L^2 REIDEMEISTER FRANZ TORSION

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INTRODUCTION

In this paper we discuss an approach to the study of closed, connected, oriented manifolds, with infinite fundamental group which have a special property which we call L^2-acyclicity. The first object of our exposition is to summarise in an accessible form what we have established about such manifolds in [2]. As a result of discussions on this work which occurred during the conference we attempted to prove some new results and these are included in Section 4.

The main object of the first three sections is to introduce the definition of a new differential invariant of an L^2-acyclic manifold (as in [2]) called L^2-RF torsion. The theory of finite von Neumann algebras is an essential ingredient in the definition.

The new results in the final section enable us to compute the L^2-RF torsion for more L^2-acyclic manifolds.

To indicate why this study should have some general interest we begin with a conjecture which is suggested by our research.

CONJECTURE: *Let M be an odd dimensional, closed, connected, oriented manifold of negative sectional curvature. Then M is an L^2 acyclic manifold.*

The evidence for this conjecture is based on the following: Using the results of Donnelly and Xavier [9] and Dodzuik [8], it can be shown that if M is an odd dimensional, closed, connected, oriented manifold, with negative sectional curvature pinched between two sufficiently close negative constants, then M is an L^2-acyclic manifold.

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1. SOME ALGEBRA & THE FUGLEDE-KADISON DETERMINANT

Let π be a discrete group and $\ell^2(\pi)$ be the Hilbert space of square summable functions on π. Let $\mathbb{C}(\pi)$ denote the group algebra (over $\mathbb{C}$) of π i.e. $\mathbb{C}(\pi)$ consists of ($\mathbb{C}$ valued) functions on the group π which have finite support, and the algebra operations are addition and convolution.

Recall the left regular representation $\phi : \pi \to \text{Unit }(\ell^2(\pi)) \equiv \{$ Unitary operators on $\ell^2(\pi)\}$ defined by $(\phi(g)f)(h) = f(gh)$, which extends linearly to a representation $\phi : \mathbb{C}(\pi) \to B(\ell^2(\pi)) \equiv \{$ Bounded operators on $\ell^2(\pi)\}$.

The weak closure $\{\equiv$ von Neumann algebra generated by $\phi(\mathbb{C}(\pi))\}$ in $B(\ell^2(\pi))$ we call the group von Neumann algebra $\mathcal{U}(\pi)$. There is a natural trace on $\mathbb{C}(\pi)$ defined by

$$\tau\left(\sum_{g\in\pi} f_g g\right) \equiv f_e, e \equiv \{\text{identity element of}\pi\}$$

which extends uniquely to a trace on $\mathcal{U}(\pi)$ also denoted by τ. Then $\tau \otimes tr_n$ is the induced trace on

$$\mathcal{U}(\pi) \otimes M_n(\mathbb{C}) \cong M_n(\mathcal{U}(\pi))$$

which we again denote by τ.

We will now discuss the Fuglede-Kadison determinant of a finite Von Neumann algebra $\mathcal{U}$.

We shall stick to our earlier convention and denote the induced trace on $M_n(\mathcal{U})$ by τ. If $A \in Gl(n,\mathcal{U})$, then $A^*A \in Gl(n,\mathcal{U})$ is positive definite and $\log(A^*A) \in Gl(n,\mathcal{U})$ is self-adjoint. The Fuglede-Kadison determinant

$$|\text{Det}_\tau| : Gl(n,\mathcal{U}) \to \mathbb{R}^*_+$$

is defined by

$$|\text{Det}_\tau|(A) \equiv \exp[\tau(\log(A^*A))/2]$$

and it has the following properties:

(1) $|\text{Det}_\tau|(H) = \exp(\tau(\log(H)))$ if $H \in Gl(n,\mathcal{U})$, $H = H^*$ and $H \geq 0$.

(2) $|\text{Det}_\tau|(\lambda I) = |\lambda|, \quad \lambda \neq 0$.

(3) $|\text{Det}_\tau|(A) = |\text{Det}_\tau|(A^*) = |\text{Det}_\tau|((A^*A)^{1/2})$ if $A \in Gl(n,\mathcal{U})$.

(4) $|\text{Det}_\tau|(AB) = |\text{Det}_\tau|(A)|\text{Det}_\tau|(B)$ if $A, B \in Gl(n,\mathcal{U})$.

