CONSTRUCTIONS PRESERVING WEAK AMENABILITY

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0. INTRODUCTION

When one considers module derivations from commutative Banach algebras it seems natural to restrict the attention to symmetric modules, that is, modules where left and right module multiplications agree. In particular one suspects that, if a commutative Banach algebra $\mathcal{A}$ has the property that every continuous module derivation from $\mathcal{A}$ into any symmetric Banach $\mathcal{A}$-module is zero, then $\mathcal{A}$ is particularly nice. Phrased in terms of Hochschild-Johnson cohomology groups, we require that $H^1(\mathcal{A}, X) = (0)$ for all symmetric Banach $\mathcal{A}$-modules X. Commutative Banach algebras with this property were called *weakly-amenable* (from here on abbreviated WA) in [1]. In that paper the authors related WA to amenability for certain classes of Banach algebras. They also showed that the only symmetric module which one has to consider is the dual of $\mathcal{A}$, that is, $\mathcal{A}$ is WA if and only if $H^1(\mathcal{A}, \mathcal{A}^*) = (0)$.

In a subsequent paper ([5]) the present author gave a characterization of WA in forms of the short exact sequence

$$\sum : 0 \to K \xrightarrow{\ i\ } \mathcal{A}^{\#}\hat{\otimes}\mathcal{A} \xrightarrow{\ \pi\ } \mathcal{A} \to 0,$$

where $\pi(a \otimes b) = ab$ and $K = \ker \pi$. (Here and throughout $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbf{C}$, the algebra obtained by formal adjunction of a unit.) The algebra $\mathcal{A}$ is WA if and only if $(K^2)^- = K$. If $\mathcal{A}$ has a b.a.i. then one may replace $\mathcal{A}^{\#}$ by $\mathcal{A}$. This parallels completely the corresponding characterization of amenability (Theorem III.21 of [9]). The result is especially useful for Banach algebras which behave nicely under the formation of projective tensor product.

To illustrate this let us show:

THEOREM 0.1. *Let G be a locally compact abelian group and let $w : G \to \mathbf{R}_+$ be a continuous submultiplicative weight. Then the Beurling algebra $L^1(G,w)$ is* WA *if and only if*

$$\sup\left\{\frac{|\phi(g)|}{w(g)w(-g)}\middle| g \in G\right\} = +\infty$$

for each non-zero measurable group homomorphism $\phi : (G,+) \to (\mathbf{C},+)$.

Proof. We identify $L^1(G,w)\hat{\otimes}L^1(G,w)$ with $L^1(G\times G, w\times w)$. Then $f \in K$ if and only if

$$\int_G f(t-s,s)ds = 0 \quad (a.e.).$$

First suppose that there is an additive, non-zero, measurable function $\phi : G \to \mathbf{C}$ so that $\phi(g) = O(w(g)w(-g))$. Let $m : g \to \mathbf{C}$ be any measurable function such that $(s,t) \mapsto \frac{|\phi(s)m(s+t)|}{w(s)w(t)}$ is essentially bounded, for example $m(s) = \frac{1}{w(-s)}$. On $L^1(G\times G, w\times w)$ define a functional D by

$$\langle f, D\rangle = \int_{G\times G} f(s,t)\phi(t)m(s+t)d(s,t).$$

Then for $f, g \in K$

$$\begin{aligned}\langle f*g, D\rangle &= \int_{G\times G}\int_{G\times G} f(s-u,t-v)g(u,v)\phi(t)m(s+t)d(u,v)d(s,t)\\ &= \int_{G\times G}\int_{G\times G} f(s-u,t)g(u,v)\{\phi(t)+\phi(v)\}m(s+t+v)d(u,v)d(s,t)\\ &= I_1 + I_2.\end{aligned}$$

Now

$$\begin{aligned}I_1 &= \int_{G\times G}\int_{G\times G} f(s-u,t)g(u,v)\phi(t)m(s+t+v)d(u,v)d(s,t)\\ &= \int_{G\times G}\int_{G\times G} f(s,t)g(u,v)\phi(t)m(s+u+t+v)d(s,t)d(u,v)\\ &= \int_{G\times G}\int_{G\times G} f(s,t)g(u,v-u)\phi(t)m(s+t+v)d(u,v)d(s,t)\\ &= 0,\end{aligned}$$

since $g \in K$. Similarly $I_2 = 0$. Obviously one can pick m so that D does not annihilate K. Conversely, suppose that $D \not\equiv 0$ on K and $D \perp K^2$. Let g', g'', h', h'' be arbitrary elements in $L^1(G, w)$. Let $f_s(t) := f(t-s)$ define the shift. Then one may verify that

$$\phi(s) = \langle (g' * g'')_{-s} \otimes (h' * h'')_s - (g' * g'') \otimes (h' * h''), D \rangle$$

defines an additive map $\phi : G \to \mathbf{C}$ such that

$$|\phi(s)| \leq \|D\| \|g' * g''\| \|h' * h''\| w(s) w(-s).$$

Since $L^1(G, w)^2$ is dense in $L^1(G, w)$ and $D \not\equiv 0$ on K, we may choose g', g'', h', h'' so that $\phi \not\equiv 0$.

