SURFACE MEASURES —
MAXIMAL FUNCTIONS AND FOURIER TRANSFORMS

Michael Cowling and Giancarlo Mauceri

Let S denote a smooth hypersurface in $\mathbb{R}^{n+1}$ with surface measure dS induced by the Lebesgue measure of $\mathbb{R}^{n+1}$. We fix a smooth nonnegative function w with compact support in $\mathbb{R}^{n+1}$ and consider the finite Borel measure μ with $d\mu = wdS$, which is carried by S. For any function f in the Schwartz space $\mathcal{S}(\mathbb{R}^{n+1})$ we denote by $M_t f$ the averages of f over the dilates of S —

$$M_t f(x) = \int_S f(x - ty)\, d\mu(y) \qquad \forall t \in \mathbb{R}^+, \quad \forall x \in \mathbb{R}^{n+1} \text{ —}$$

and by $M_* f$ the associated maximal function —

$$M_* f(x) = \sup_{t>0} |M_t f(x)| \qquad \forall x \in \mathbb{R}^{n+1}.$$

Our purpose is to determine the range of p's for which an ***a priori*** estimate of the form

$$\|M_* f\|_p \le C\|f\|_p \qquad \forall f \in \mathcal{S}(\mathbb{R}^{n+1}),$$

holds; this estimate entails that the sublinear operator M_* extends to a bounded operator on the Lebesgue space $L^p(\mathbb{R}^{n+1})$, hereafter abbreviated to L^p. In the last decade, since Stein's work on the "spherical maximal function" [S1], [SW], this problem has attracted considerable attention [B], [CM1], [CM2], [G], [SS1], [SS2]. It turns out that, at least when $p < 2$, the range of p's for which the maximal operator M_* is bounded on L^p is determined by the decay at infinity of the Fourier transform $\hat{\mu}$ of the measure μ.

THEOREM 1. *If for some α, $1/2 < \alpha \leq n/2$*

$$|\hat{\mu}(\lambda\sigma)| \leq C(1+\lambda)^{-\alpha} \qquad \forall\sigma \in S^n, \quad \forall\lambda \in \mathbb{R}^+,$$

then the maximal operator M_* is bounded on L^p if $p > 1 + 1/2\alpha$.

The proof of this theorem can be found in [CM1]. Later Rubio de Francia [R] proved that the theorem holds for any compactly supported Borel measure μ.

It has been known for a long time that the decay at infinity of $\hat{\mu}$ is related to the curvature of the surface S [Hl], [Hz], [L]. In particular Littman [L] proved the following result.

THEOREM 2. *If at every point the hypersurface S has at least k nonvanishing principal curvatures then*

$$|\hat{\mu}(\lambda\sigma)| \leq C(1+\lambda)^{-k/2} \qquad \forall\sigma \in S^n, \quad \forall\lambda \in \mathbb{R}^+.$$

Thus if at every point S has at least k nonvanishing curvatures, where $k \geq 2$, Theorem 1 applies and M_* is L^p-bounded for $p > 1 + 1/k$. However if for some σ in S^n the Fourier transform $\hat{\mu}(\lambda\sigma)$ decays of order less than 1/2 as λ tends to $+\infty$ (as might be the case if at some point less than 2 principal curvatures are different from zero), Theorem 1 no longer applies. Indeed examples show that in this case M_* may fail to be bounded even on L^2 [C]. Since M_* is obviously bounded on L^∞ it follows by interpolation that M_* cannot be bounded on L^p for any $p < 2$. Nevertheless, even when $\hat{\mu}$ fails to decay sufficiently fast at infinity, one can prove L^p-boundedness of the maximal operator M_* for some $p > 2$. Indeed in [CM1] the authors proved the following theorem.

THEOREM 3. *Let u be a nonnegative bounded Borel function on S such that $\mu\{x \in S : u(x) = 0\} = 0$. Suppose that there exist positive real numbers α, β, ϵ such that*

(i) $|(u^\alpha \mu)\hat{}(\lambda\sigma)| \leq C(1+\lambda)^{-1/2-\epsilon} \qquad \forall \sigma \in S^n, \quad \forall \lambda \in \mathbb{R}^+.$

(ii) $u^{-\beta}$ *is integrable with respect to the measure* μ.

Then M_* is bounded on L^p for $p > 2(1+\alpha/\beta)$.

The basic idea of the proof of Theorem 3 is that by (i) and Theorem 1 the maximal operators M_*^z corresponding to the measures $d\mu_z = u^z d\mu$ are bounded on L^2 when $\text{Re}z = \alpha$, while from (ii), the operators M_*^z are bounded on L^∞ when $\text{Re}z = \beta$. Thus, by complex interpolation, $M_* = M_*^0$ is L^p-bounded if $p > 2(1+\alpha/\beta)$.