(5) $|\mathrm{Det}_\tau|(A) = |\mathrm{Det}_\tau|(UAU^*)$ if $A, U \in Gl(n,\mathcal{U})$ and U is unitary.

(6) $|\mathrm{Det}_\tau|(A) \leq |A|$ if $A \in Gl(n,\mathcal{U})$.

For us the most important fact about this determinant is that it has a non-trivial extension (i.e. not the algebraic one) to certain singular operators, namely those which are injective but whose range is not closed. If it were not for this the ensuing discussion would have little content. The point is that using the spectral representation $A^*A = \int \lambda d\mu(E_\lambda)$ the preceding definition still applies with the understanding that $|\mathrm{Det}_\tau|(A) = 0$ when $\int \ln\lambda d\mu(E_\lambda)$ is divergent. With obvious modifications all of the preceding properties continue to hold together with the following computationally useful result: for $B \in M_n(\mathcal{U})$, $B \geq 0$,

(7) $|\mathrm{Det}_\tau|(B) = \lim_{\epsilon\to 0}|\mathrm{Det}_\tau|(B+\epsilon)$

Note that [10] discusses the preceding results only for II_1 factors. The generalisation to the finite case is given in [6] except for the discussion of singular operators. However it is not hard to see that the argument in [10] for this extension holds also for finite von Neumann algebras.

Finally, a Hilbert $\mathcal{U}(\pi)$ module is defined to be a closed, π invariant subspace of $\oplus_{x\in X}\ell^2(\pi)$. For more details, we refer the reader to our paper [2].

2. L^2-RF TORSION

Let M be a closed, connected, oriented manifold and $\pi \equiv \pi_1(M)$ denote the fundamental group of M. Let $\widetilde{M} \to M$ denote the universal cover of M. We only consider Riemannian metrics on $\widetilde{M}$ which are induced from M. The space of L^2 differential p-forms M on $\widetilde{M}$ denoted $\Omega^p_{(2)}(\widetilde{M})$ is defined to be the Hilbert space completion of $\{\omega \in \Omega^p(\widetilde{M}) : \int_{\widetilde{M}} \omega \wedge *\omega < \infty\}$ where $\Omega^p(\widetilde{M})$ denotes the space of differential p-forms on $\widetilde{M}$. Let Δ_p denote the Laplace operator $d\delta + \delta d$ acting on p-forms. We consider weak (distributional) solutions to the equation $\Delta_p\omega = 0$ $(*)$ for $\omega \in \Omega^p_{(2)}(\widetilde{M})$. Since Δ_p is (an elliptic operator, it follows from standard theory that a weak solution to $(*)$ is also a strong solution to $(*)$, and by elliptic regularity results, is also smooth. It can similarly be shown that ker (Δ_p) is closed subspace of $\Omega^p_{(2)}(\widetilde{M})$, since Δ_p is π equivariant, it is a result of Atiyah [1] that ker (Δ_ρ) is a Hilbert $\mathcal{U}(\pi)$ module of finite π-dimension. Define

$$b^j_{(2)}(\widetilde{M}) \equiv \tau(\ker\Delta_j)$$

Let K be a triangulation of M and $\widetilde{M}$ be the induced triangulation of $\widetilde{M}$. Let $C^j_{(2)}(\widetilde{K})$ denote the space of j- cochains which are L^2 i.e. $C^j_{(2)}(\widetilde{K})$ is the Hilbert space completion of

$$\{f \in C^j(\widetilde{K}) : \sum_{\sigma = j \text{ simplex}} |f(\sigma)|^2 < \infty\}.$$

The coboundary operator d_K on $C^j(\widetilde{K})$ is easily seen to induce a bounded operator (also denoted by d_K)

$$d^j_K : C^j_{(2)}(\widetilde{K}) \to C^{j+1}_{(2)}(\widetilde{K})$$

which satisfies $d^{j+1}_K \circ d^j_K = 0$ i.e. $C_{(2)}((\widetilde{K}), d_K)$ is a complex of Hilbert $\mathcal{U}(\pi)$ modules. Define as in Dodzuik [7], the L^2- cohomology of this complex to be

$$H^j_{(2)}(\widetilde{K}) \equiv \ker d^j_K / \text{range } d^{j-1}_K$$

which is also a Hilbert $\mathcal{U}(\pi)$ complex by Dodzuiks theorem [7], ker (Δ_j) is isomorphic to $H^j_{(2)}(\widetilde{K})$ as Hilbert $\mathcal{U}(\pi)$ modules, i.e.

$$b^j_{(2)}(\widetilde{M}) = \tau(H^j_{(2)}(\widetilde{K})) \equiv b^j_{(2)}(\widetilde{K})(*)$$

and $b^j_{(2)}(\widetilde{M})$ are homotopy invariants of M.

