# ON THE THERMODYNAMICS OF CURVES AND OTHER CURLICUES

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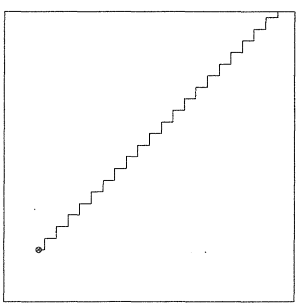
This lecture was to be a description of the work of Mendès France and of various of his collaborators on how one may felicitously and instructively attach thermodynamic quantities to plane curves. However, our interest in that project cooled somewhat as we became interested in questions arising from the preparation of the pictures that were to illustrate the ideas to be presented. As a result we give only a summary introduction to the work on thermodynamics and divert our efforts to a description and reformulation of the work of Berry and Goldberg [1] on renormalisation of certain curves containing fantastic curls and twists.

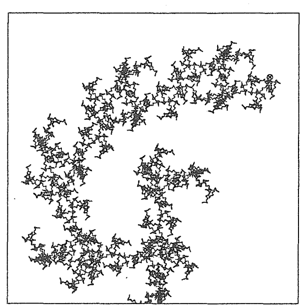
## 1. Thermodynamics of curves

Our initial remarks arise from work of Michel Mendès France and his collaborators (and are detailed in [5]).

### 1.1 Linear and superficial curves; Dimension

Plainly it would be congenial to be able to formalise the notion that the 'line-like' curve below really is 1-dimensional, whilst the ubiquitous curve tends towards 2-dimensionality.





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Dekking and Mendès France [2] propose the following strategy. One draws with broad strokes, fattening the given curve $\Gamma$ by forming the set $\Gamma(\epsilon) = \{y : \text{distance}\,(y,x) < \epsilon; x \in \Gamma\}$. Next consider the area $\Gamma(\epsilon, R)$ common to this 'epsilonised' curve and a disk of radius $R$ (centred at the origin, say). Take the quotient

$$\dim \Gamma(\epsilon, R) = \log \Gamma(\epsilon, R) / \log R$$

considering its limit as $R \nearrow \infty$ and $\epsilon \searrow 0$. For an unbounded curve the limit $\dim \Gamma$ does not depend on the position of the curve or on the scale to which it is drawn and

$$1 \leq \dim \Gamma \leq 2 .$$

For a more general definition let Diameter $S$ be the supremum of the distances separating points of a set $S$ in the plane and let $\epsilon S := \{\epsilon x : x \in S\}$. Denote by $\Gamma_r$ the beginning part of $\Gamma$ of length $r$. Then in the limit as $r \nearrow \infty$ and $\epsilon \searrow 0$, the quotient

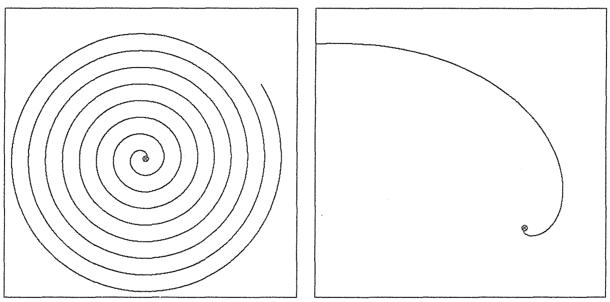
$$\log \text{Area } \epsilon^{-1}\Gamma_r(\epsilon) / \log \text{Diameter } \epsilon^{-1}\Gamma_r(\epsilon)$$

yields upper and lower dimensions satisfying

$$1 \leq \underline{\dim}\, \Gamma \leq \overline{\dim}\, \Gamma \leq 2 .$$

If $\underline{\dim}\, \Gamma = \overline{\dim}\, \Gamma$, then the common value $\dim \Gamma$ is the *dimension* of $\Gamma$, and coincides with various classical dimension functions appropriate to bounded and unbounded curves respectively.

Abusing language, we say that $\Gamma$ is *superficial* if $\underline{\dim}\, \Gamma > 1$. Not trivially, this coincides with $\lim_{r\to\infty} r / \text{Diameter } \Gamma_r = \infty$ if $\Gamma$ is unbounded, and $\lim_{\epsilon\to 0} \text{Area } \Gamma(\epsilon)/\epsilon = \infty$ if $\Gamma$ is bounded. The Archimedean spiral $\rho = \theta$ is superficial. In the alternative case the curve is said to be *linear*; the spiral $\rho = \exp \theta$ is linear.



*Archimedean spiral* $\rho = \theta$ *Exponential spiral* $\rho = e^{\theta}$

One says that an unbounded curve $\Gamma$ is *resolvable* if, for some $\epsilon > 0$

$$\underline{\lim}_{r\to\infty} \text{Area } \Gamma_r(\epsilon)/r > 0 .$$

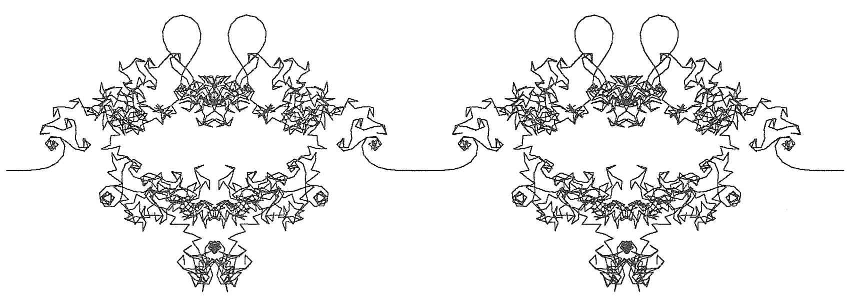
It turns out that the spiral $\rho = \theta^{\alpha}$ is resolvable if $\alpha \geq 1$, and not if $\alpha < 1$.

1.2 Picturesque sums

Given a function $u_t$ (which need be defined only at the nonnegative integers, so a given sequence will do) consider a particle starting at the origin at time $t = 0$. It moves with velocity constant in magnitude, but changing direction at $t = 0, 1, 2, \ldots$; the new directions are given by $2\pi u_t \bmod 2\pi$, taking values in $[0, 2\pi]$.
Plainly we may view the particle as moving in the complex plane $\mathbb{C}$ rather than in $\mathbb{R}^2$; in which case, up to a sensible normalisation, its path is a polygon $\Gamma(u)$ with vertices at

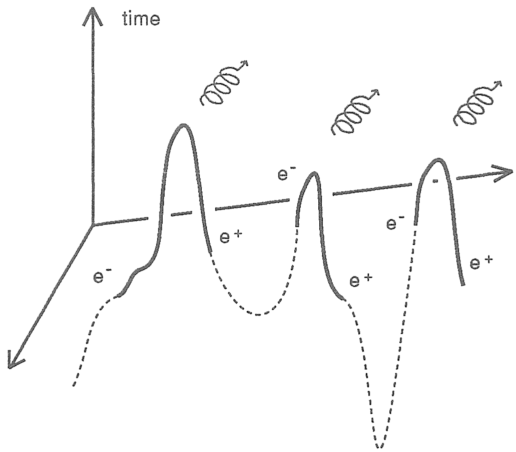
$$z_n = \sum_{h=0}^{n-1} \exp(2\pi i u_h)\,, \qquad n = 0\,,1\,,2\,,\ldots\,.$$

The core observation of [DMF] is that a sequence $(u_h)$ is 'equidistributed modulo one' if and only if the curve $\Gamma(mu)$ is superficial for each positive integer $m$.
It turns out that quite fascinating pictures of the initial $n$ segments $\Gamma_n(u)$ of these curves can readily be drawn with aid of computer. (A number of the more attractive examples appear as decoration on the succeeding pages.) However the finiteness of time, and of the paper on which the curves are to be represented, frequently makes it difficult to actually see the superficiality of the complete curves. This is especially so in the unbounded case. Nor do there seem to be simple criteria whereby one can generally decide in advance whether a curve in fact *is* bounded. The principal portion of this note is dedicated to an analysis of the relatively tractable curves belonging to the class of fairly interesting sequences $(\tau h^2)$, with $\tau \in \mathbb{R}\backslash\mathbb{Q}$. The secret lies in understanding the classical complete Gauß sums, namely when $\tau \in \mathbb{Q}$.



1.3 Entropy

In his address on receiving the Nobel Prize, Feynman [6] moots, on Wheeler's suggestion, the notion that all electrons and positrons have the same mass and (up to sign) the same charge because there is a unique directed curve in space-time whose intersections with $\mathbb{R}^3$ are (from time to time) the manifestations of all electrons and positrons.




*The electron.*

With this thought in mind, and back in $\mathbb{R}^2$, we say that a time $t$ is a straight line. Such a line is said to *find a finite curve* $\Gamma_r$ *in state* $N_r(t)$ if it intersects the curve in $N_r(t)$ points. Averaging over all times, we write

$$\overline{N}_r = \int_{t\cap\Gamma_r\neq\emptyset} N_r(t)\,dt \Big/ \int_{t\cap\Gamma_r\neq\emptyset} dt\ .$$

Because $N_r(t)$ is almost always well defined, plainly $\overline{N}_r$ is well defined.
Let $K_r$ be the convex hull of $\Gamma_r$ and denote by $|\partial K_r|$ the length of its boundary. By a theorem of Steinhaus (see [9] p.31) any 'natural' choice of the probability

$$dt \Big/ \int_{t\cap\Gamma_r\neq\emptyset} dt$$

yields

$$2|\Gamma_r| = 2r = \overline{N}_r|\partial K_r|\,.$$

Having fixed a natural probability measure, let $t_k$ denote the measure of those times $t$ which find $\Gamma_r$ in state $k$. Then the entropy of $\Gamma_r$ is defined as

$$-\sum_k t_k \log t_k$$

and this is maximal (as may be seen by elementary computation) if we have selected the equilibrium measure yielding

$$t_k = e^{-\beta_r k}(e^{\beta_r} - 1)\,.$$

Then

$$\sum_k k t_k = \overline{N}_r = 2|\Gamma_r|/|\partial K_r|$$

entails

$$\beta_r = \log\left(2|\Gamma_r|/(2|\Gamma_r| - |\partial K_r|)\right)$$

and we may attribute to $\Gamma_r$ the intrinsic entropy

$$S(\Gamma_r) = \log(2|\Gamma_r|/|\partial K_r|) + \beta_r/(e^{\beta_r} - 1)\,.$$

Whilst the lead-up to this conclusion is somewhat woolly the definition itself makes sense for all rectifiable curves. In particular $S \geq 0$ for all $\Gamma$, and $S = 0$ if and only if $\Gamma$ is a straight line. If $\Gamma$ is an algebraic curve of degree $\nu$ then $S(\Gamma) \leq \log \nu + 1$; the higher the entropy, the higher the degree.