REMARK. When G is discrete, this theorem is proved in [5].

If $\mathcal{A}$ is non-commutative we may use the result of [1] to define weak amenability for $\mathcal{A}$.

DEFINITION 0.2. Let $\mathcal{A}$ be a Banach algebra. We call $\mathcal{A}$ *weakly amenable* (WA) if $H^1(\mathcal{A}, \mathcal{A}^*) = (0)$.

It is known that all C^*-algebras are WA [8, Corollary 4.2] and that $L^1(G)$ is WA when G is a SIN group ([10]).

One might hope that a description in terms of the short exact sequence Σ may also prove useful in the non-commutative case. When $\mathcal{A}$ has an identity there is such a description: We replace $\mathcal{A}^{\#}$ by $\mathcal{A}$ in Σ. Consider the map on $\mathcal{A} \hat{\otimes} \mathcal{A}$ given by $(a \otimes b)^0 = b \otimes a$ $(a, b \in \mathcal{A})$. Put $A = \{u \in \mathcal{A} \hat{\otimes} \mathcal{A} | u^0 = -u\}$, and $S = \{u \in \mathcal{A} \hat{\otimes} \mathcal{A} | u^0 = u\}$. Then $\mathcal{A} \hat{\otimes} \mathcal{A} = A \oplus S$, and $\mathcal{A}$ is WA if and only if

$$((A \cap K) \cdot K \oplus \mathcal{A} \otimes e)^- = A \cap K + S.$$

This formula is neither as pretty nor probably as useful as the corresponding one in the commutative case. It illustrates that the situation is considerably more complicated in the non-commutative case.

Having focussed on the homological invariant $\mathcal{A} \mapsto H^1(\mathcal{A}, \mathcal{A}^*)$, it is an immediate task to determine which constructions preserve it. We set forth to investigate conditions under which the taking of extensions, embedded ideals, and homomorphic images preserve WA. We shall also illustrate that under certain conditions WA is preserved by the formation of Banach algebraic free products. The latter will give a unified approach to establishing WA for discrete group algebras and C^*-algebras.

It is easy to see that, if $D : \mathcal{A} \to \mathcal{A}^*$ is a derivation of a C^*-algebra or a group algebra, then D satisfies $\langle a, D(b)\rangle + \langle b, D(a)\rangle = 0$ $(a, b \in \mathcal{A})$. (Here and throughout $\langle -, -\rangle$ will denote the canonical pairing between $\mathcal{A}$ and $\mathcal{A}^*$.) Derivations with this property are called *cyclic*. Clearly inner derivations are cyclic. The corresponding subgroup of $H^1(\mathcal{A}, \mathcal{A}^*)$ will be denoted by $H^1_\lambda(\mathcal{A})$. It is the first order cyclic cohomology group in the theory of cyclic cohomology developed by A. Connes in [3], and independently from the point of view of homology by B.L. Tzygan in [12].

DEFINITION 0.3. A Banach algebra $\mathcal{A}$ is called *cyclicly amenable*, (CA), if

$$H^1_\lambda(\mathcal{A}) \;=\; (0).$$

The property CA is in general not nearly as restrictive as WA. We shall show that, if $\mathbf{F}_X$ is the free semigroup on a set X, then $\ell^1(\mathbf{F}_X)$ is CA. Consequently every Banach algebra is a quotient of a CA Banach algebra. However, for important classes of Banach algebras (e.g. C^*-algebras and group algebras) the two concepts coincide. Since the hereditary properties of CA are somewhat better than of WA this will be important.

1. PRELIMINARIES

This chapter consists mainly of definitions of the concepts involved. Throughout $\mathcal{A}$ denotes a Banach algebra and $\mathcal{A}^\# = \mathcal{A} \oplus \mathbf{C}$ its unitization. All linear maps are continuous unless otherwise stated. Although the proper setting of the concepts dealt with is that of full (Hochschild-Johnson) cohomology we shall limit ourselves to dimensions $n = 0, 1$. We remind the reader that, if M is a Banach $\mathcal{A}$-module, then $H^0(\mathcal{A}, M) = \{m \in M | a \cdot m = m \cdot a \ \ (a \in \mathcal{A})\}$. In particular $H^0(\mathcal{A}, \mathcal{A}^*)$ is the space of (bounded) traces on $\mathcal{A}$.