We define an L^2-acyclic manifold to be a closed, connected oriented manifold M such that $b^j_{(2)}(\widetilde{M}) = 0$ for all $j \geq 0$.

The condition that M is an L^2-acyclic manifold is easily seen to be equivalent via $(*)$ to the condition that

$$d_K + d^*_K : C^{\text{odd}}_{(2)}(\widetilde{K}) \to C^{\text{even}}_{(2)}(\widetilde{K})$$

is a weak isomorphism i.e. $\ker(d_K + d^*_K) = 0$ and range $(d_K + d^*_K)$ is dense. Here d^*_K is the L^2 adjoint of d_K.

It follows that we can construct a $\mathcal{U}(\pi)$ module isomorphism of $C^{\text{odd}}_{(2)}(\widetilde{K})$ with $C^{\text{even}}_{(2)}(\widetilde{K})$ using the obvious bases of these free Hilbert $\mathcal{U}(\pi)$ modules. The operator

$d_K + d_K^*$ may be regarded then as an element of $\mathcal{U}(\pi)' \otimes M_n(\mathbb{C})$ where n is the rank of $C_{(2)}^{\text{odd}}$. Now this latter algebra is also a finite von Neumann algebra (as $\mathcal{U}(\pi)$ is anti-isomorphic to its commutant $\mathcal{U}(\pi)'$ which is generated by the right regular representation acting on $\ell^2(\pi)$). We shall continue to abuse the notation and denote by τ the trace on $\mathcal{U}(\pi)' \otimes M_n(\mathbb{C})$. Finally we may now define the L^2 RF-torsion by

$$T_{(2)}(M) \equiv |\mathrm{Det}_\tau|(d_K + d_K^*)$$

THEOREM [2]: *$T_{(2)}(M)$ is independent of the choice of the C^1 triangulations K of M, i.e. $T_{(2)}(M)$ is differential invariant of M.*

The proof is in the spirit of that of ordinary torsion, but with new technical difficulties arising from the fact that $d_K + d_K^*$ is only a weak isomorphism.

3. L^2-ACYCLIC MANIFOLDS

We now show that there are many L^2-acyclic manifolds, and also that the class of L^2-acyclic manifolds is closed under certain simple geometric extensions.

Recall that an A-foliated manifold is closed connected oriented manifold with a nowhere zero, closed 1-form.

EXAMPLE: Any manifold M which fibres over the circle.

THEOREM: [2] *If M is an A-foliated manifold, then M is an L^2-acyclic manifold.*

This theorem is proved essentially by supersymmetry ideas using a Witten type argument [15], and some L^2 estimates.

We digress at this point to recall some definitions (cf [13]). First we note that a Riemannian metric on a connected closed manifold M is said to be locally homogeneous if given any two points p and q in M there are open neighbourhoods of U and V of p and q respectively and an isometry $(U,p) \to (V,q)$. A closed manifold admits a geometric structure if M admits a locally homogeneous Riemannian metric on $X \equiv \widetilde{M}$ has a group of isometries G which acts transitively on X with a compact isotropy group. We then say (following Thurston) that M admits a geometric structure modelled on (X, G). Thurston [14]has classified all the three dimensional geometries; there are eight of them:

(i) Euclidean three space.

(ii) Hyperbolic three space.

(iii) $S^2 \times \mathbb{R}$.

(iv) The product of hyperbolic 2 space with $\mathbb{R}$.

(v) The universal cover of $SL(2, \mathbb{R})$.

(vi) The three dimensional Hiesenberg group.

(vii) The three dimensional solvable Lie group.

(viii) S^3.

In each case G is the group of isometries of the space.

Finally we recall that a countably generated discrete group π is said to be amenable if there is a finitely additive, left invariant measure on π.

