ULTRAPRIME GROUP ALGEBRAS

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Let A be a Banach algebra and for each pair, (a,b), of elements of A define a map $M_{a,b} : A \to A$ by $M_{a,b}(x) = axb$. Then A is said to be *prime* if $M_{a,b} \neq 0$ whenever a and b are non-zero, and to be *ultraprime* if there is a constant, $K > 0$, such that $\|M_{a,b}\| \geq K\|a\|\|b\|$ for every (a,b), see [5]. The centre of A will be denoted $Z(A)$.

Let G be a discrete group and let $\ell^1(G)$ denote its group algebra. It is easily checked that a function, f, belongs to $Z(\ell^1(G))$ if and only if f is constant on the conjugacy classes of G, that is, if and only if $f(x) = f(yxy^{-1})$ for every $x,y \in G$. Hence δ_e, the point mass at the identity element, belongs to $Z(\ell^1(G))$, and $Z(\ell^1(G))$ has dimension greater than one if and only if G has a finite conjugacy class other than $\{e\}$. These observations will be useful in the proof of the following.

PROPOSITION. (i) *If $Z(\ell^1(G))$ has dimension greater than one, then $\ell^1(G)$ is not prime.*

(ii) *If the dimension of $Z(\ell^1(G))$ equals one then $\ell^1(G)$ is ultraprime.*

Proof. (i) $Z(\ell^1(G))$ is semisimple because it has a faithful $*$-representation on $\ell^2(G)$. Hence, if $Z(\ell^1(G))$ has dimension greater than one, then there are two distinct points, p and q, in the space X of multiplicative linear functionals on $Z(\ell^1(G))$. Since X is Hausdorff there are open sets $U,V \subset X$ such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. By Theorem 1.8 in [4], $Z(\ell^1(G))$ is a regular Banach algebra and so there exist $a,b \in Z(\ell^1(G))$ such that $\hat{a}(p) = 1$ and $\hat{a}$ is zero outside U, $\hat{b}(q) = 1$ and $\hat{b}$ is zero outside V. Hence a and b are non-zero and ab = 0. Since $a,b \in Z(\ell^1(G))$, $M_{a,b} = 0$ and so $\ell^1(G)$ is not prime.

(ii) The second part will be proved in a sequence of lemmas.

LEMMA 1. *Let* G *be an infinite group and* H_i, $i = 1,2,..,n$ *be subgroups with infinite index in* G. *Then any set of the form* $\bigcup\limits_{i=1}^{n} A_i$, *where each* A_i *is the union of a finite number of cosets of* H_i, *is a proper subset of* G.

Proof. The proof is by induction on n. The statement of the lemma is true when $n = 1$ because G cannot be the union of a finite number of cosets of a subgroup, H_1, which has infinite index in G.

Suppose the lemma is true when $n = k$. For each $i = 1,2,..., k + 1$ let A_i be a union of finitely many cosets of a subgroup H_i of infinite index in G. In particular, there exist $x_1,...,x_\ell \in G$ with $A_{k+1} = \bigcup\limits_{j=1}^{\ell} H_{k+1}x_j$, where H_{k+1} has infinite index. Then we may choose $y \in G$ such that $y^{-1} \notin A_{k+1}$. Since A_{k+1} is a union of cosets, it follows that $hy^{-1} \notin A_{k+1}$ for every $h \in H_{k+1}$.

Suppose, for a contradiction, that $G = \bigcup\limits_{i=1}^{k+1} A_i$. Then $hy^{-1} \in \bigcup\limits_{i=1}^{k} A_i$ for each $h \in H_{k+1}$. Hence $H_{k+1} \subseteq \bigcup\limits_{i=1}^{k} A_i y$ and so $A_{k+1} = \bigcup\limits_{j=1}^{\ell} H_{k+1}x_j \subseteq \bigcup\limits_{j=1}^{\ell} (\bigcup\limits_{i=1}^{k} A_i y)x_j = \bigcup\limits_{i=1}^{k} B_i$, where each B_i is the union of a finite number of cosets of H_i. It follows that $G = \bigcup\limits_{i=1}^{k} C_i$, where each C_i is the union of a finite number of cosets of H_i, which contradicts the induction hypothesis. It follows that the result is true for each $n \geq 1$.

LEMMA 2. *Let* G *be an infinite group which has no finite conjugacy classes other then* $\{e\}$. *Then for every finite set* A *contained in* $G\setminus\{e\}$, *there is* $x \in G$ *such that* $x^{-1}Ax \cap A = \emptyset$.