DEFINITION 1.1. Let I be a closed two-sided ideal of $\mathcal{A}$. We say that I has *property* ET (with respect to $\mathcal{A}$) if the restriction map $H^0(\mathcal{A}, \mathcal{A}^*) \to H^0(\mathcal{A}, I^*)$ is surjective, that is, if every $m \in I^*$, satisfying $a \cdot m = m \cdot a$ $(a \in \mathcal{A})$, can be extended to a trace on $\mathcal{A}$.

The following concept is important for extension of linear maps.

DEFINITION 1.2. Suppose $\mathcal{A}$ has a bounded approximate identity $(e_\gamma)_{\gamma \in \Gamma}$. Then $(e_\gamma)_{\gamma \in \Gamma}$ is called *quasi-central* if $e_\gamma M - M e_\gamma \to 0$ for each multiplier M on $\mathcal{A}$.

It is well known, see for instance Appendix 3 of [11], that if $\mathcal{A}$ has a quasi-central bounded approximate identity $(e_\gamma)_\Gamma$, then $be_\gamma - e_\gamma b \to 0$ for b belonging to any Banach algebra containing $\mathcal{A}$ as an ideal. Examples of Banach algebras with quasi-central bounded approximate identities are Arens regular Banach algebras with bounded approximate identities, in particular C^*-algebras.

We shall now give various results concerning hereditary properties of WA and CA. The proofs are all fairly straightforward, and will be omitted in most cases. For a more detailed account, see [6].

PROPOSITION 1.3. *Let I be a closed ideal in a Banach algebra $\mathcal{A}$. Assume that I has a quasi-central bounded approximate identity. Then I has the* ET*-property with respect to $\mathcal{A}$ and every derivation $D : I \to I^*$ can be lifted to a derivation $\tilde{D} : \mathcal{A} \to \mathcal{A}^*$.*

"Proof". The extensions alluded to in the proposition are made by passing to weak* limits of nets of the form $(e_\gamma \cdot f)_{\gamma \in \Gamma}$ where $(e_\gamma)_{\gamma \in \Gamma}$ is a quasi-central bounded approximate identity and f is a functional on I. The quasi-centrality, in the case of a derivation together with Cohen factorization, will ensure that the extended map has the desired property.

PROPOSITION 1.4. *Let G be a discrete group and let N be a normal subgroup. Then the kernel of the canonical homomorphism $\ell^1(G) \to \ell^1(G/N)$ has the* ET *property.*

Proof. We represent elements in $\ell^1(G)$ as formal sums $\sum_{g\in G} \lambda(g)g, \lambda(g) \in \mathrm{C}$, $\sum_{g\in G} |\lambda(g)| < \infty$. The kernel I is

$$I = \mathrm{clspan}\{h - h' | h \equiv h' \pmod N\}.$$

Let $m \in H^0(\ell^1(G), I^*)$, and suppose $s, t \in \ell^1(G)$ satisfy $st \equiv ts \pmod N$. Then

$$\begin{aligned} m(s^{n+1}ts^{-n} - ts) &= \sum_{i=0}^{n} m(s^{i+1}ts^{-i} - s^i ts^{-i+1}) \\ &= \sum_{i=0}^{n} m(st - ts) = (n+1)m(st - ts), \end{aligned}$$

so that $m(st - ts) = 0$. Define an equivalence on G by $h \sim g \Leftrightarrow g \in \gamma^{-1}h\gamma N$ for some $\gamma \in G$, and let C be a set consisting of exactly one element from each equivalence class. Define an element $\tau \in \ell^\infty(G)$ by

$$\tau(g) = m(g - \gamma^{-1}c\gamma) \quad (g \in G),$$

where $c \in C$ and $g \in \gamma^{-1}c\gamma N$. This is well defined: if $\gamma_1^{-1}c\gamma_1 \equiv \gamma_2^{-1}c\gamma_2 \pmod N$, then

$$m(\gamma_1^{-1}c\gamma_1 - \gamma_2^{-1}c\gamma_2) = m(c - (\gamma_1\gamma_2^{-1}c)\gamma_2\gamma_1^{-1}) = 0.$$

Obviously τ extends m, since we may choose the same γ for g and g' in the definition of τ where $g \equiv g' \pmod N$, and τ is equally easily shown to be a trace. Suppose $h_1 = ghg^{-1}$. Then

$$\begin{aligned} \tau(h) - \tau(h_1) &= m(h - \gamma^{-1}c\gamma) - m(ghg^{-1} - g\gamma^{-1}c\gamma g^{-1}) \\ &= m(h - \gamma^{-1}c\gamma) - m(g(h - \gamma^{-1}c\gamma)g^{-1}) = 0. \end{aligned}$$

We shall now define two types of Banach algebraic free products.