THEOREM: [2] *Let M be a closed, connected, oriented three dimensional manifold with infinite fundamental group. Assume either that*

1. *$\pi_1(M)$ is amenable.*
2. *M admits a geometric structure.*

Then M is an L^2-acyclic manifold.

Proof:

1. is proved via a result of Cheeger and Gromov and L^2 Poincaré duality.
2. is proved via a case by case study.

This made us conjecture in [2] that any closed connected oriented 3-manifold, with infinite fundamental group, is an L^2-acyclic manifold.

A fibre bundle $F \to M \to B$ is said to be special if in the long exact sequence in homotopy

$$\to \pi_2(B) \xrightarrow{\partial} \pi_1(F) \xrightarrow{i} \pi_1(M) \to \pi_1(B) \to$$

we have $\ker i = \text{range } \partial = 0$. This implies that there is a fibre bundle of universal covers

$$\widetilde{F} \to \widetilde{M} \to \widetilde{B}.$$

EXAMPLES: If $\pi_2(B) = 0$, or M is a flat bundle i.e. $M = \widetilde{B} \times_\rho F$ where $\rho : \pi_1(B) \to \mathrm{Diff}(F)$ is a representation, then M is special.

THEOREM [2] *Let $F \to M \to B$ be a special fibre bundle of closed, connected oriented manifolds. If F is an L^2-acyclic manifold, then so is M.*

This theorem is proved using quasi-isometry invariance and also long exact sequences of L^2-cohomology.

4. SOME NEW RESULTS

In this section we extend the analysis of [2] by proving a number of results which enable us to compute the L^2 RF-torsion of manifolds of the form $M \times S^1$. The main results are 4.2 and 4.9. For the algebraic ideas we refer to [16].

LEMMA 4.1: *Let G be an element of $\mathcal{U} \otimes M_{r+s}$ which has the form (relative to some basis) $\begin{pmatrix} A & D \\ B & C \end{pmatrix}$ where $A \in \mathcal{U} \otimes M_r$ and $C \in \mathcal{U} \otimes M_r$ are invertible and $B \in \mathcal{U} \otimes M_{r \times s}$, $D \in \mathcal{U} \otimes M_{s \times r}$. Then G is also the product*

$$\begin{pmatrix} 1 & DC^{-1} \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} A - DC^{-1}B & 0 \\ 0 & C \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ C^{-1}B & 1 \end{pmatrix} \qquad (*)$$

where the first and last operators are commutators.

Proof: The result follows by multiplying out the product $(*)$ and observing that

$$\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 2X & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \right]$$

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} = \left[\begin{pmatrix} 1 & 2X \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \right]$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ is the group commutator. We recall some of the continuity properties of the Fuglede Kadison determinant (see [10]):

1. $|\mathrm{Det}_\tau|(A) = \lim_{\epsilon \to 0} |\mathrm{Det}_\tau|(|A| + \epsilon)$
2. $|\mathrm{Det}_\tau(H_1) \geq |\mathrm{Det}_\tau|(H_2)$ if $H_1 \geq H_2 \geq 0$
3. $\lim_{n \to \infty} |\mathrm{Det}_\tau|(A_n) \leq |\mathrm{Det}_\tau|(A)$, where A_n tends uniformly to A.
4. $\lim_{n \to \infty} |\mathrm{Det}_\tau|(H_n)$ if $H_n \geq H \geq 0$ and H_n tends to H uniformly.

Before we can introduce our results we need some notation and definitions from [2]. Let

$$C : 0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots \xrightarrow{d} 0$$

be a complex over $\mathcal{U}$ where each C^j is a free finitely generated Hilbert $\mathcal{U}(\pi)$ module and d is a bounded $\mathcal{U}(\pi)$ module map. We call (C, d) a Hilbert $\mathcal{U}(\pi)$ complex. We write δ for the adjoint of d. We note that the commutant of $\mathcal{U}(\pi)$ acting on this complex is isomorphic to $\mathcal{U}(\pi)' \otimes M_n(\mathbb{C})$ so we may identify both d and δ as elements of this algebra. We may choose a set $e = (e_1, e_2, \ldots, e_n)$ of generators for (C, d) as a free $\mathcal{U}(\pi)$ module which are pairwise orthogonal and with respect to which d and hence δ may be represented by an explicit matrix over $\mathcal{U}'$. In this case we refer to the triple (C, d, e) as an L^2-RF complex. We write $m(d, e)$ or $m(\delta, e)$ to denote this matrix. Such a complex is said to be L^2-acyclic if all its L^2-cohomology groups vanish. In that case $d + \delta$ is injective but not necessarily invertible.