### 1.4 Temperature

The reciprocal of $\beta$ is traditionally identified (up to a scaling constant) with the absolute temperature $T$. Then the curve $\Gamma_r$ has *temperature*

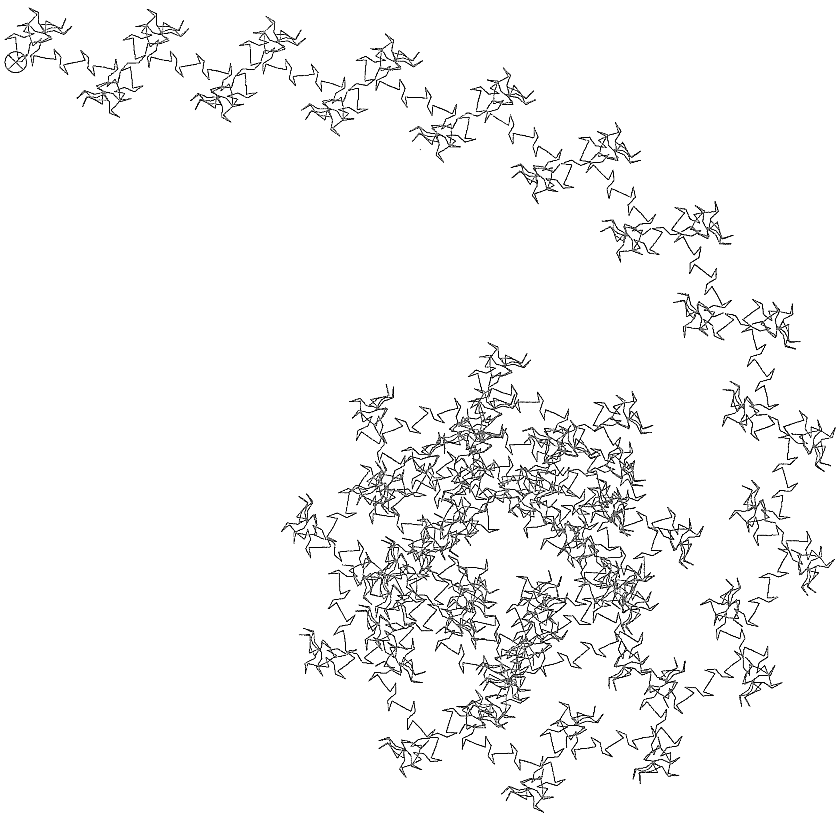
$$T_r = \left(\log \frac{2|\Gamma_r|}{2|\Gamma_r| - |\partial K_r|}\right)^{-1}\,.$$

Plainly $T_r$ is always nonnegative and is zero if and only if $2|\Gamma_r| = |\partial K_r|$, when $\Gamma_r$ is a straight line. Thus only straight lines exist at $T = 0$, and then $S = 0$.

Pursuing the somewhat questionable thermodynamic analogy might lead one to identify the length $r = |\Gamma_r|$ with the *volume* $V_r$ of the curve. Then the *pressure* seems appropriately defined by $P_r = |\partial K_r|^{-1}$, so that the greater the pressure, the more confined the curve. We have

$$2P_r V_r = \left(1 - e^{-1/T_r}\right)^{-1}\,,$$

so that at high temperature, indeed congenially, $2P_r V_r \sim T_r$.



"$\pi$ curlicue" – $\Gamma_{4000}(\pi\, h^2)$ .

### 1.5 More entropy and chaos

For infinite curves $\Gamma$ it appears useful to describe the quotient

$$\log \overline{N}_r / \log r = \log\bigl(2|\Gamma_r|/|\partial K_r|\bigr)/\log r \sim 1 - \log|\partial K_r|/\log|\Gamma_r|,$$

as $r \to \infty$, as the *entropy* of the curve. Then, always,

$$0 \leq \underline{\text{ent}}\,\Gamma \leq \overline{\text{ent}}\,\Gamma \leq 1 .$$

For a resolvable unbounded curve there is the interesting relation to the notion of dimension introduced earlier:

$$\text{ent}\ \Gamma = 1 - 1/\dim\Gamma .$$

Generally, one has

$$\overline{\dim}\,\Gamma \leq 1/(1-\overline{\text{ent}}\,\Gamma) \quad \text{and} \quad \underline{\dim}\,\Gamma \leq 1/(1-\underline{\text{ent}}\,\Gamma) .$$

Thus an infinite curve with zero entropy has dimension 1. However, a nonresolvable curve such as $y = \sin x$ has dimension 1 yet entropy 1. A fast-winding spiral, such as $\rho = \log\theta$ has entropy 1, appropriate to its chaotic behaviour. The spirals $\rho = \theta^\alpha$ have entropy $1/(1+\alpha)$. Unbounded algebraic curves have entropy zero. A slow spiral such as $\rho = e^\theta$ which mimics an organised biological phenomenon, the growth of sea shells, has entropy zero.
It appears that if an unbounded curve has entropy greater than $\frac{1}{2}$ it must be self-intersecting. Amusingly, the non-intersecting paperfolding curves that star in "FOLDS!" [3], and which seem both random yet organised, are exactly half-way to chaos, with entropy a half.

## 2. Generalised Gauß sums

### 2.1 Exact formulae

The $\theta$-function $\theta(\tau) = \sum_{-\infty}^{\infty} exp(\pi i \tau h^2)$ has, for $\tau \in \mathbb{R}$, the interesting functional equation

$$\theta(-1/\tau) = \sqrt{|\tau|}\exp(-\operatorname{sgn}(\tau)\,\pi i/4)\theta(\tau) .$$

Also of course, for all $\tau$,

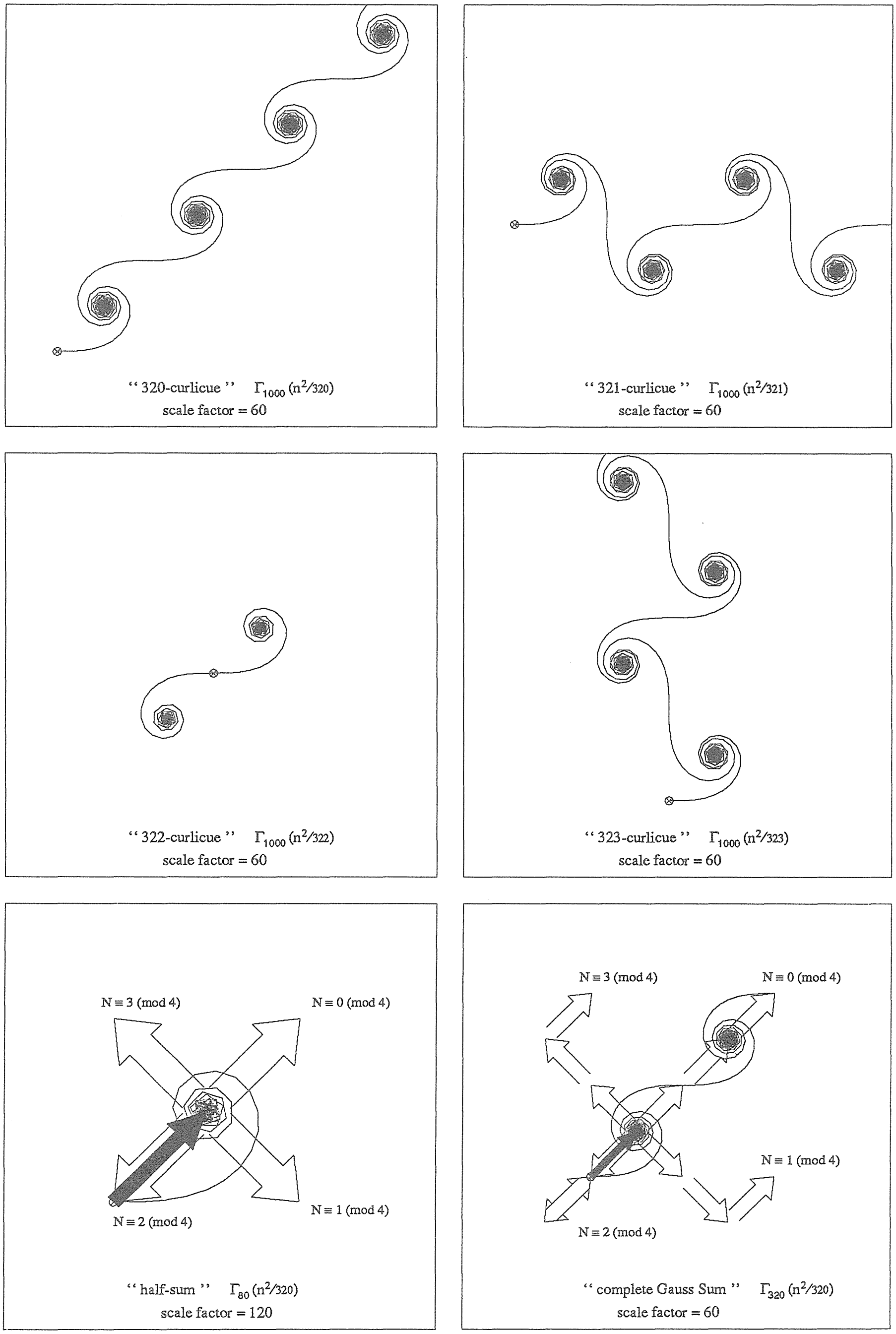
$$\theta(\tau+2) = \theta(\tau) .$$

The complete Gauss sums

$$\sum_{h=0}^{N-1} \exp(2\pi i h^2/N)$$

sum to

$$2\left(1+(-i)^N\right) \times \frac{1+i}{4}\sqrt{N} .$$




Modulo 4 variation of the complete Gauß Sums.

The different parts comprising this formula correspond to fairly obvious symmetries in the graphs alluded to at §1.2 above; for example because $(N-h)^2 \equiv h^2 \bmod N$, and according to the congruence class of $N$ modulo 4 .