DEFINITION 1.5. Let $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ be a family of Banach algebras. A *Banach algebraic free product* of $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ is a Banach algebra $\mathcal{F}$, satisfying the following universal property. There are isometries $i_\gamma : \mathcal{A}_\gamma \to \mathcal{F}$ so that, whenever $\varphi_\gamma : \mathcal{A}_\gamma \to \mathcal{B}$ is a family of bounded homomorphisms into a Banach algebra $\mathcal{B}$ with

$$\sup\{\|\varphi_\gamma\| | \gamma \in \Gamma\} \leq 1,$$

there is a *unique* bounded homomorphism $\varphi : \mathcal{F} \to \mathcal{B}$ so that the diagrams

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\varphi} & \mathcal{B} \\ i_\gamma \nwarrow & & \nearrow \varphi_\gamma \\ & \mathcal{A}_\gamma & \end{array} \qquad (\gamma \in \Gamma)$$

all commute.

For existence and uniqueness (up to isomorphism), see [2]. We use the notation $\mathop{\hat{*}}_{\gamma\in\Gamma} \mathcal{A}_\gamma$ for $\mathcal{F}$. Likewise, we may define a *unital Banach algebraic free product* of a family of unital Banach algebras $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ as a unital Banach algebra $\mathcal{U}$ which satisfies the same universal properties as above, now for unital Banach algebras and unital homomorphisms. $\mathcal{U}$ may be represented as $(\mathop{\hat{*}}_{\gamma\in\Gamma} M_\gamma)^\#$ where M_γ is a (maximal) two-sided ideal of $\mathcal{A}_\gamma$ such that $\mathcal{A}_\gamma = \mathbf{C}e_\gamma \oplus M_\gamma$, where e_γ is the unit of $\mathcal{A}_\gamma$. The usual arguments show that unital Banach algebraic free products are unique up to isomorphism.

EXAMPLE 1.6. We wish to give a representation of Banach algebraic free products. Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras. Then $\mathcal{A} \hat{*} \mathcal{B}$ is defined as

$$\mathcal{A} \hat{*} \mathcal{B} = \bigoplus_{i=1}^{\infty} A_n \bigoplus_{i=1}^{\infty} B_n,$$

where

$$A_n = \mathcal{A} \hat{\otimes} \mathcal{B} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{B} \ldots \quad (n \text{ factors})$$

and

$$B_n = \mathcal{B} \hat{\otimes} \mathcal{A} \hat{\otimes} \mathcal{B} \otimes \ldots \quad (n \text{ factors})$$

as Banach spaces. Here $\oplus$ denotes ℓ^1-direct sum. The algebra product on $\mathcal{A} \hat{*} \mathcal{B}$ is given by tensor product modulo the relations $a_1 \otimes a_2 = a_1 a_2$, $b_1 \otimes b_2 = b_1 b_2$ ($a_i \in \mathcal{A}$, $b_i \in \mathcal{B}$, $i = 1, 2$). For further details, see [2].

In general, we define

$$\mathcal{A}_1 \hat{*} \ldots \hat{*} \mathcal{A}_n \quad \text{as} \quad (\mathcal{A}_1 \hat{*} \ldots \hat{*} \mathcal{A}_{n-1}) \hat{*} \mathcal{A}_n \quad (n \geq 2),$$

and for an arbitrary family, we define

$$\hat{*}_{\gamma \in \Gamma} \mathcal{A}_\gamma = \underset{\longrightarrow F \in \mathcal{P}_0(\Gamma)}{\lim} \big(\hat{*}_{\gamma \in F} \mathcal{A}_\gamma \big),$$

where $\mathcal{P}_0(F)$ is the set of finite subsets of Γ and the direct limit is taken with respect to the inclusion order on $\mathcal{P}_0(F)$ and the canonical embeddings

$$\hat{*}_{\gamma \in F_1} \mathcal{A}_\gamma \hookrightarrow \hat{*}_{\gamma \in F_2} \mathcal{A}_\gamma \quad (F_1 \subseteq F_2).$$

EXAMPLE 1.7. If G_1 and G_2 are two groups, then the unital Banach algebraic free product of $\ell^1(G_1)$ and $\ell^1(G_2)$ is $\ell^1(G_1 \cdot G_2)$, where $G_1 \cdot G_2$ is the free group product of the groups G_1 and G_2, whereas $\ell^1(G_1) \hat{*} \ell^1(G_2) = \ell^1(G_1 \hat{*} G_2)$ where $G_1 \hat{*} G_2$ is the free semigroup product of the two (semi-) groups G_1 and G_2.

2. WEAK AND CYCLIC AMENABILITY OF EXTENSIONS AND FREE PRODUCTS

Given a derivation $D : \mathcal{A} \to \mathcal{A}^*$ it is natural to ask if D can be extended to a derivation $\tilde{D} : \mathcal{A}^\# \to (\mathcal{A}^\#)^*$. The answer is given by the following.