THEOREM 4.2: *Let $0 \to (C', d', e') \to (C, d, e) \to (C'', d'', e'') \to 0$ be a short exact sequence of L^2-acyclic $\mathcal{U}$-RF complexes. Then*

$$T_{(2)}(C, d, e) = T_{(2)}(C', d', e')T_{(2)}(C'', d'', e'').$$

Proof: Let H^* be the orthogonal complement of C'^*, in C^*, with a basis e_H chosen to satisfy the following conditions

(1) that it extend the basis e' of C' to a basis of C which is $\mathcal{U}$ related to the basis e of C, and hence

(2) that e_H projects onto the basis e'' of C''.

The differential

$$d : C^{\text{odd}} \to C^{\text{even}}$$

has a matrix relative to this basis of the form

$$d = \begin{pmatrix} d' & 0 \\ pd & d_H \end{pmatrix}$$

Here $p : C^{\text{even}} \to C'^{\text{even}}$ is the orthogonal projection. As $d^2 = 0$, we see that $d_H^2 = 0$ and hence (H, d_H, e_H) is a Hilbert $\mathcal{U}$-RF complex. The following diagram of $\mathcal{U}$-RF complexes is commutative.

$$\begin{array}{ccccccccc} 0 & \to & C' & \to & C & \to & C'' & \to & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & H & \to & 0 \end{array}$$

It follows that there is a natural isomorphism taking bases to bases of (H, d_H, e_H) and (C'', d'', e'). We see similarly that the adjoint map

$$\delta : C^{\text{odd}} \to C^{\text{even}}$$

can be decomposed as

$$\delta = \begin{pmatrix} \delta' & 0 \\ p\delta & \delta_H \end{pmatrix}$$

as C' is a subcomplex of C. It follows that

$$m(d+\delta, e) = \begin{pmatrix} m(d'+\delta', e' & 0 \\ m(p(d+\delta), e) & m(d_H+\delta_H, e_H) \end{pmatrix} \qquad (*)$$

We now prove a lemma which will enable us to complete the proof of the theorem

LEMMA 4.3: *If*

$$G = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

is a matrix of the sort considered in $(*)$*, then*

$$|\mathrm{Det}_\tau|(G) = |\mathrm{Det}_\tau|(A)|\mathrm{Det}_\tau|(C)$$

Proof: We recall that $|\mathrm{Det}_\tau|(G)^2 = |\mathrm{Det}_\tau(G^*G)| = \lim_{\epsilon\to 0}|\mathrm{Det}_\tau|(G^*G+\epsilon)$ by property (1) above of the Fuglede-Kadison determinant, i.e.

$$G^*G+\epsilon = \begin{pmatrix} A^*A+B^*B+\epsilon & B^*C \\ C^*B & C^*C+\epsilon \end{pmatrix}$$

Applying the lemma 4.1, this equals

$$\begin{pmatrix} A^*A+B^*B+\epsilon - B^*C(C^*C+\epsilon)^{-1}C^*B & 0 \\ 0 & C^*C+\epsilon \end{pmatrix}$$

modulo commutators.

Hence

$$|\mathrm{Det}_\tau|(G^*G+\epsilon) = |\mathrm{Det}_\tau|(A^*A+B^*B+\epsilon - B^*C(C^*C+\epsilon)^{-1}C^*B)|\mathrm{Det}_\tau|(C^*C+\epsilon)$$

Since $C(C^*C) = (CC^*)C$ it follows that $C(C^*C+\epsilon) = (CC^*+\epsilon)C$ and hence that $(CC^*+\epsilon)^{-1}C = C(C^*C+\epsilon)^{-1}$.

So we see that $B^*C(C^*C+\epsilon)^{-1}C^*B = B^*(CC^*+\epsilon)^{-1}CC^*B \leq B^*B$ since by the spectral theorem we have $(CC^*+\epsilon)^{-1}CC^* \leq 1$.