The actual computation of such sums is facilitated by a convenient decomposition, whereby if $a$ and $b$ are relatively prime and $N = ab$, then

$$\sum_{h=0}^{N-1} \exp(2\pi i h^k/N) = \sum_{h=0}^{a-1} \exp(2\pi i h^k b'/a) \sum_{h=0}^{b-1} \exp(2\pi i h^k a'/b)\,,$$

where $bb' \equiv 1 \bmod a$ and $aa' \equiv 1 \bmod b$. Moreover, there is a further reduction

$$\sum_{h=0}^{p^\ell-1} \exp(2\pi i h^k q/p^\ell) = \begin{cases} p^{k-1} \sum_{h=0}^{p^{\ell-k}-1} \exp(2\pi i h^k q/p^{\ell-k}) & \text{if } \ell > k \\ p^{\ell-1} & \text{if } \ell \leq k. \end{cases}$$

Ultimately, one need only know the sums

$$\sum_{h=0}^{p-1} \exp(2\pi i h^k q/p)\,.$$

## 2.2 The graph $\Gamma_N(\tau h^2)$.

The graphs of $\Gamma_N(h^2/N)$ display a spiral structure reminiscent of the Cornu spiral of physical optics. That is no accident, for the latter is the graph of the function

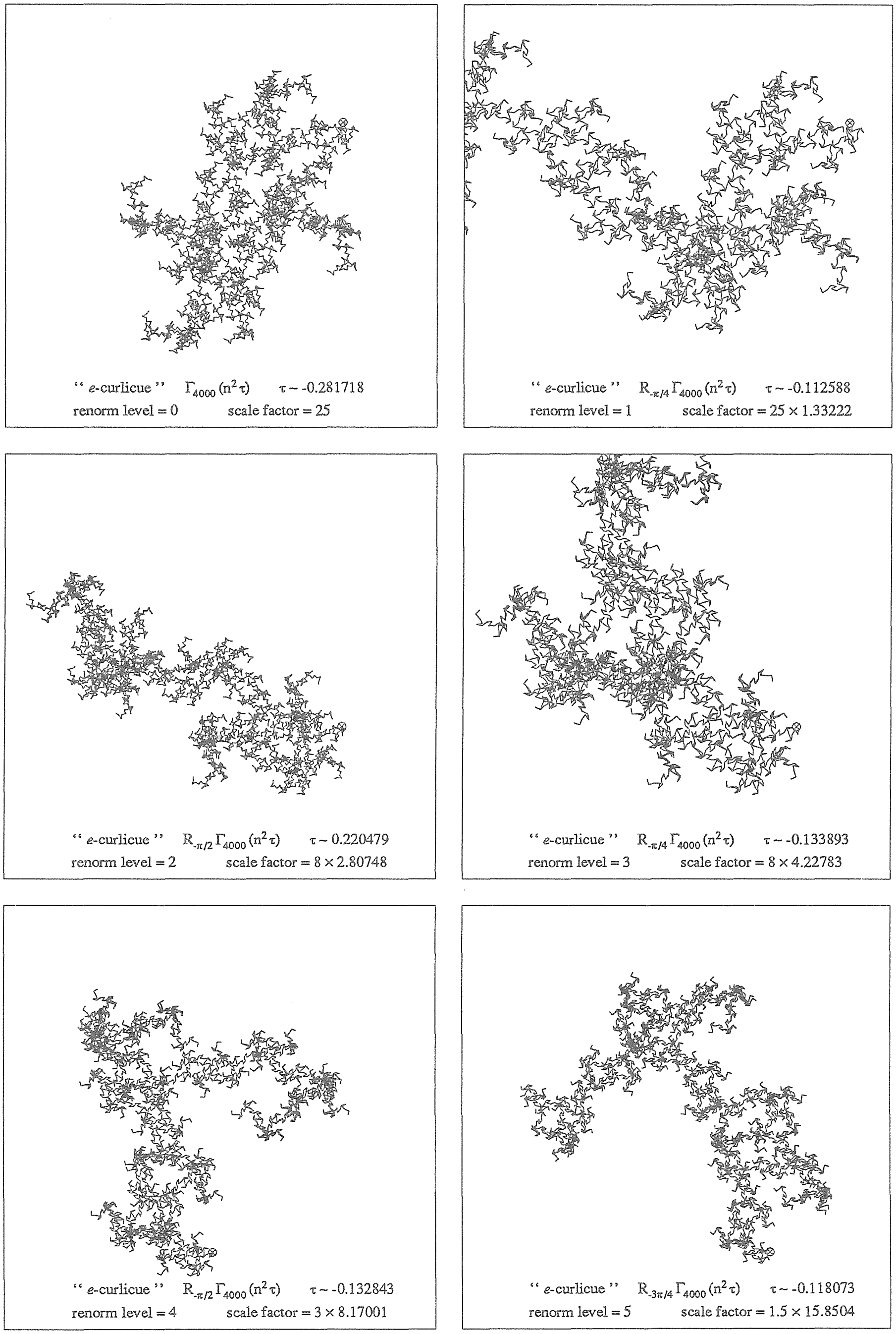
$$f(x) = \int_{-\infty}^{x} \exp(2\pi i t^2)\, dt$$

and, after a change of variables the complete Gauß sums can be viewed as approximations to this integral. This fact is exploited by Lehmer [7] to obtain sharp estimates for the size of the spirals occurring in the graphs.

To understand the spiral structure, or *curlicues*, that occur note that the angle at the $(h+1)$-st vertex of the graph $\Gamma_N(\tau h^2)$ is given by the phase difference $((h+1)^2 - h^2)2\pi\tau = (2h+1)2\pi\tau$ between successive terms. Thus the average of the angles at the ends of the $h$-th segment is $4h\pi\tau$.

Now take $\tau = 1/N$, so that we are dealing with complete Gauß sums. We shall suppose that $N$ is not small. For small values of $h$ (or when $h$ is roughly an integer multiple of $N/2$) the phase difference is quite small so the graph appears to be only gently curving. As $h$ increases the phase difference initially increases, giving a spiralling effect. When the phase difference exceeds $\pi/2$ each segment tends to fold back onto the previous, resulting in a filling in of the spiral. Then, as $h$ approaches $N/4$ one gets sharp spikes as each successive edge almost reverses the previous one. Once $h$ exceeds $N/4$ the curlicue proceeds to unwind in much the way it was formed. Then, as $h$ approaches $N/2$ the graph straightens out again prior to commencing the formation of a further spiral.

Of course the significant contribution to the actual sum comes from those parts in which the phase is varying only a litle. It is this that constitutes the stationary phase approximation and accounts for the possibility of renormalising as briefly described below.




Renormalisations of $\tau = e$.

For general $\tau$ it is evident that the shape of the initial segments of the graph will tend to be that yielded by graphs $\Gamma_N(ph^2/q)$ corresponding to sharp rational approximations $p/q$ of $\tau$. All good such approximations arise as truncations (convergents) of the regular continued fraction expansion of $\tau$ and very good approximations occur when the first partial quotient omitted is relatively large.

The pictures decorating the present manuscript illustrate the points just made. Employing a convenient notation whereby we denote a continued fraction by listing its partial quotients we have, for example,

$$\pi = [3\,;7\,,15\,,1\,,292\,,1\,,\ldots\ldots],$$

enabling us to predict mini-spirals arising from the approximation $[3\,;7] = 22/7$ and a ferocious, initially dominating, spiral arising from the approximation $[3\,;7\,,16] = 355/113$. Similarly, if $\alpha$ is the real zero of the polynomial $\alpha^3 - \alpha^2 - \alpha - 1$ then

$$\alpha = [1\,;1\,,5\,,4\,,2\,,305\,,\ldots\ldots],$$

and we can predict mini-curlicues spiralling according to $\alpha \approx [1\,;1\,,5\,,4\,,2]$. On the other hand one has

$$e - 1 = [1\,;1\,,2\,,1\,,1\,,4\,,1\,,1\,,6\,,1\,,1\,,8\,,\ldots\ldots],$$

which cannot yield exciting approximations and suggests virtually random behaviour of the corresponding graph.

Our graphs bear out these predictions (which is unsurprising since the 'predictions' were made *post facto*).

## 2.3 Renormalisation

Denote the graph, as described at §1.2, of the sum

$$S_N(\tau) = \frac{1}{2} + \sum_{h=1}^{N-1} \exp(2\pi i \tau h^2) + \frac{1}{2},$$

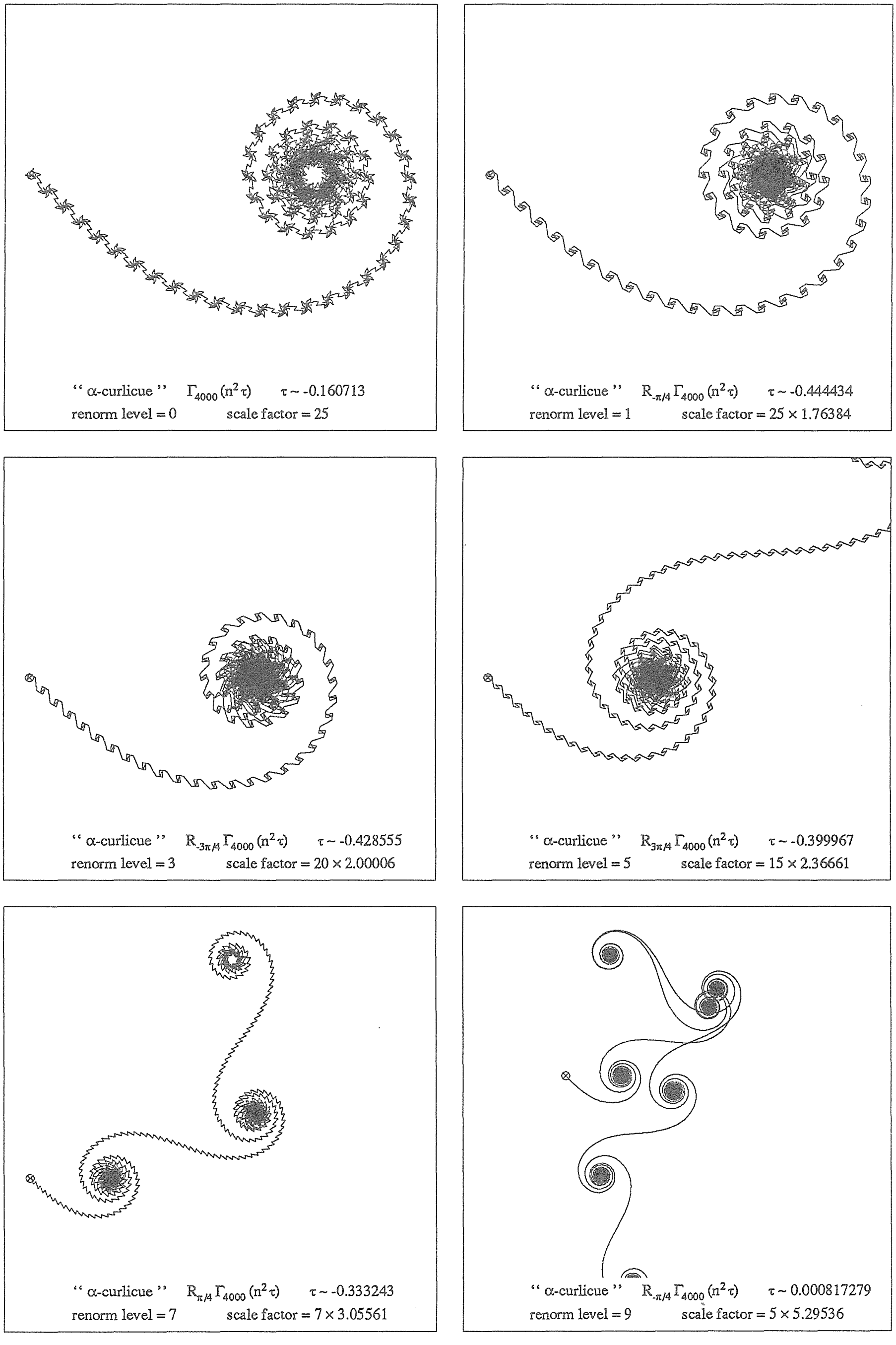
by $\Gamma_N(\tau h^2)$. Our splitting the first term, which translates the bulk of the graph by half a unit to the left, is a convenience to better use the functional equation for the $\theta$-function. We lose no generality in supposing that $-\frac{1}{2} < \tau \le \frac{1}{2}$. Then