PROPOSITION 2.1. *Let $D : \mathcal{A} \to \mathcal{A}^*$ be a derivation. Then D can be extended to a derivation $\tilde{D} : \mathcal{A}^\# \to (\mathcal{A}^\#)^*$, if and only if there is a constant $K \geq 0$ so that*

$$\left| \sum_{i=0}^{n} (\langle a_i, D(b_i) \rangle + \langle b_i, D(a_i) \rangle) \right| \leq K \left\| \sum_{i=0}^{n} a_i b_i \right\|$$

($a_i, b_i \in \mathcal{A}$; $i = 1, \ldots, n$).

Proof. Suppose D can be extended. Then

$$\begin{aligned}\left|\sum_{i=0}^{n}(\langle a_i, D(b_i)\rangle + \langle b_i, D(a_i)\rangle)\right| &= \left|\left\langle \mathbf{1}, \tilde{D}\left(\sum_{i=0}^{n} a_i b_i\right)\right\rangle\right| \\ &\leq \|\mathbf{1}\|\|\tilde{D}\|\left\|\sum_{i=0}^{n} a_i b_i\right\|.\end{aligned}$$

Conversely, suppose the inequalities hold and define a linear functional by $\varphi(ab) = \langle a, D(b)\rangle + \langle b, D(a)\rangle$ $(a, b \in \mathcal{A})$. The inequalities imply that φ an be extended to a bounded linear functional on $\mathcal{A}$. We identify $(\mathcal{A}^{\#})^*$ with $\mathbf{C} \oplus \mathcal{A}^*$. Then the natural action is given by $(\lambda\mathbf{1} + a) \cdot (\mu, f) = (\lambda\mu + \langle a, f\rangle, \lambda f + a \cdot f)$ $(\lambda, \mu \in \mathbf{C}; a \in \mathcal{A}; f \in \mathcal{A}^*)$. Put

$$\tilde{D}(\lambda\mathbf{1} + a) = (\varphi(a), D(a)) \quad (\lambda \in \mathbf{C}, a \in \mathcal{A}).$$

A routine calculation verifies that $\tilde{D}$ is a bounded derivation, extending D.

COROLLARY 2.2. *If $\mathcal{A}$ has a bounded two-sided approximate unit, then every derivation $D : \mathcal{A} \to \mathcal{A}^*$ can be extended to a derivation $\tilde{D} : \mathcal{A}^{\#} \to (\mathcal{A}^{\#})^*$.*

Proof. Let $(e_i)_{i\in I}$ be a bounded two-sided approximate identity for $\mathcal{A}$ and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$. Then

$$\begin{aligned}\left|\sum_{i=0}^{n}(\langle a_k, D(b_k)\rangle + \langle b_k, D(a_k)\rangle\right| &= \lim_i \left|\sum_{i=0}^{n}(\langle e_i a_k, D(b_k)\rangle + \langle b_k e_i, D(a_k)\rangle)\right| \\ &= \lim_i \left|\sum_{i=0}^{n}\langle e_i, D(a_k b_k)\rangle\right| \\ &\leq \limsup_i \|e_i\|\|D\|\left\|\sum_{i=0}^{n} a_k b_k\right\|.\end{aligned}$$

COROLLARY 2.3. *If $D : \mathcal{A} \to \mathcal{A}^*$ is a cyclic derivation, then D can be extended to $\tilde{D} : \mathcal{A}^{\#} \to (\mathcal{A}^{\#})^*$. In particular, if there are no bounded traces on $\mathcal{A}$, then every derivation $D : \mathcal{A} \to \mathcal{A}^*$ can be extended.*

Proof. Clearly cyclic derivations can be extended. If $\mathcal{A}$ has no bounded traces, it follows from the long exact sequence of A. Connes, [3],

$$\ldots \to 0 \to H^1_\lambda(\mathcal{A}) \to H^1(\mathcal{A}, \mathcal{A}^*) \to H^0_\lambda(\mathcal{A}) \to H^2_\lambda(\mathcal{A}) \to \ldots$$

that every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is cyclic.

In view of Corollary 2.3, and for later purposes, it is of interest to know for which Banach algebras all derivations into the dual module are cyclic. Concerning this, we have:

LEMMA 2.4. *Let $\mathcal{A}$ be a C^*-algebra or a discrete group algebra. Then every derivation $D : \mathcal{A} \to \mathcal{A}^*$ is cyclic.*

Proof. We only prove the case when $\mathcal{A}$ is a C^*-algebra. The other case is similar. By Corollary 2.2 we may assume that $\mathcal{A}$ is unital. Hence we must prove that $\langle 1, D(a)\rangle = 0$ $(a \in \mathcal{A})$. If a is normal this is an obvious statement about commutative C^*-algebras, so it clearly holds for any $a \in \mathcal{A}$.

EXAMPLE 2.5. Let $\mathcal{A}^2 = (0)$ and choose $f \in \mathcal{A}^* \backslash (0)$. It is immediate to verify that the map $D : \mathcal{A} \to \mathcal{A}^*$, given by $Da = f(a)f$ $(a \in \mathcal{A})$, is a derivation, and since $\langle a, D(b)\rangle + \langle b, D(a)\rangle = 2f(a)f(b)$ it is clear that D cannot be extended to $\mathcal{A}^\#$.