The rôle of the function u in the statement of Theorem 3 is to mitigate the effect of the points of S where the curvature vanishes. Thus we shall call it a "mitigating factor". This result raises two natural questions:

(1) for every hypersurface S is it possible to find a mitigating factor u such that, for some exponent α, $(u^\alpha d\mu)\hat{}$ has optimal decay, i.e.

$$|(u^\alpha d\mu)\hat{}(\lambda\sigma)| \leq C(1+\lambda)^{-n/2} \qquad \forall \sigma \in S^n, \quad \forall \lambda \in \mathbb{R}^+ ?$$

(2) for any hypersurface S, how can we choose the mitigating factor to optimize the range of p's for which we can prove L^p-boundedness of M_* using Theorem 3?

We address question (1) first. Since the role of the mitigating factor is to compensate for the lack of curvature of S, a natural choice for u is the Gaussian curvature κ of S.

We recall that, if S is locally the graph of a function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, its principal curvatures are, up to a nonvanishing factor, the eigenvalues of the Hessian matrix $H\phi$ of

ϕ. Thus the Gaussian curvature κ is, up to a nonvanishing factor, the determinant of $H\phi$. In [CM1] the authors were able to exhibit an example of a class of surfaces in $\mathbb{R}^3$ for which $(\kappa^{1/2}ds)\hat{}$ has optimal decay. In [SS1] Sogge and Stein proved the following theorem.

THEOREM 4. *Let S be a smooth hypersurface in $\mathbb{R}^{n+1}$. Then*

$$|(\kappa^{2n}d\mu)\hat{}(\lambda\sigma)| \leq C(1+\lambda)^{-n/2} \qquad \forall\sigma \in S^n, \quad \forall\lambda \in \mathbb{R}^+.$$

It follows from Theorems 3 and 4 that if the Gaussian curvature of S does not vanish of infinite order at any point of S, then M_* is L^p-bounded for all p larger than a critical index $p_0(S)$. The critical index depends on the order of vanishing of κ and can be arbitrarily large. For general hypersurfaces it is not yet clear whether κ^{2n} is the lowest power of the curvature that yields optimal decay of the Fourier transform of the surface carried measure. However for convex surfaces this result has been considerably improved [CDMM].

THEOREM 5. *Let S be a compact convex hypersurface in $\mathbb{R}^{n+1}$ of class C^Q, all of whose tangent lines have order of contact at most q, where $q < Q$, and let κ denote the Gaussian curvature of S. If u is a nonnegative C^{Q-1} function on S with the property that $0 \leq u \leq \kappa^{1/2}$, then*

$$|(u\mu)\hat{}(\lambda\sigma)| \leq C(1+\lambda)^{-n/2} \qquad \forall\sigma \in S^n, \quad \forall\lambda \in \mathbb{R}^+,$$

provided that $n \leq Q-2$, $nq \leq 2(Q+n-1)$ and $q \leq Q-2$.

The proof of this theorem requires obtaining uniform estimates of the decay of oscillatory integrals depending on parameters. Indeed, by taking a partition of unity on S, and using suitable coordinate systems, $(u\mu)\hat{}(\lambda\sigma)$ can be written as the sum of two oscillatory integrals of the form

$$I(\lambda) = \int_{\mathbb{R}^n} \exp(i\lambda\phi_\sigma(x))v_\sigma(x)\,dx, \qquad \forall\lambda \in \mathbb{R}^+,$$

plus a term which is negligible as $\lambda \to +\infty$. Here the phase function $\phi_\sigma : \mathbb{R}^n \to [0, +\infty)$ is a convex C^Q-function and has a single critical point in the support of the compactly supported amplitude function $v_\sigma : \mathbb{R}^n \mapsto [0, +\infty)$. As the direction σ varies in the unit sphere S^n, the functions ϕ_σ and v_σ vary continuously in C^Q and C^{Q-1} respectively, and one must obtain estimates of $I(\lambda)$ which are uniform in σ. The oscillatory integral I is controlled by the volume integral V_σ —

$$V_\sigma(t) = \int_{\{x:\phi_\sigma(x)\leq t\}} v_\sigma(x)\,dx \qquad \forall t \in \mathbb{R}^+.$$

Indeed it is easy to see that

$$I(\lambda) = \int_{\mathbb{R}} \exp(i\lambda t)\,dV_\sigma(t) \qquad \forall\lambda \in \mathbb{R}^+.$$

