbf{E}} c_n E_n$ where, by renumbering if necessary, we are supposing that all E_n are not zero. Then η is a one-one, bounded, algebra homomorphism, and its image is dense in A because of [18, p.82].

(iii) $\Rightarrow$ (i) Consider the adjoint mapping of η, $\eta^* : \varphi \mapsto \tilde{\varphi}$, $\Phi_A \to \Phi_{\ell^1(\mu)} \cong \mathbb{N}$ given by $\tilde{\varphi}\left(\sum_{n=1}^{\infty} c_n \delta_n\right) = \sum_{n=1}^{\infty} c_n \varphi(\delta_n)$, if $\sum_{n=1}^{\infty} |c_n|\mu_n < \infty$. The mapping η^* is continuous, and it is also injective. For if $\varphi_1, \varphi_2 \in \Phi_A$ with $\sum_{n=1}^{\infty} c_n[\varphi_1(\delta_n) - \varphi_2(\delta_n)] = 0$ for every $(c_n)_{n=1}^{\infty} \in \ell^1(\mu)$, then taking into account that $\{\varphi_1(\delta_n) - \varphi_2(\delta_n)\}_{n=1}^{\infty}$ is a bounded sequence, we obtain that $\varphi_1(\delta_n) = \varphi_2(\delta_n)$ $(n \geq 1)$, so $\varphi_1 = \varphi_2$. It follows that Φ_A is countable and, moreover, $\{\varphi\} = (\eta^*)^{-1}(\{\eta^*(\varphi)\})$ is an open subset for all $\varphi \in \Phi_A$, i.e. Φ_A is discrete.

Finally, since $\eta(\delta_n)$ is an idempotent for all $n \geq 1$, we have proved (i).

Note that in the above theorem we have proved the implication (iii) $\Rightarrow$ (i) without using the condition $A^{\perp} = (0)$.

The presence of property (iii), among the equivalent properties of Theorem (3.4), shows that the simple manner in which we have constructed, in the proof of Theorem (3.1), an AS relatively weakly compact on vertical lines is, in an evident sense, canonical.

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