We now turn to a description of hereditary properties of WA and CA under extensions. Consider the short exact sequence of Banach algebras and continuous homomorphisms

$$0 \to I \xrightarrow{j} \mathcal{A} \xrightarrow{\theta} \mathcal{B} \to 0.$$

We shall identify I with a closed two-sided ideal in $\mathcal{A}$. As the following proposition shows, CA and WA behave almost identically with respect to extensions.

PROPOSITION 2.6. *Consider the short exact sequence above. Then*

(i) *If $\mathcal{B}$ is* CA, *then I has* ET.

(ii) *If $\mathcal{A}$ is* WA (CA) *and I has* ET, *then $\mathcal{B}$ is* WA (CA).

(iii) *If $\mathcal{A}$ is* WA (CA) *and I has a quasi-central bounded approximate identity, then I is* WA (CA).

(iv) *If $\mathcal{B}$ is* WA *and I is* WA *(I is* CA *and $(I^2)^- = I$), then $\mathcal{A}$ is* WA (CA).

"Proof". The proofs of the various statements consist mostly of appropriate diagram chases. As an example let us prove (i):

Suppose $\mathcal{B}$ is CA. Let $f \in I^*$ satisfy $a \cdot f - f \cdot a = 0$ $(a \in \mathcal{A})$. We must prove that f can be extended to a trace on $\mathcal{A}$. Let $\tilde{f} \in \mathcal{A}^*$ be any extension of f. Let $\delta : \mathcal{A} \to \mathcal{A}^*$ be the inner derivation generated by $\tilde{f}$ and define a derivation $D : \mathcal{B} \to \mathcal{B}^*$ by $\theta^* D\theta = \delta$. This map is well defined. Let $a, a' \in \mathcal{A}$ and $i, i' \in I$. Then

$$\begin{aligned}\langle a' + i', \delta(a+i)\rangle &= \langle a', \delta(a)\rangle + \langle i, \delta(a)\rangle + \langle a' + i', \delta(i)\rangle \\ &= \langle a', \delta(a)\rangle + \langle i', f \cdot a - a \cdot f\rangle - \langle i, f \cdot (a' + i') - (a' + i') \cdot f\rangle \\ &= \langle a', \delta(a)\rangle.\end{aligned}$$

Clearly D is a cyclic derivation, and since θ is an open map, D is bounded. The property CA of $\mathcal{A}$ then implies that there is $g \in \mathcal{B}^*$ so that

$$(\tilde{f} - \theta^*(g)) \cdot a - a \cdot (\tilde{f} - \theta^*(g)) = 0 \quad (a \in \mathcal{A}),$$

that is, $\tilde{f} - \theta^*(g)$ is a trace on $\mathcal{A}$. Since $\theta^*(g)(I) = (0)$ it follows that $\tilde{f} - \theta^*(g)$ extends the given functional $f \in I^*$.

EXAMPLE 2.7. One may ask, whether the condition $(I^2)^- = I$ is necessary in Proposition 2.6(iv). The following example shows that some condition is needed. Let

$R = \mathbf{C}$, with trivial algebra product. Then R is clearly CA. However, the split extension given by

$$0 \to R \to R \times R \to R \to 0$$

is not CA.

A simple example shows that if $\mathcal{A}$ and $\mathcal{B}$ are two Banach algebras, then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is never WA. The theorem to follow shows that this is caused by the asymmetry between the first and second entries in bilinear forms $\langle \cdot, D(\cdot)\rangle$ arising from non-cyclic derivations.

THEOREM 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two* CA *Banach algebras and let $\mathcal{F}$ be their (possibly unital) Banach algebraic free product. Then $\mathcal{F}$ is* CA.

"Proof". We shall illustrate the proof by showing that, if $\ell^1(G_1)$ and $\ell^1(G_2)$ are CA, then $\ell^1(G_1 \cdot G_2)$ is CA, where $G_1 \cdot G_2$ is the free group product of G_1 and G_2. Elements of $G_1 \cdot G_2$ will be called *words*, composed of *letters* from the two groups G_1 and G_2. Each word $w \in G_1 \cdot G_2$ has a unique representation $w = h_1 \dots h_n$, where no two adjacent letters h_i and h_{i+1} $(i = 1, \dots, n-1)$ are from the same group. We shall always represent words in this standard form.