$$S_N(\tau) \approx |2\tau|^{-1/2} \exp(\pi i \operatorname{sgn}(\tau)/4) S_{\lfloor 2N\tau \rfloor}(-1/2\tau).$$

The error in this approximate equality is of order $|2\tau|^{-1/2}$, independent of $N$. In other words, if we set

$$\tau' = ((1/2\tau)) - 1/2\tau$$

where $((\chi))$ denotes the nearest even integer to $\chi$, then, up to rescaling and a turn of $\pi/4$, the $N$ segments comprising the curve $S_N(\tau)$ are well approximated by the $\lfloor 2N\tau \rfloor$ segments of the curve $S_{\lfloor 2N\tau \rfloor}(\tau')$.




Renormalisations of $\tau = \alpha$, where $\alpha^3 - \alpha^2 - \alpha - 1 = 0$.

2.4 Why one may renormalise

Berry and Goldberg [1] obtain the renormalisation by using the Poisson summation formula to replace the sum by a series of integrals which are then approximated by the method of stationary phase. Our equivalent, and we hope insightful, procedure is in effect first to use the stationary phase approximation, and then the exact formula for the complete Gauß sums. Let $N_1, N_2, \ldots$ be the monotonic increasing sequence of those integers for which the phase is closest to stationary: that is, the phase difference is as close as possible to being integral; thus $2\tau N_h \approx h$. Similarly denote by $M_1, M_2, \ldots$ a monotonic increasing sequence of integers so that $2\tau M_h \approx h + \frac{1}{2}$; since the phase is rapidly changing in their neighbourhood, precise values for the $M_h$ are not critical. We have

$$S_N(\tau) = \sum_{n=0}^{M_0-1} \exp(2\pi i n^2 \tau) + \sum_{h>0} \sum_{n=M_{h-1}}^{M_h-1} \exp(2\pi i n^2 \tau^2).$$

A typical term yields

$$\sum_{n=M_{h-1}}^{M_h-1} \exp(2\pi i n^2 \tau) = \sum_{n=M_{h-1}-N_h}^{M_h-N_h-1} \exp(2\pi i (N_h+n)^2 \tau).$$

Using the fact that the values of the $M_h$ are not well defined, set

$$H \approx \frac{1}{2}(N_h - N_{h-1}) \approx M_h - N_h - 1 \approx N_h - M_{h-1};$$

then $H \approx (4\tau)^{-1}$. Noting that $\exp(2\pi i(2nN_h)\tau) \approx 1$ and recalling the value of the complete Gauß sum, the typical term above is approximately equal to

$$\exp(2\pi i N_h^2 \tau) \sum_{n=-H}^{H} \exp(2\pi i n^2 \tau) \approx \exp(2\pi i N_h^2 \tau)(1+i)/2\sqrt{\tau}.$$

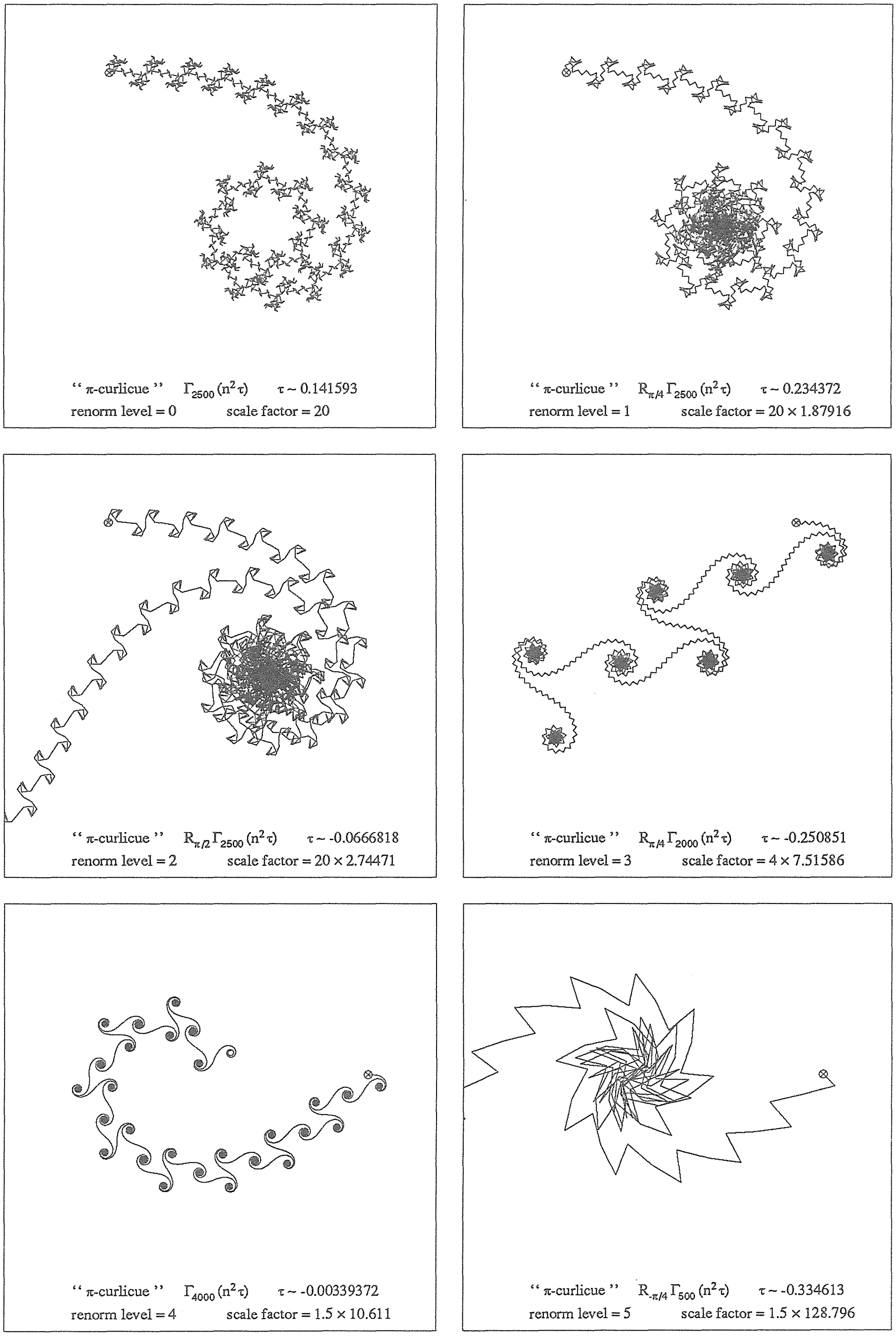
The apparently wild estimations employed here are less drastic than may at first appear. For example, on writing $\tau = (k+\tau')^{-1}$ with $k \in \mathbb{Z}$ and $|\tau'| < \frac{1}{2}$, it is easy to see that the graphs of

$$\sum_{n=-H}^{H} \exp(2\pi i n^2 \tau) \qquad \text{and} \quad \sum_{n=-H}^{H} \exp(2\pi i n^2/k)$$