In terms of the hypersurface S this fact has a simple geometric interpretation. For fixed σ in S^n denote by $p(\sigma)$ the point of S whose inward unit normal is σ and by $C(\sigma, t)$ the cap at $p(\sigma)$ of height t, t in $\mathbb{R}^+$,

$$C(\sigma, t) = \{p \in S : \; (p - p(\sigma)) \cdot \sigma \leq t\}.$$

If u is a nonnegative measurable function on S denote by $V(u, \sigma, t)$ —

$$V(u, \sigma, t) = \int_{C(\sigma,t)} u(p)\,d\mu(p) \text{ —}$$

the u-volume of the cap. Then

$$|(u\mu)\hat{}(\lambda\sigma)| \leq C\{V(u,\sigma,\lambda^{-1}) + V(u,-\sigma,\lambda^{-1})\} + \text{higher order terms.}$$

(see [CDMM] Theorem 5.1 for a more precise statement). When u is nonvanishing and $d\mu = dS$ this estimate was proved by Bruna, Nagel and Wainger [BNW]. The second key result in [CDMM] is the estimate

$$V(\kappa^{1/2},\sigma,t) \leq Ct^{n/2} \qquad \forall t \in \mathbb{R}^+. \tag{2}$$

By combining (1) and (2) one easily gets the desired estimate of $(u\mu)\hat{}$.

Examples show that Theorem 5 is sharp: there are smooth convex hypersurfaces for which no measure $\kappa^a d\mu$, with α less than 1/2, has optimal Fourier transform decay [CM2]. In the nonconvex case it is still an open problem to determine the lowest α for which $(u\mu)\hat{}$ has optimal decay for all smooth function u such that $0 \leq u \leq \kappa^\alpha$. It is known that α must be at least 2.

The last part of this note is a contribution toward a solution of question 2: can we choose a different mitigating factor so as to optimize the range of p's for which we can prove L^p-boundedness of the maximal operator? Notice that in order to apply Theorem 3 we do not need full decay of $(u\mu)\hat{}$. Any decay of order better than 1/2 will suffice. Littman's result (Theorem 2) suggests that we consider mitigating factors which are products of powers of principal curvatures of S.

THEOREM 6. *Let S be a hypersurface satisfying the assumptions of Theorem 5. Let $k_1,\ldots,k_n$ denote the principal curvatures of S, and let $\theta_1,\ldots,\theta_n$ be nonnegative numbers* *whose sum θ is less than or equal to 1. If u is a C^{Q-1}-function on S with the property that*

$$0 \le u \le (k_1^{\theta_1} \cdots k_n^{\theta_n})^{1/2},$$

then $$|(u\mu)\hat{}(\lambda\sigma)| \le C\lambda^{-[(1/2-1/q)\theta+n/q]} \qquad \forall\lambda \in \mathbb{R}^+, \quad \forall\sigma \in S^n,$$

provided that $\max(n,q) \le Q-2$ *and* $\theta(q/2-1) \le Q-1$.

Proof. By Theorem 5.1 of [CDMM], it is sufficient to show that if

$$V(\underline{k}^{\theta/2}, \sigma, t) = \int_{C(\sigma,t)} (k_1^{\theta_1} \cdots k_n^{\theta_n})^{1/2} d\mu,$$

then, for some C independent of σ in S^n,

$$V(\underline{k}^{\theta/2}, \sigma, t) \le Ct^{(1/2-1/q)\theta+n/q} \qquad \forall t \in \mathbb{R}^+.$$

(The restrictions on Q imply that the contributions of the error terms in Theorem 5.1 of [CDMM] may be neglected). Let $\pi_0 = 1$, and let $\pi_j = k_1 \cdots k_j$ be the product of the first j principal curvatures, $j = 1, \ldots, n$.

We shall first estimate $V(\pi_j^{1/2}, \sigma, t)$. Let p be the point of S whose inward unit normal is σ. Choose a coordinate system in $\mathbb{R}^{n+1}$ "based at p" by choosing an orthonormal frame $\{\tau_0, \tau_1, \ldots, \tau_n\}$ at p such that $\tau_1, \ldots, \tau_n$ span the tangent space at p and τ_0 points in the direction of σ. As in [CDMM] we shall denote by ϕ_σ the C^Q-function defined in a neighborhood of the origin in $\mathbb{R}^n$ whose graph is a subset of S. By rescaling, if necessary, we may assume that ϕ_σ is defined on $B(2)$, the ball of radius 2 in $\mathbb{R}^n$, for every σ in S^n. If $\xi = (\xi_1, \ldots, \xi_n)$ is a vector in $\mathbb{R}^n$ we shall write $\xi = (\xi', \xi'')$ where $\xi' = (\xi_1, \ldots, \xi_j)$ and