Let $D : \ell^1(G_1 \cdot G_2) \to \ell^\infty(G_1 \cdot G_2)$ be a derivation. By assumption there are functionals $m_i \in \ell^\infty(G_i)$ $(i = 1, 2)$ such that

$$\langle g_i', D(g_i'')\rangle = m_i(g_i''g_i' - g_i'g_i'') \quad (g_i', g_i'' \in G_i).$$

First we extend m_i to functionals $\tilde{m}_i \in \ell^\infty(G_i \cdot G_2)$ so that

$$\langle w, D(g_i)\rangle = \tilde{m}_i(g_i w - w g_i) \quad (w \in G_1 \cdot G_2, g_i \in G_i, \quad i = 1, 2).$$

To be definite, let us extend m_1. Let w be any word in $G_1 \cdot G_2 \backslash G_1$. Let h be the first letter of w and write $w = hw'$.

Define

$$\lambda_1(w) = \begin{cases} 0, & \text{if } h \in G_2 \\ \langle w', D(h)\rangle & \text{if } h \in G_1 \ . \end{cases}$$

Now, let $g \in G_1$. If $h \in G_2$ we get

$$\begin{aligned}\langle w, D(g)\rangle &= \lambda_1(gw)\\ &= \lambda_1(gw - wg),\end{aligned}$$

since the first letter of wg is from G_2. If $h \in G_1$, then the first letter of w' is from G_2. So in this case we get

$$\begin{aligned}\langle w, D(g)\rangle &= \langle hw', D(g)\rangle\\ &= \langle w', D(gh)\rangle - \langle w'g, D(h)\rangle\\ &= \lambda_1(ghw' - w'gh) - \lambda_1(hw'g - w'gh)\\ &= \lambda_1(gw - wg),\end{aligned}$$

by what we have just proved about words beginning with a letter from G_2. Let $\tilde{m}_1$ be the join of m_1 and λ_1.

Hence, by subtracting the inner derivation generated by $\tilde{D}_2$ we may assume that D has the form

$$D(g_1) = m \cdot g_1 - g_1 \cdot m \quad (g_1 \in G_1) \tag{1}$$

$$D(g_2) = 0 \qquad\qquad (g_2 \in G_2) \tag{2}$$

for a certain function $m \in \ell^\infty(G_1 \cdot G_2)$. To prove that D is inner, we must show that m has a decomposition

$$m = \mu_1 + \mu_2, \tag{3}$$

where $\mu_1 \cdot g_1 - g_1 \cdot \mu_1 = \mu_2 \cdot g_2 - g_2 \cdot \mu_2 = 0$ ($g_i \in G_i$, $i = 1, 2$). Let

$$S_n^i = \{w \in G_1 \cdot G_2 | w \text{ has } n \text{ letters, first letter is from } G_i\}.$$

We regard S_n^i as subsets of $\ell^1(G_1 \cdot G_2)$. Let $\alpha_1\beta_1 \ldots \alpha_n\beta_n \in S_{2n}^1$. Since $D(G_2) = \{0\}$ and D is cyclic we have $\langle \beta_n, D(\alpha_1\beta_1 \ldots \alpha_n)\rangle = 0$.

Using the derivation identity and (1) and (2) we get

$$0 = \left\langle \beta_n, \sum_{i=1}^{n} \alpha_1 \ldots \beta_{i-1} \left[m \cdot \alpha_i - \alpha_i \cdot m \right] \beta_i \ldots \alpha_n \right\rangle$$
$$= \sum_{i=1}^{n} m(\alpha_i \ldots \beta_n \alpha_1 \ldots \beta_{i-1}) - m(\beta_i \ldots \beta_n \alpha_1 \ldots \alpha_i),$$

so that

$$\sum_{i=1}^{n} m(\alpha_i \ldots \beta_n \alpha_1 \ldots \beta_{i-1}) = \sum_{i=1}^{n} m(\beta_i \ldots \beta_n \alpha_1 \ldots \alpha_i). \tag{4}$$

We now define a bounded linear map on $\ell^1(G_1 \cdot G_2)$ by $\sigma(t_1 \ldots t_n) = t_2 \ldots (t_n t_1)$ for $n \geq 2$, where the bracket indicates that the right hand side does not have the standard form if n is odd, and put $\sigma(G_1 \cup G_2) = \{0\}$.

The functions μ_1 and μ_2 will be defined stepwise by gradually extending their domains. On $S^1_{2n} \cup S^2_{2n}$ we first define μ_1 on range(id-σ) by

$$\mu_1 \cdot (\mathrm{id} - \sigma) = m \cdot (\mathrm{id} - \sigma) p_1$$

where p_1 is the projection on S^1_{2n} along S^2_{2n}. This is well-defined, for suppose $(\mathrm{id} - \sigma)(w_1 + w_2) = 0$, where $w_i \in S^i_{2n}$ $(i = 1, 2)$. Since σ interchanges S^1_{2n} and S^2_{2n}, this means that $\sigma^2(w_i) = w_i$ $(i = 1, 2)$. Hence

$$\begin{aligned} m \cdot (\mathrm{id} - \sigma) p_1(w_1 + w_2) &= m(w_1 - \sigma(w_1)) \\ &= \frac{1}{n} \sum_{i=0}^{n-1} m(\sigma^{2i} w_1 - \sigma^{2i+1} w_1) \\ &= 0, \end{aligned}$$

by the identity (4). Then we extend μ_1 to all of $S^1_{2n} \cup S^2_{2n}$ by the Hahn-Banach theorem.