It follows that

$$A^*A + B^*B + \epsilon - B^*C(C^*C+\epsilon)^{-1}C^*B \geq A^*A$$

and so

$$A^*A + B^*B + \epsilon - B^*C(C^*C+\epsilon)^{-1}C^*B$$

converges uniformly to A^*A. Now by property 4 above of the Fuglede-Kadison determinant, we have

$$\lim_{\epsilon\longrightarrow 0}|\mathrm{Det}_\tau|(A^*A + B^*B + \epsilon - B^*C(C^*C+\epsilon)^{-1}C^*B) = |\mathrm{Det}_\tau|(A^*A) = |\mathrm{Det}_\tau|(A)^2$$

Also by property 1 above of the determinant, we see that

$$\lim_{\epsilon\longrightarrow 0}|\mathrm{Det}_\tau|(C^*C+\epsilon) = |\mathrm{Det}_\tau|(C)^2$$

We now apply lemma 4.3 to deduce that

$$|\mathrm{Det}_\tau|(m(d+\delta,\epsilon,e)) = |\mathrm{Det}_\tau|(m(d'+\delta',e'))|\mathrm{Det}_\tau|(m(d_H+\delta_H,e_H))$$

which suffices to prove the theorem.

Our next result needs some preliminary discussion. We refer to bounded $\mathcal{U}$ module cochain maps f between $\mathcal{U}$-RF complexes (C,d,e) and (C',d',e') as simply maps. Then two such maps f and g are said to be L^2 homotopic if there is a sequence of maps $D^j \in L(C^j, C'^{j-1})$ such that $d'_{j-1}D^j + D^{j+1}d_j = f_j - g_j$ for all j. If there is a map h from (C',d',e') to (C,d,e) such that both $f\circ h$ and $h\circ f$ are L^2 homotopic to the identity map then f is called an L^2 homotopy equivalence. Finally we may define the mapping cone (C_f, d_f, e_f) of a map f as follows:

$$C_f^j \equiv C^j \oplus C'^{j-1},$$

$$(d_f)_j \equiv (d_j + (f_j - d'_{j-1}))$$

$$(e_f)_j \equiv e_j \cup e'_{j-1}$$

From Lemma 1.30 of [2] we know that if f is an L^2 homotopy equivalence of $\mathcal{U}$-RF complexes then the mapping cone complex is also an L^2-acyclic $\mathcal{U}$-RF complex and hence it has an L^2-RF torsion which we denote $T_{(2)}(f, e, e')$. We now note the following fact:

COROLLARY 4.4: *Let $f : (C, d, e) \to (C', d', e')$ be an L^2 homotopy equivalence of L^2-acyclic $\mathcal{U}$-RF complexes. Then $T_{(2)}(C', d', e') = T_{(2)}(f, e, e')T_{(2)}(C, d, e)$.*

It follows that if an L^2 homotopy equivalence f satisfies $T_{(2)}(f, e, e') = 1$ then the L^2-RF torsions of the complexes (C, d, e) and (C', d', e') are equal.

Our next result begins with a simple observation. Let $\pi = \pi_1 \times \pi_2$. Hence

$$\ell^2(\pi) \cong \ell^2(\pi_1) \otimes_{\mathbb{C}} \ell^2(\pi_2)$$

naturally as Hilbert spaces.

LEMMA 4.5: *If $\mathcal{M}$ is a free Hilbert $\mathcal{U}(\pi_1)$-module of rank m, and $\mathcal{N}$ is a free Hilbert $\mathcal{U}(\pi_2)$ module of rank n, then $\mathcal{M} \otimes_{\mathbb{C}} \mathcal{N}$ is in a natural way a free Hilbert $\mathcal{U}(\pi)$ module of rank mn.*

Proof: It is enough to show that $(\ell^2(\pi_1))^m \otimes (\ell^2(\pi_2))^n$ has a natural Hilbert $\mathcal{U}(\pi)$ module structure. But

$$(\ell^2(\pi_1))^m \otimes (\ell^2(\pi_2))^n \cong (\ell^2(\pi))^{mn}$$

naturally as Hilbert spaces. Hence we can naturally endow the tensor product with a free Hilbert $\mathcal{U}(\pi)$ module structure of rank mn.

REMARK: If $(\mathcal{N}, d)$ is a free Hilbert $\mathcal{U}(\pi)$ complex and $\mathcal{M}$ is a free Hilbert $\mathcal{U}(\pi)$ module then lemma 4.5 and induction proves that $\mathcal{M} \otimes \mathcal{N}$ is a free Hilbert $\mathcal{U}(\pi)$ complex.