Proof. For each $y \in A$, let $H_y = \{g \in G \,|\, g^{-1}yg = y\}$. Then, since the conjugacy class of y is infinite, H_y is a subgroup with infinite index in G. Let $A_y = \{g \in G \,|\, g^{-1}yg \in A\}$. Then A_y is the union of a finite number of cosets of H_y.

Suppose that, for every $x \in G$, $x^{-1}Ax \cap A \neq \emptyset$. Then for every $x \in G$, there is

$y \in A$ such that $x^{-1}yx \in A$, that is, for every $x \in G$, there is $y \in A$ such that $x \in Ay$. It follows that $G = \bigcup_{y \in A} A_y$, but this contradicts lemma 1. Therefore there is $x \in G$ such that $x^{-1}Ax \cap A = \emptyset$.

LEMMA 3. *Let G be an infinite group which has no finite conjugacy classes other that $\{e\}$. Then for every pair, C, D, of finite sets contained in G, there is an $x \in G$ such that the map*

$$\pi_x : C \times D \to G \; ; \; (c,d) \mapsto cxd$$

is an injection.

PROOF. Suppose that $\pi_x(c_1,d_1) = \pi_x(c_2,d_2)$ for some x, c_i and d_i where $(c_1,d_1) \neq (c_2,d_2)$. Then $c_1 x d_1 = c_2 x d_2$, which is equivalent to $x^{-1}c_2^{-1}c_1 x = d_2 d_1^{-1}$. If we let $A = \{c_2^{-1}c_1, d_2 d_2^{-1} | c_i \in C$ and $d_i \in D, \; i = 1,2\} \setminus \{e\}$, then this shows that $\pi_x : C \times D \to G$ is an injection if $x^{-1}Ax \cap A = \emptyset$. Now, since A is finite there is, by lemma 2, an $x \in G$ such that $x^{-1}Ax \cap A = \emptyset$. Therefore, for this x, π_x is an injection.

Proof of Proposition (ii). Now let G be an infinite group such that the dimension of $Z(\ell^1(G))$ is one. Then G has no finite conjugacy classes other than $\{e\}$. Let $a, b \in \ell^1(G)$ with finite support, and denote their supports by C and D respectively. By lemma 3, there is $x \in G$ such that the map $(c,d) \mapsto cxd$ is an injection of $C \times D$ into G. It follows that $\|a * \delta_x * b\|_1 = \|a\|_1 \|b\|_1$ and so $\|M_{a,b}\| = \|a\|_1 \|b\|_1$. Since the functions with finite support are dense in $\ell^1(G)$, this holds for all pairs, $a,b \in \ell^1(G)$. Therefore $\ell^1(G)$ is ultraprime.

Remark. The group algebra of a discrete group is weakly amenable, see [3], and so the above proposition confirms a special case of a conjecture by the author that every prime, weakly amenable Banach algebra is ultraprime . This is a non-commutative

version (and strengthening) of the conjecture that an amenable integral domain must be the algebra of complex numbers, $\mathbb{C}$.

There is some evidence that the commutative version of the conjecture is true. Essentially the only method used to date to show that a commutative Banach algebra, A, is amenable is to produce a continuous homomorphism from $L^1(G)$, for some locally compact group abelian G, into A which has dense range. If A is the target of such a homomorphism and has dimension greater than two, then A has divisors of zero. Since the empty set and singletons are sets of synthesis for $L^1(G)$, A will have at least two distinct multiplicative linear functionals, and the construction of divisors of zero then goes as in part (i) of the proposition. From the other direction, the standard examples of integral domains, such as weighted convolution algebras on the positive half-line, have non-zero continuous derivations into their duals and so are not weakly amenable. Further, there are methods for constructing derivations (usually discontinuous) from integral domains, see [2], section 4, for example.

However, there is not much evidence for the non-commutative version of the conjecture other than the examples of ultraprime algebras given in section 3 of [5] and in the above proposition.

The conjecture for commutative algebras has a consequence for automatic continuity, but its non-commutative extension does not have any obvious consequence. If $\mathbb{C}$ were the only amenable integral domain, then no commutative, amenable Banach algebra could have a closed prime ideal with codimension other than one. Since every ideal with finite codimension in a commutative amenable Banach algebra is closed and has a bounded approximate identity, it would then follow from theorem 4.2 in [1] that derivations from amenable commutative Banach algebras are automatically continuous. The automatic continuity of derivations from amenable algebras which are not commutative would not follow immediately if it were known that every amenable prime algebra was ultraprime, because not enough is known

about derivations from amenable ultraprime Banach algebras. Indeed, it is not known in general whether derivations from $\ell^1(G)$ are continuous if $\ell^1(G)$ is amenable and ultraprime.

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