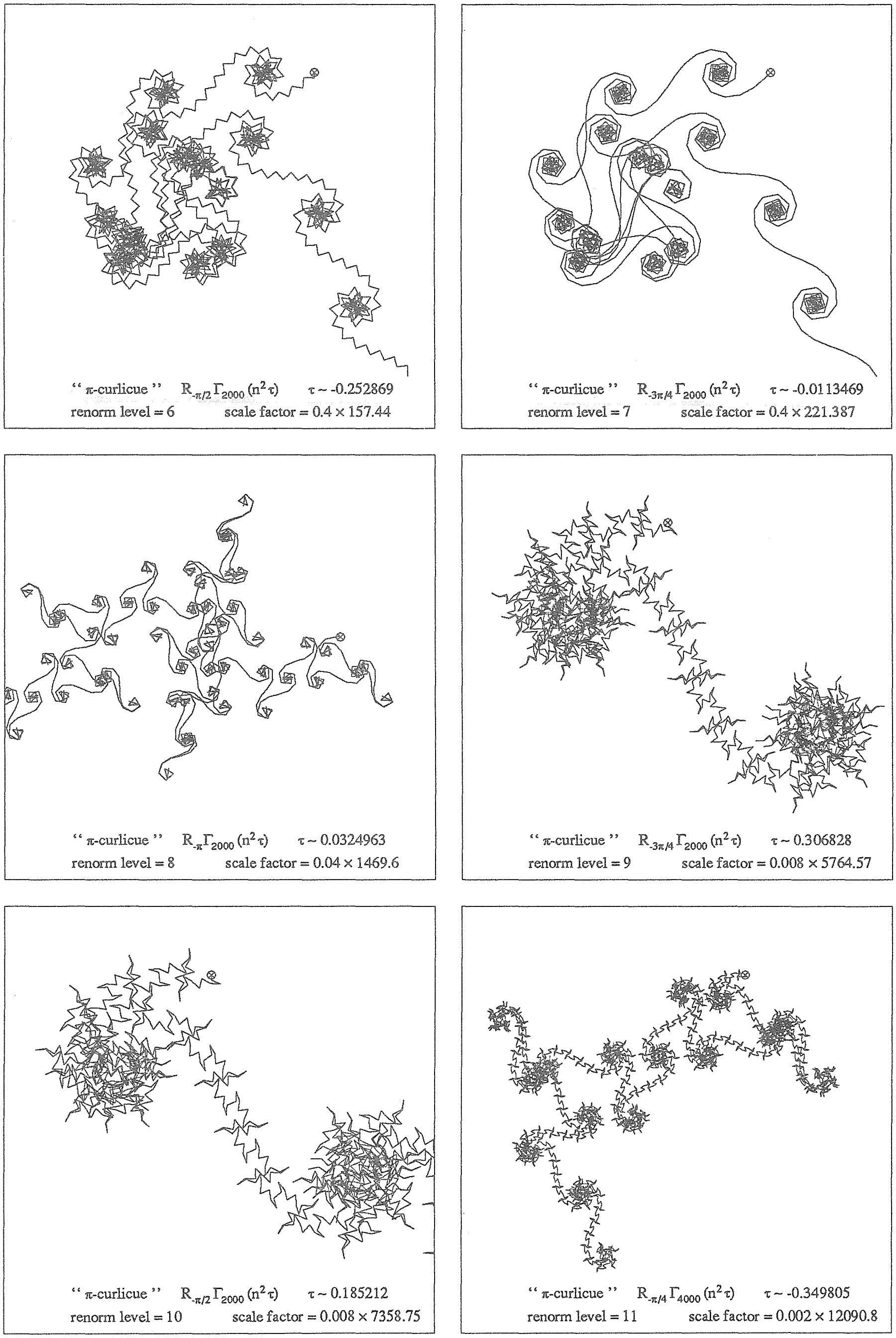
are quite close. Indeed, the term by term deviation becomes significant only for $m^2$ large and this is exactly when the phase difference has become large; so the total deviation is quite small. Thus the complete Gauß sum is a good estimate. Moreover, particularly for $k$ not too small, the approximation $(\sqrt{\tau})^{-1} \approx \sqrt{k}$ is not at all bad.
Finally, set, as above, $\tau' = ((1/2\tau)) - 1/2\tau$. Then $\tau \approx (((1/2\tau)) + \tau')/4((1/2\tau))^2$ and $N_h \approx 2h((1/2\tau))$. Hence

$$\exp(2\pi i N_h^2 \tau) \approx \exp(2\pi i \tau' h^2).$$

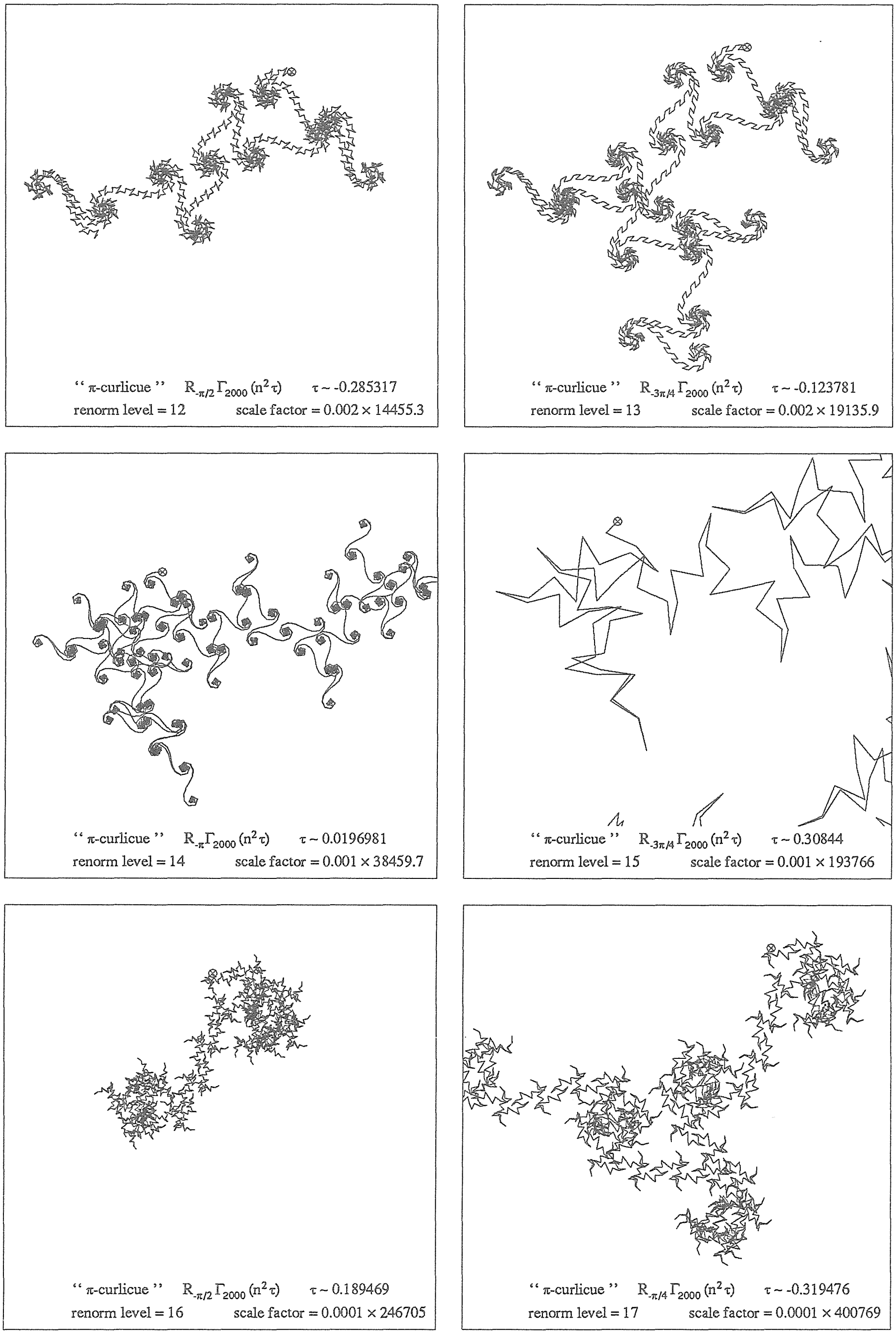
This last observation, and summing over $h$, recovers the renormalisation transformation.




Renormalisations of $\pi$, levels 0–5.




Renormalisations of $\pi$, levels 6–11 .




Renormalisations of $\pi$, levels 12–17.

### 2.5 Remarks on the renormalisations

The vertices of the renormalised graph tend to lie at the centres of the curlicues in the graph of the original sum. Approximately $1/2|\tau|$ terms of the original sum combine into a single term of the renormalised sum. The length of each renormalised term is approximately $1/\sqrt{2|\tau|}$. The renormalised graph is rotated by $\pm\pi/4$, according to the sign of $\tau$.

### 2.6 Even continued fractions

It is easy to see that the sequence $(\tau_i)$, where $\tau = \tau_0$ and $\tau_{i+1} = \tau_i'$, arises from a continued fraction expansion of the shape

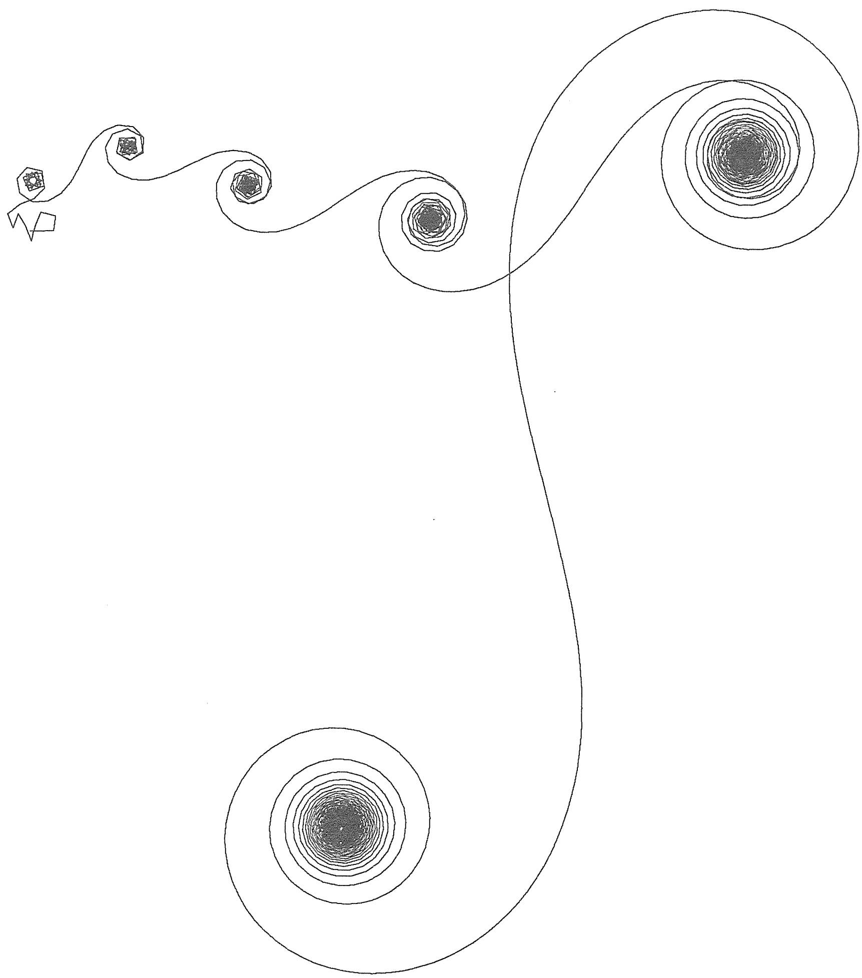
$$0 + \cfrac{1}{2a_1 + \cfrac{1}{2a_2 + \cfrac{1}{2a_3 + \ddots}}} = [0\,;\,2a_1\,,\,2a_2\,,\,2a_3\,,\,\ldots\ldots]\,.$$

Then $2a_{h+1} = (((-1)^h\tau_h))$. Given the usual regular continued fraction for $\tau$ (as one is e.g. if $\tau = e$, for it is well known that $e - 1 = [1\,;1\,,2\,,1\,,1\,,4\,,1\,,1\,,6\,,1\,,1\,,8\,,\ldots\ldots]$ — and this is a fact easier to remember than it may be to find reference to an expansion for $e$ correct to many decimal places), it becomes an interesting question to find techniques for converting regular continued fractions to such 'even' continued fractions. We are exploring this problem as an independent matter, and shall be writing about it more expansively.
One noticeable, and quite unsurprising feature of these expansions — in regular continued fraction expansions one knows that some 42% of the entries in the expansion of a random real number are 1 — is lengthy strings of entries equal to $\pm 2$. As is evident from our remarks above, in those cases the normalisation is almost to no effect. Accordingly, Berry and Goldman consider collapsed renormalisations in which a sequence of such 'ineffectual' renormalisations is combined into one. Nevertheless (because we were not quite energetic enough to incorporate this refinement in our picture sequences), the accompanying figures which illustrating our remarks make no attempt to use this efficiency; our renormalisations have not been usefully collapsed.