LEMMA 4.6: *Let $\mathcal{M}$ and $\mathcal{N}$ be as in the lemma above. If $f \in L(\mathcal{M}, \mathcal{M})$ and $g \in L(\mathcal{N}, \mathcal{N})$ then $f \otimes g \in L(\mathcal{M} \otimes_{\mathbb{C}} \mathcal{N}, \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N})$ is in a natural way a Hilbert $\mathcal{U}(\pi)$ module homomorphism.*

Proof: It is enough to consider $\mathcal{M} = (\ell^2(\pi_1))^m$ and $\mathcal{N} = (\ell^2(\pi_2))^n$. Let ϕ denote the natural isomorphism between $(\ell^2(\pi))^{mn}$ and $(\ell^2(\pi_1))^m \otimes (\ell^2(\pi_2))^n$ described in the previous lemma. Then

$$\phi \circ f \otimes g \circ \phi^{-1} \in L((\ell^2(\pi))^{mn}, (\ell^2(\pi))^{mn})$$

We will denote this map by just $f \otimes g$.

COROLLARY 4.7: *If $\mathcal{M}_j$ is a Hilbert $\mathcal{U}(\pi_j)$-module ($j = 1, 2$) then $\mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_2$ is in a natural way a Hilbert $\mathcal{U}(\pi)$ module.*

Proof: By definition $\mathcal{M}_j$ is finitely generated and projective. Let $\mathcal{F}_j$ be a free Hilbert $\mathcal{U}(\pi_j)$ module and $p_j \in L(\mathcal{F}_j, \mathcal{F}_j)$ be self adjoint projections such that range $p_j = \mathcal{M}_j$. By the lemma above $p_1 \otimes p_2$ is naturally a Hilbert $\mathcal{U}(\pi)$ module homomorphism. Hence range $(p_1 \otimes p_2) = \mathcal{M}_1 \otimes_{\mathbb{C}} \mathcal{M}_2)$ is in a natural way a Hilbert $\mathcal{U}(\pi)$ module.

LEMMA 4.8: *Let $0 \to \mathcal{N}' \xrightarrow{i} \mathcal{N} \xrightarrow{p} \mathcal{N}'' \to 0$ be a short exact sequence of free Hilbert $\mathcal{U}(\pi_2)$ modules and $\mathcal{M}$ be a free Hilbert $\mathcal{U}(\pi_1)$ module, then*

$$0 \to \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N}' \overset{1\otimes i}{\to} \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N} \overset{1\otimes p}{\to} \mathcal{M} \otimes_{\mathbb{C}} \mathcal{N}'' \to 0$$

is a short exact sequence of Hilbert $\mathcal{U}(\pi)$ modules.

Proof: We will check injectivity of $1 \otimes i$. If $m \otimes n \in \ker(1 \otimes \mathrm{i})$ then $m \otimes i(n) = 0$ is equivalent to $m = 0$ or $n = 0$. Surjectivity of $1 \otimes p$ is obvious. Also by the above lemmas all maps and modules can be considered to be Hilbert $\mathcal{U}(\pi)$ modules or homomorphisms.

PROPOSITION 4.9: *Let (C, d, e) be an L^2-acyclic $\mathcal{U}(\pi_1)$-RF complex and (C', d', e') be a $\mathcal{U}(\pi_2)$-RF complex. Then*

(1) $(C \otimes C', d \otimes 1 + 1 \otimes d', e \otimes e')$ is an L^2-acyclic $\mathcal{U}(\pi)$-RF complex.

(2) $T_{(2)}(C \otimes C', d \otimes 1 + 1 \otimes d', e \otimes e') = T_{(2)}(C, d, e)^{\chi(C')}$

Proof: (1) By our previous lemmas it is clear that $(C \otimes C', d \otimes 1 + 1 \otimes d', e \otimes e')$ is a $\mathcal{U}(\pi)$-RF complex. We will prove that it is acyclic by induction on the length of C'.

If $C' : 0 \to C'^0 \to 0$ has length one, then

$$H^j_{(2)}(C \otimes C') \cong H^j_{(2)}(C) \otimes C'^0 = 0$$

since C'^0 is a free Hilbert module. Assume that $C \otimes C'$ is L^2-acyclic for all $\mathcal{U}'$-RF complexes C' of length $\leq n$.