Define μ_2 on $S^1_{2n} \cup S^2_{2n}$ by means of the equation $m = \mu_1 + \mu_2$. On S^1_{2n+1} $(n \geq 1)$ we put

$$\mu_1(w) = \mu_1(\sigma(w)) \quad (w \in S^1_{2n+1}).$$

This is well-defined since $\sigma = S^1_{2n+1} \to S^2_{2n}$ and μ_1 has already been defined on S^2_{2n}. Similarly define on S^2_{2n+1}:

$$\mu_2(w) = \mu_2(\sigma(w)) \quad (w \in S^2_{2n+1}).$$

Finally, put $\mu_1 = m$ on $S^1_1 \cup S^2_2 \cup \{e\}$ and use thereafter the equation $m = \mu_1 + \mu_2$ to define μ_1 and μ_2 on all of $G_1 \cdot G_2$.

By construction μ_1 and μ_2 satisfy the prescribed commutator relations. However, they may not be bounded as functions on $G_1 \cdot G_2$. But we do have the relation

$$\langle w', D(w'')\rangle = \mu_2(w''w' - w'w'')$$

so the variation of μ_2 on each conjugacy class is bounded by $\|D\|$. If necessary, we may subtract a suitable function which is constant on conjugacy classes (i.e. an (unbounded) trace) to ensure that D is generated by a bounded functional.

Now a rather straightforward induction shows that if $\mathcal{F}$ is a (unital) Banach algebraic free product of a family $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ of CA Banach algebras, then $\mathcal{F}$ is CA, provided one can choose generators f_γ for any given derivations $D_\gamma : \mathcal{A}_\gamma \to \mathcal{A}^*_\gamma$ such that

$$\sup\left\{ \frac{\|D_\gamma\|}{\|f_\gamma\|} \middle| \gamma \in \Gamma \right\} < \infty.$$

This will certainly be the case if all $\mathcal{A}_\gamma$'s are commutative. This gives us the corollaries.

COROLLARY 2.9. *Every Banach algebra $\mathcal{A}$ is a homomorphic image of a* CA *Banach algebra, which can be chosen to be unital and/or separable if $\mathcal{A}$ is.*

Proof. It is easy to see that the discrete convolution algebra $\ell^1(\mathbb{N})$ is CA. By the remark following the proof of Theorem 2.8 every discrete convolution algebra on a free semigroup $\mathbb{F}_X$, being the Banach algebraic free product $\hat{\ast}_{x\in X} \ell^1(\mathbb{N}_x)$ ($\mathbb{N}_x$ a copy of $\mathbb{N}$) is CA. Since every Banach algebra is a homomorphic image of an appropriate $\ell^1(\mathbb{F}_X)$, where X may be taken to be countable if $\mathcal{A}$ is separable, the result follows.

COROLLARY 2.10. (B.E. Johnson) *Every convolution algebra on a discrete group is* WA.

Proof. By the remark following Theorem 2.8 it is true for all free groups. Invoke Proposition 1.4.

COROLLARY 2.11. (U. Haagerup) *All C^*-algebras are* WA.

Proof. Let $\mathcal{A}$ be a C^*-algebra. We may assume that $\mathcal{A}$ is unital. Let $(\mathcal{A}_\gamma)_{\gamma\in\Gamma}$ be the family of unital commutative C^*-subalgebras of $\mathcal{A}$. Then $\hat{\ast}_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is CA. Using Theorem 2.6 of [4] and the Grothendieck-Haagerup inequality [7] one can show that the kernel of the canonical map $\hat{\ast}_{\gamma\in\Gamma} \mathcal{A}_\gamma \to \mathcal{A}$ has the ET property. Since all derivations from a C^*-algebra are cyclic, the result follows.

REMARK. As the two corollaries show, it is pertinent to find methods to prove that some given ideal has the ET property. Suppose for convenience that $\mathcal{A}$ is unital and let $m \in H^0(\mathcal{A}, I^*)$. Let $\mathcal{K} = \{f \in \mathcal{A}^* | f_{|I} = m,\ \|f\| = \|m\|\}$. Then $\mathcal{K}$ is non-empty, weak*-compact and convex. Let $g \in \mathrm{Inv}(\mathcal{A})$ and denote by T_g the inner automorphism given by g. Since $m \in H^0(\mathcal{A}, I^*)$ we have the inclusion $T_g^*(\mathcal{K}) \subseteq \mathcal{K}$. Hence, if one could find a common fixed point of some appropriate subgroup of $\mathrm{Inv}(\mathcal{A})$ we would have obtained a trace extension of m.

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