### 2.7 Computational techniques

In compensation, we have taken care to ensure that our drawings are true: an interesting feature of some of the graphs which inspired our interest in the matters discussed above is that they are relatively unreliable for $N$ at all large.
We found it useful to realise that graphs $\Gamma_N(u_h)$ could readily be drawn by our Apple LaserWriterPlus™ using it both as a Postscript® computer and as printer*. Indeed, the Postscript language is admirably suited to drawing a sequence of lines of constant length but with varying directions; our graphs were often drawn faster than a page of usual TeX

*We recommend this argument for use at bodies such as the ARC.



“The Loch Ness Monster” [8]. $(u_h = (\log h)^4)$.

text. They have been computed simultaneously with the text in the present presentation. In creating the graphs, our Macintosh™ had little more use than as a text processor preparing the text file comprising the program. Our desire to maintain numerical accuracy involved some minor tricks. Ultimately, however, to prepare the numerical data to be placed in the program files we required no more than the 15 decimal digit accuracy provided by the spreadsheet Excel®.

## 3. More curves

We conclude with the undigested display of a variety of curves $\Gamma_N(u)$, our principal purpose being to entertain. Nonetheless, we should remark that the curves $\Gamma_N(h(\log h)^k)$ are readily explained by phase considerations of the genre discussed at §2.2 above, and indeed provide a rather vivid illustration of those principles. We should also mention that the final blobs ultimately spread to cover the entire plane in accordance with the theorem of Dekking and Mendès France mentioned at §1.2. The curve $\Gamma_N(h^{3/2})$ virtually shouts a theorem waiting to be proved (see Deshouillers [4]); talk of a picture being worth quite a few words. The Kummer sums $\Gamma_N(h^k/N)$ surprise by their variety. Friends with birthdays in the latter part of the year (January and February are rather dull) can be honoured by being presented with a personalised graph $\Gamma_N(d\,h^m/y)$ according to their birthdate $d{:}\,m{:}\,y$. Prime birth years $y$ tend to be more attractive and to avoid unfortunate cancellation with $d$.

## References

[1] M. V. Berry and J. Goldberg, 'Renormalisation of curlicues', *Nonlinearity* **1** (1988), 1–26

[2] F. M. Dekking and M. Mendès France, 'Uniform distribution modulo one: a geometrical viewpoint', *J. für Math.* **329** (1981), 143–153

[3] Michel Dekking, Michel Mendès France and Alf van der Poorten, 'Folds!', *The Mathematical Intelligencer* **4** (1982), 130–138; II: 'Symmetry disturbed', *ibid* 173–181; III: 'More morphisms', *ibid* 190–195

[4] Jean-Marc Deshouillers, 'Geometric aspects of Weyl sums', Elementary and Analytic Theory of Numbers (Warsaw 1982) *Banach Center Publ.* **17** (PUN Warsaw 1985), 75–82

[5] Yves DuPain, Teturo Kamae and Michel Mendès France, 'Can one measure the temperature of a curve?', *Archive for Rational Mechanics and Analysis* **94** (1986), 155–163

[6] R. Feynman, *La nature de la physique*, Collection Points Seuil 1980

[7] D. H. Lehmer, 'Incomplete Gauss sums', *Mathematika* **23** (1976), 125-135

[8] J. H. Loxton, 'Captain Cook and the Loch Ness Monster', *James Cook Mathematical Notes*, (27) **3** (1981), 3060–3064

[9] Luis A. Santaló, *Integral geometry and geometric probability*, Encyclopædia of Mathematics and its Applications **1** (Addison Wesley, 1976)

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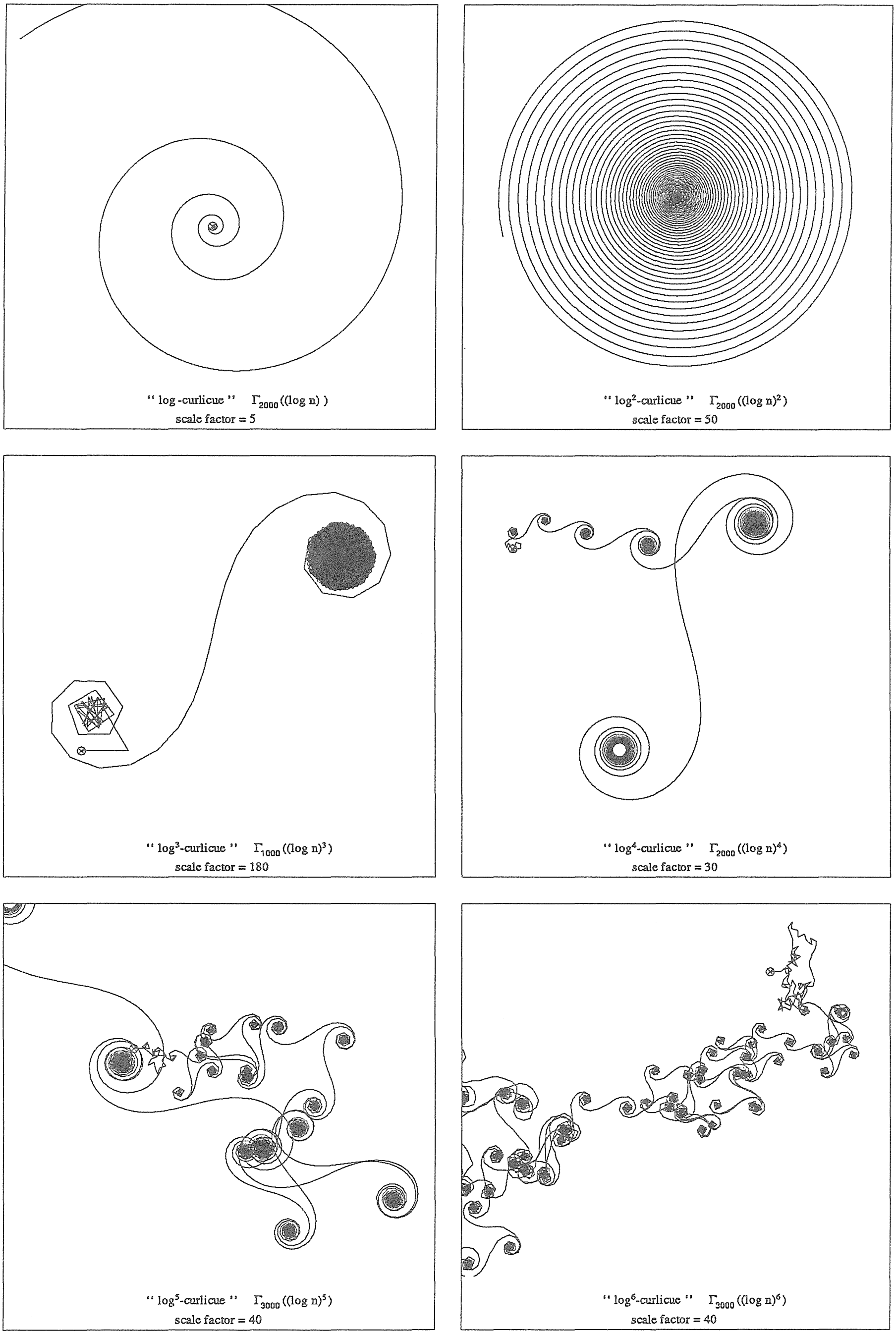
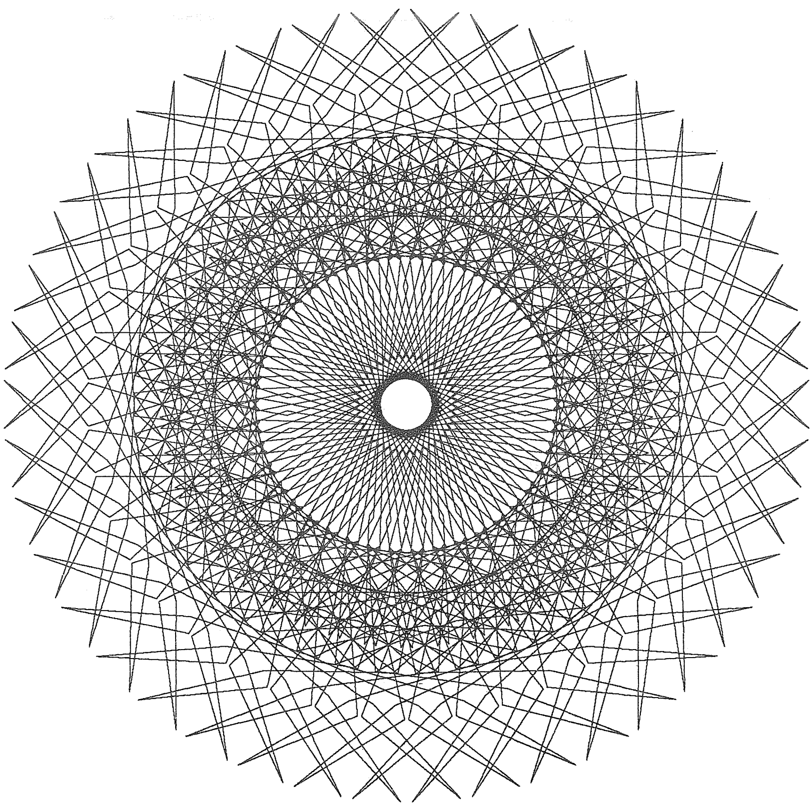
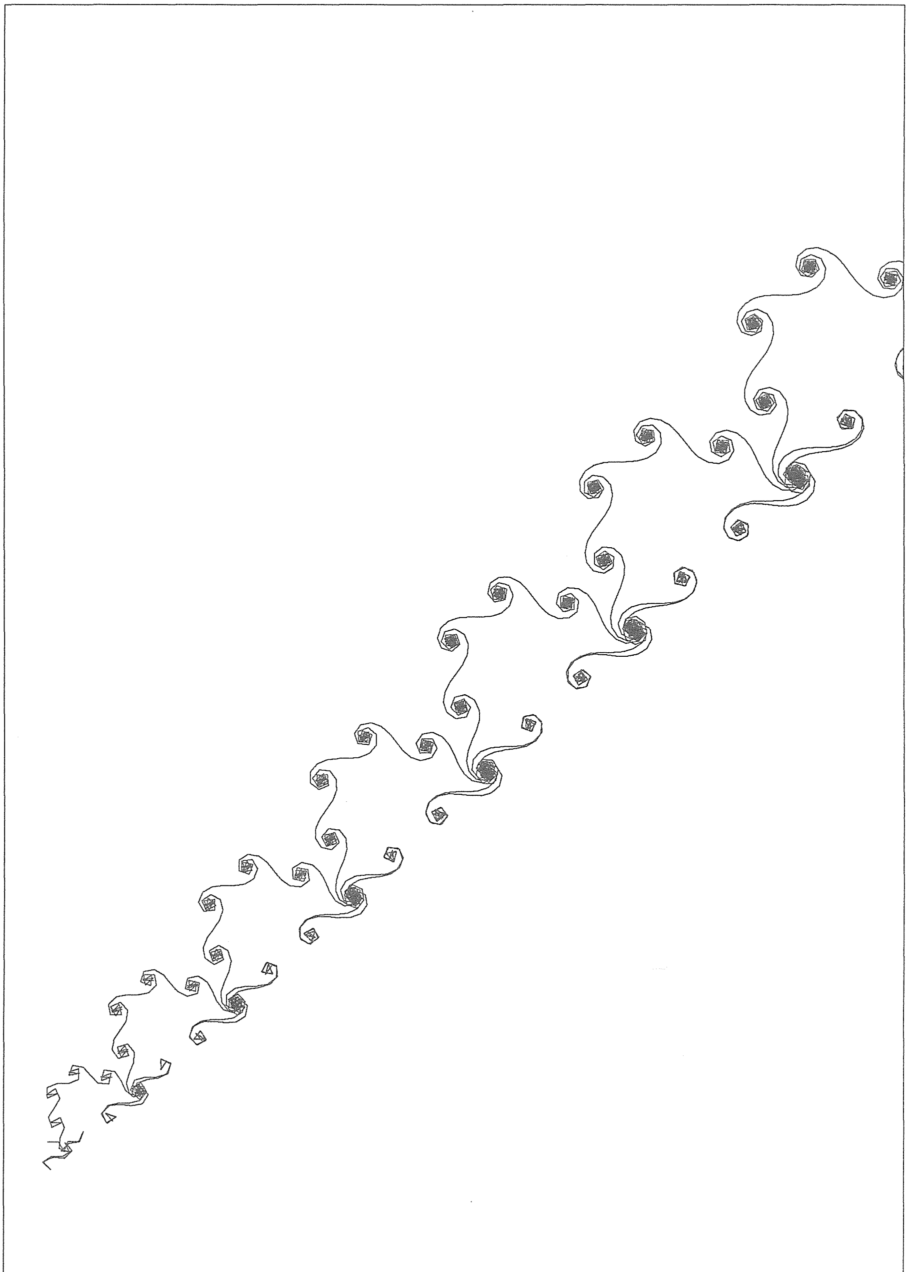