If $C' : 0 \to C'^0 \xrightarrow{d'} \dots \xrightarrow{d'} C'^n \to 0$ is such that (C', d', e') is a $\mathcal{U}(\pi_2)$-RF complex of length equal to $n+1$, we define the complexes

$$B : 0 \to C'^0 \to 0$$

$$B' : 0 \to C'^1 \xrightarrow{d'} C^2 \to \dots \to C^n \to 0$$

Then B and B' are $\mathcal{U}(\pi_2)$-RF complexes of length $\leq n$. Also we have the short exact sequence of $\mathcal{U}'$-RF complexes

$$0 \to B \to C' \to B' \to 0$$

Tensoring over $\mathbb{C}$ with the complex C; and by an extension of lemma 4.8, we obtain a short exact sequence of $\mathcal{U}(\pi)$-RF complexes

$$0 \to C \otimes B \to C \otimes C' \to C \otimes B' \to 0 \qquad (*)$$

which gives rise to a long exact sequence in L^2-cohomology by Cheeger and Gromov [4]. By our induction hypotheses

$$0 = H^j_{(2)}(C \otimes B) = H^j_{(2)}(C \otimes B')$$

for all $j \geq 0$. Hence from the long exact sequence we see that $H^j_{(2)}(C \otimes C') = 0$ for all $j \geq 0$.

(2) Let $\mathcal{F}(\pi_2)$ denote the semigroup of all $\mathcal{U}(\pi_2)$-RF complexes. We will define functions $f_j : \mathcal{F}(\pi_2)) \to \mathbb{R}$ by

$$f_1(C', d', e') \equiv T_{(2)}(C \otimes C', d \otimes 1 + 1 \otimes d', e \otimes e')$$

$$f_2(C', d', e') \equiv T_{(2)}(C, d, e)^{\chi(C')}$$

We shall prove that $f_1 = f_2$ by induction on the length of C'.

If $C' : 0 \to C'^0 \to 0$ is of length one, $f_1 = f_2$ trivially. So assume that $f_1 = f_2$ for all $\mathcal{U}(\pi_2)$-RF complexes of length $\leq n$. If

$$C' : 0 \to C'^0 \xrightarrow{d'} \dots \xrightarrow{d'} C'^n \to 0$$

is a $\mathcal{U}(\pi_2)$-RF complex of length $n+1$, we define the $\mathcal{U}(\pi_2)$-RF complexes B and B' as in (1) above and we get the short exact sequence of L^2-acyclic $\mathcal{U}(\pi_1) \otimes \mathcal{U}(\pi_2)$-RF complexes as in (*) above

$$0 \to C \otimes B \to C \otimes C' \to C \otimes B' \to 0$$

Hence by theorem 4.2

$$T_{(2)}(C \otimes C') = T_{(2)}(C \otimes B) T_{(2)}(C \otimes B')$$

By our induction hypothesis this equals

$$= T_{(2)}(C)^{\chi(B)} T_{(2)}(C)^{\chi(B')}$$

$$= T_{(2)}(C)^{\chi(B)+\chi(B')}$$

$$= T_{(2)}(C)^{\chi(C')}$$

COROLLARY 4.10: *Let M be an L^2-acyclic manifold and N a closed, connected oriented manifold. We know by preceding results that $M \times N$ is L^2 -acyclic. Then we have that*

$$T_{(2)}(M \times N) = T_{(2)}(M)^{\chi(N)}$$

Proof: Let K and L be triangulations if M and N respectively. Then $C_{(2)}(\widetilde{K \times L}) = C_{(2)}(\widetilde{K}) \otimes C_{(2)}(\widetilde{L})$ where we use the Hilbert tensor product. We can now apply the preceding theorem to see that

$$T_{(2)}(M \times N) = T_{(2)}(M)^{\chi(C_{(2)}(\widetilde{L})}$$

Now by theorem 1 in Cohen [4] we see that $\chi(C_{(2)}(\widetilde{L})) = \chi(L)$

COROLLARY 4.11: *Let M be a closed, connected, oriented manifold. Then by preceding results $M \times S^1$ is an L^2-acyclic manifold. Also $T_{(2)}(M \times S^1) = 1$.*

COROLLARY 4.12: $T_{(2)}(\mathbb{T}^k) = 1$ *when $k > 0$.*

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