$\xi'' = (\xi_{j+1}, \ldots, \xi_n)$. Define Ω_1 and Ω_2 by the formulae

$$\Omega_1(\sigma, t, \xi'') = \{\xi' : (\xi', \xi'') \in B(2),\ \phi_\sigma(\xi', \xi'') \leq t\} \qquad \forall \xi'' \in \mathbb{R}^{n-j},$$

$$\Omega_2(\sigma, t) = \{\xi'' : \Omega_1(\sigma, t, \xi'') \neq \emptyset\}.$$

Then

$$\begin{aligned} V(\pi_j^{1/2}, \sigma, t) &= \int_{C(\sigma,t)} (k_1 \cdots k_j)^{1/2} d\mu \\ &= \int_{\{\xi \in B(2), \phi_\sigma(\xi) \leq t\}} (\det{}' H\phi_\sigma(\xi))^{1/2} w_\sigma(\xi)\, d\xi \end{aligned}$$

where $\det' H\phi_\sigma$ is the determinant of the first j rows and columns of the Hessian matrix $H\phi_\sigma$ and w_σ is of class C^{Q-1}, uniformly with respect to σ. Thus

$$V(\pi_j^{1/2}, \sigma, t) = \int_{\Omega_2(\sigma,t)} \int_{\Omega_1(\sigma,t,\xi'')} (\det{}' H\phi_\sigma(\xi', \xi''))^{1/2} w_\sigma(\xi', \xi'')\, d\xi' d\xi''.$$

For every ξ'' in $\Omega_2(\sigma, t)$, let

$$\psi_\sigma(\xi', \xi'') = \phi_\sigma(\xi', \xi'') - \min\{\phi_\sigma(\eta, \xi'') : \eta \in \Omega_1(\sigma, t, \xi'')\}.$$

Then by Proposition 4.4 of [CDMM],

$$\begin{aligned} \text{(3)} \qquad & \int_{\Omega_1(\sigma,t,\xi'')} (\det{}' H\phi_\sigma(\xi', \xi''))^{1/2} w_\sigma(\xi', \xi'')\, d\xi' \\ & \leq C_1 \|w_\sigma\|_\infty \ \sup\{|\psi_\sigma(\eta, \xi'')|^{j/2} : \eta \in \Omega_1(\sigma, t, \xi'')\} \\ & \leq C_2 t^{j/2}. \end{aligned}$$

On the other hand, since the tangent lines to S have order of contact at most q, there exist a positive constant m, independent of σ in S^n, such that $\phi_\sigma(\xi) \geq m|\xi|^q$, for all ξ in B(2). Thus $\{\xi : \xi \in B(2),\ \phi_\sigma(\xi) \leq t\} \subseteq B((t/m)^{1/q})$ and therefore

$$\text{(4)} \qquad \int_{\Omega_2(\sigma,t)} d\xi'' \leq \int_{\{\xi'' \in B(2) : |\xi''| \leq (t/m)^{1/q}\}} d\xi'' \leq C_3 t^{(n-j)/q}.$$

Combining estimates (3) and (4) we get

$$V(\pi_j^{1/2},\sigma,t) \le C_4 t^{(1/2-1/q)j+n/q} \qquad \forall j \in \{0,\ldots,n\}.$$

Next we estimate $V(\underline{k}^{\theta/2},\sigma,t)$. By permuting the ordering of the curvatures, if necessary, we may assume that $\theta_1 \ge \theta_2 \ge \cdots \ge \theta_n \ge 0$. Let $\alpha_n = \theta_n$, $\alpha_j = \theta_j - \theta_{j+1}$, $j=1,\ldots,n-1$ and $\alpha_0 = 1 - \sum_{j=1}^{n} \alpha_j$. Then $k_1^{\theta_1}\cdots k_n^{\theta_n} = \pi_1^{\alpha_1}\cdots\pi_n^{\alpha_n}$.

By simple application of Hölder's inequality to the conjugate exponents $\alpha_0^{-1},\alpha_1^{-1},\ldots,\alpha_n^{-1}$ we get

$$\begin{aligned}\int_{C(\sigma,t)} \underline{k}^{\theta/2}\,d\mu &\le \prod_{j=0}^{n}\Big(\int_{C(\sigma,t)} \pi_j^{1/2}\,d\mu\Big)^{\alpha_j}\\ &\le C_5 \prod_{j=0}^{n} t^{[(1/2-1/q)j+n/q]\alpha_j}\\ &= C_5 t^{(1/2-1/q)\theta+n/q}\end{aligned}$$

since $\sum_{j=0}^{n}\alpha_j = 1$ and $\sum_{j=1}^{n} j\alpha_j = \sum_{j=1}^{n}\theta_j = \theta$.

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School of Mathematics
University of New South Wales
P.O. Box 1,
Kensington, N.S.W. 2033
Australia

Dipartimento di Matematica,
Università di Genova,
Via L.B. Alberti 4,
I-16132 Genova,
Italy

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