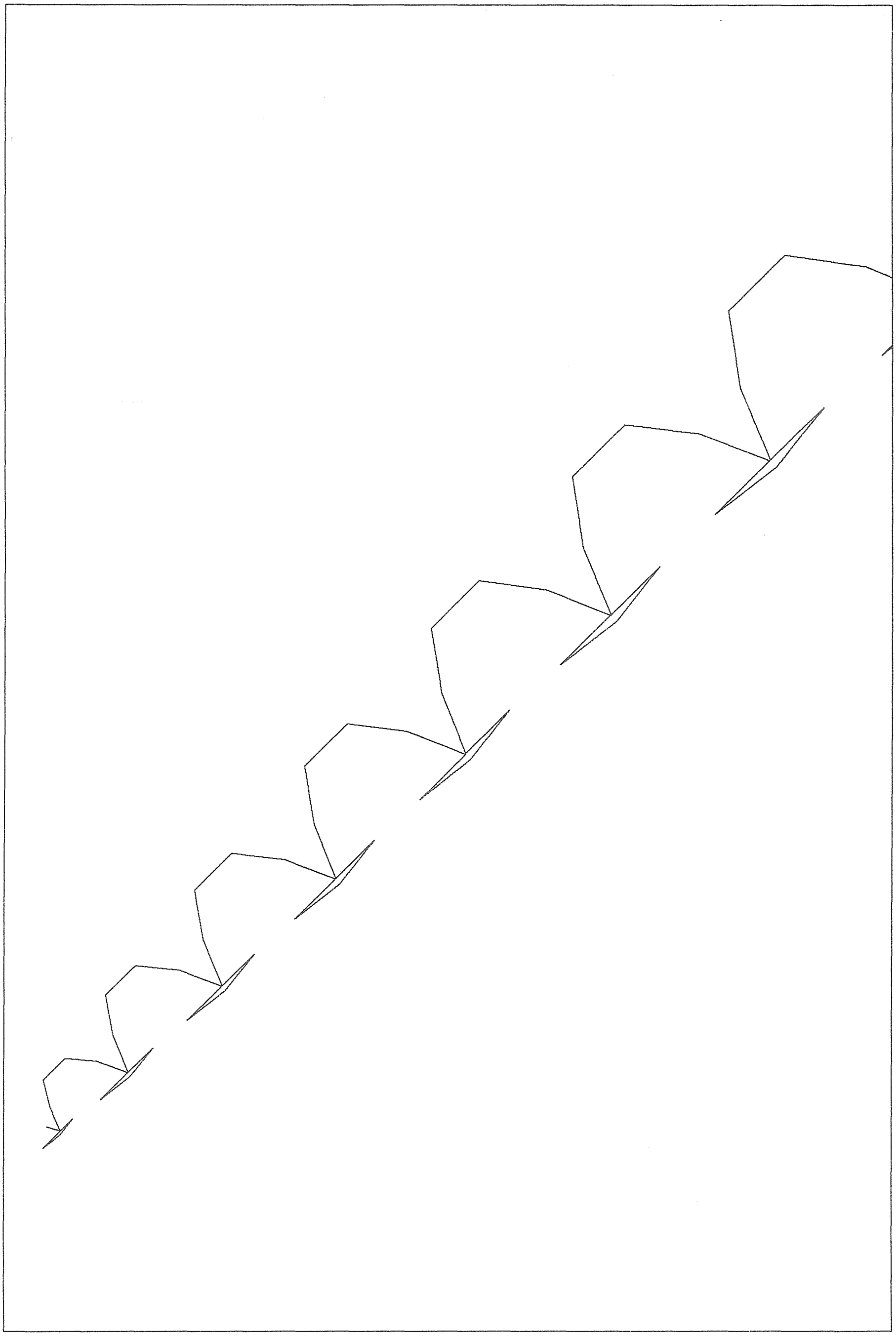

Logarithmic Powers.



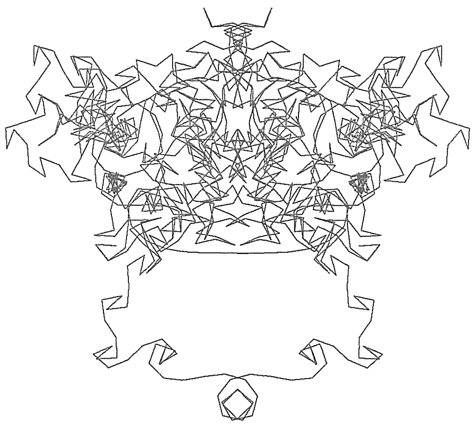
“Spirographic” $(u_h = h^7/1050)$ .

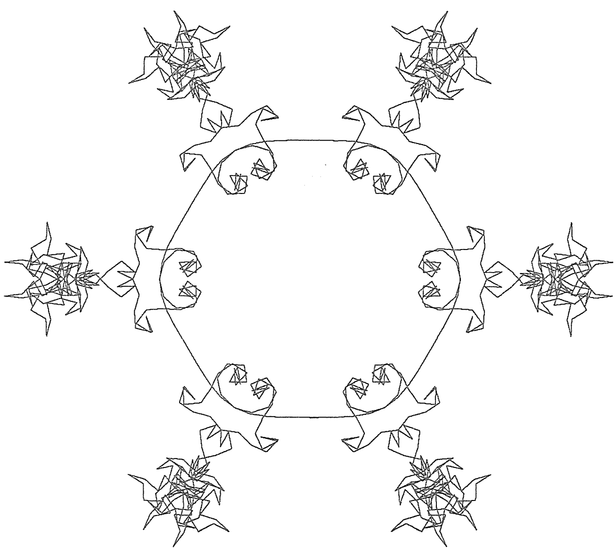


"Arches" A theorem waiting to be proved. $(u_h = h^{\frac{3}{2}})$ .

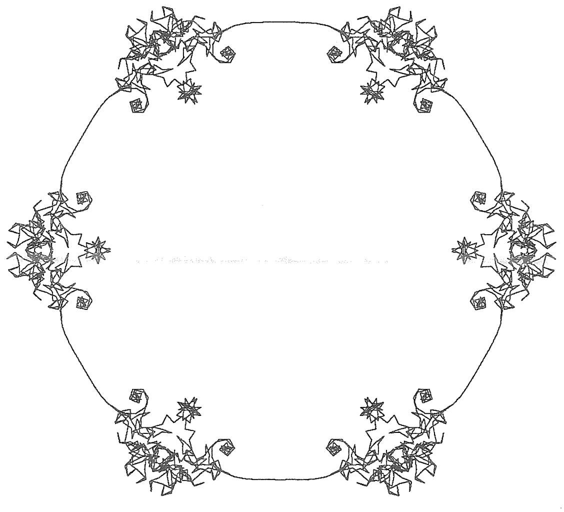


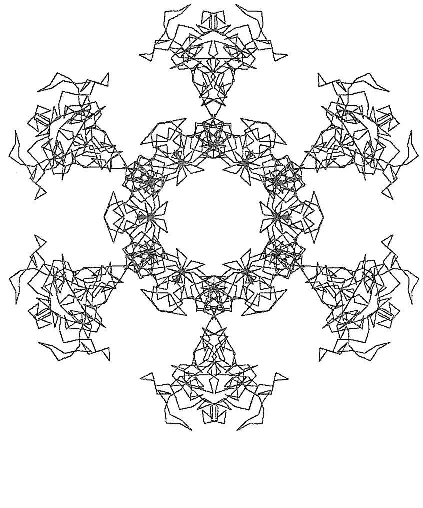
Renormalised “Arches”.



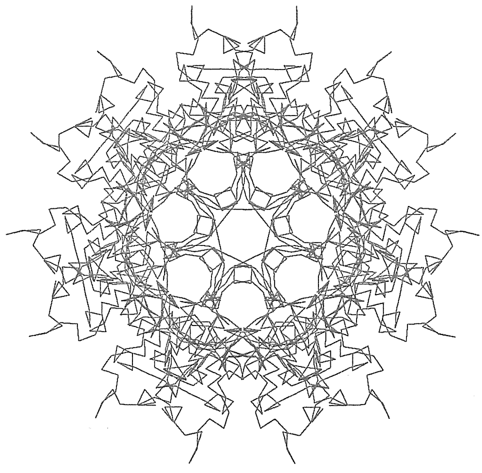


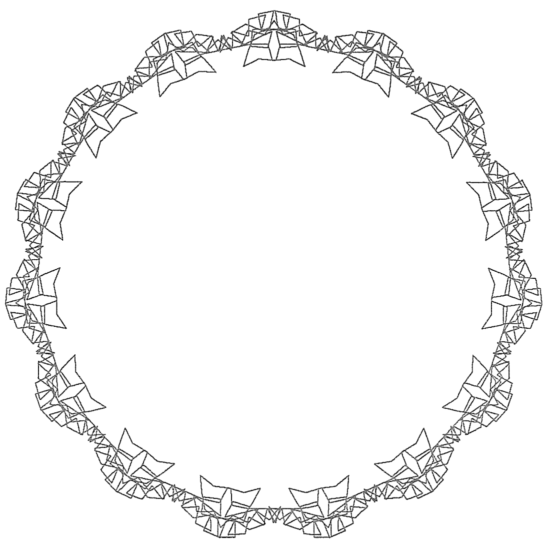
"Bull" $(u_h = h^3/1013)$ and "Bullring" $(u_h = h^3/1002)$ .



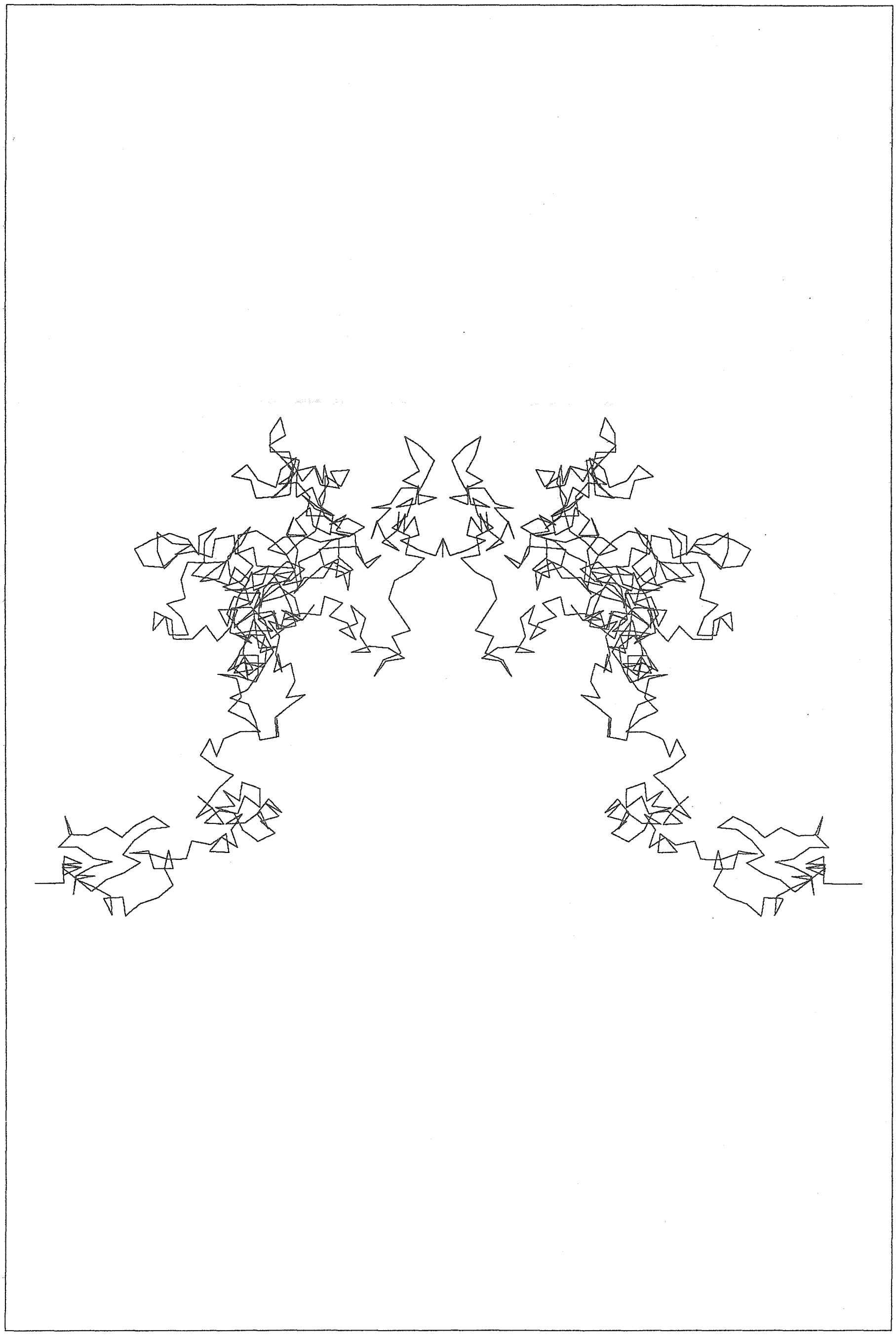


Pretty Periodic Curves: $(u_h = h^3/1986)$ and $(u_h = h^5/1986)$.

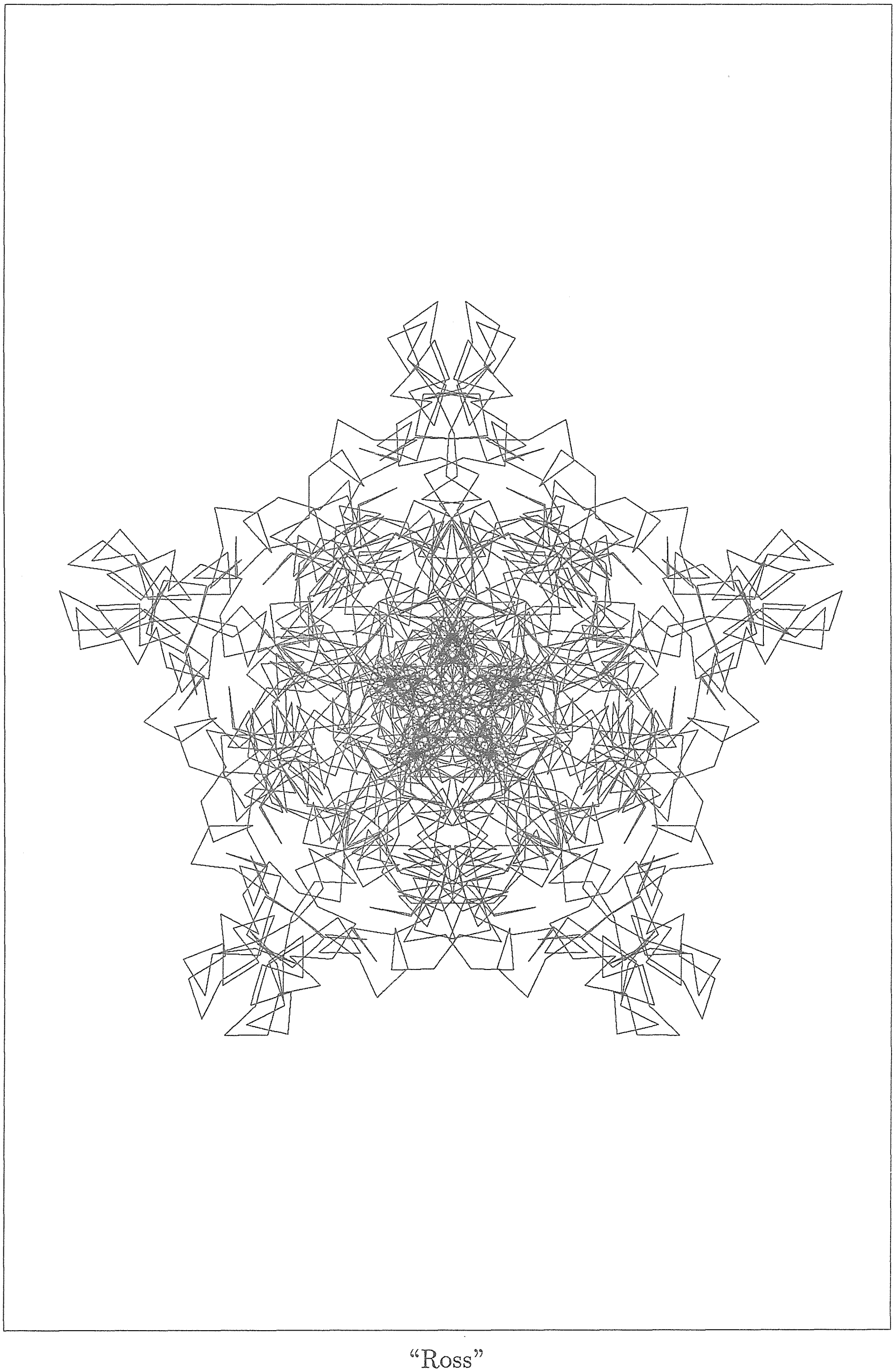




“Bicentennial” ($u_h = h^7/1988$) ; “Harbour Bridge” ($u_h = h^3/1005$) .



"Alf"



“Ross”