MONOTONE OPERATORS AND OBSTACLE PROBLEMS

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1. INTRODUCTION

Let Ω be a bounded open set in R^n and let ψ be a continuous function on $\partial\Omega$ (the boundary of Ω). Consider first of all the following problem.

Does there exist a continuous function u on the closure $\bar{\Omega}$ of Ω, agreeing with ψ on $\partial\Omega$, C'' on Ω and satisfying the non-linear partial differential equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, u(x), Du(x)) - b(x, u(x), Du(x)) = 0 \tag{1}$$

for $x \in \Omega$. The a_i and b are given functions.

If the a_i and b are sufficiently smooth and satisfy an ellipticity condition as well as certain growth conditions and if $\partial\Omega$ is sufficiently smooth, then such a function u is known to exist. One way of proving this is to use fixed point theory.

When such a function u exists and ϕ is a suitably smooth function, vanishing on $\partial\Omega$, then repeated integration and integration by parts gives

$$\sum_{i=1}^{n} \int_{\Omega} a_i(x, u, Du) \frac{\partial \phi}{\partial x_i} dx + \int_{\Omega} b(x, u, Du)\phi dx = 0. \tag{2}$$

Equation (2) still makes sense when the a_i, b and u are less smooth. u is called a weak solution when (2) holds for all ϕ.

When showing that a weak solution exists (under appropriate conditions on a_i, b and $\partial\Omega$), fixed point theory does not seem to work. But a valuable tool is provided by the theory of monotone and pseudo-monotone operators. This theory is based on the following two results in R^n.

(A) The Brouwer fixed point theorem.

(B) If K is a compact convex, non-empty subset of R^n and $x \in R^n \sim K$, then there is a $y \in K$, such that

$$(u - y) \cdot (y - x) \geq 0$$

for all $u \in K$.

The theory of monotone operators was begun by George Minty in 1962 and since then has been considerably developed and expanded into the theory of pseudo- monotone operators. The theory is fairly easily developed from (A) and (B) but applying it to problems like (2) requires quite a lot of hard work. It was not easy, however, to develop the theory in such a way that it could be applied extensively.

The theory has such wider applications than to problems like (2). If we put

$$\alpha = \inf \int_\Omega |Dv(x)|^2 dx = \inf \int_\Omega \left[\sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}(x) \right)^2 \right] dx, \tag{3}$$

where the infimum is taken over the set E of all Sobolev functions on Ω, agreeing with ψ on $\partial\Omega$ in a suitable way and if we assume that $E \neq \emptyset$, then it is well-known that there exists a $u \in E$, such that

$$\int_\Omega |Du(x)|^2 dx = \alpha \tag{4}$$

and that

$$\Delta u(x) = 0; i.e. \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(x) = 0 \tag{5}$$

for $x \in \Omega$.

Now let θ be a function defined on a non-empty subset F of Ω. Put

$$\beta = \inf \int_\Omega |Dv(x)|^2 dx, \tag{6}$$

where now the infimum is taken over the set E^* of all Sobolev functions v, agreeing with ψ on $\partial\Omega$ in a suitable way and such that

$$v(x) \geq \theta(x) \tag{7}$$

for all $x \in F$ (except for certain exceptional values of x, which will be specified later).

This is a simple example of an obstacle problem. If one assumes that $E^* \neq \phi$, then one can show that there exists a $u \in E^*$, with

$$\int_\Omega |Du(x)|^2 dx = \beta. \tag{8}$$

But in this case, u does not satisfy an equation, not even weakly. An elementary argument shows that

$$\sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i}(x) \frac{\partial \phi}{\partial x_i}(x) dx \geq 0 \tag{9}$$

for all Sobolev functions ϕ on Ω with $\phi = 0$ on $\partial\Omega$ (i.e. $\phi \in W_0^{1,2}(\Omega)$) and with

$$\phi(x) \geq \theta(x) - u(x) \tag{10}$$

for all $x \in F$ (excluding the exceptional values).

(9) is a simple example of a variational inequality. Variational inequalities can come from many sources other than obstacle problems.

2. GENERAL OBSTACLE PROBLEMS

Let α and γ be real numbers, such that $\alpha > 1$ and $\sup\{\alpha - 1, 1\} < \gamma < a$. Let a_i and b be functions on $\Omega \times R \times R^n$, which satisfy the following conditions.

(i) Each a_i and b is a Caratheodory function; i.e., for almost all $x \in \Omega$, $a_i(x, \cdot, \cdot)$ and $b(x, \cdot, \cdot)$ are continuous on $R \times R^n$, while for all $z \in R$ and $p \in R^n$, $a_i(\cdot, z, p)$ and $b(\cdot, z, p)$ are measurable on Ω.

(ii) There exists a constant $\mu > 0$ and such that

$$|a(x,z,p)| \leq \mu |p|^{\alpha-1} + \mu |z|^{\gamma-1} + \mu, \tag{11}$$

$$|b(x,z,p)| \leq \mu |p|^{\alpha-1} + \mu |z|^{\alpha-1} + \mu \tag{12}$$

and

$$\begin{aligned} p \cdot a(x,z,p) &+ zb(x,z,p) \\ &\geq |p|^{\alpha} - \mu |z|^{\gamma} - \mu \end{aligned} \tag{13}$$

for all $x \in \Omega$, $z \in R$ and $p \in R^n$.

(iii) $[a(x,z,p) - a(x,z,q)] \cdot (p - q) > 0$ for all $z \in \Omega$, $z \in R$ and $p, q \in R^n$ with $p \neq q$.

Consider the following general obstacle problem.

Does there exist a $u \in W^{1,a}(\Omega)$ agreeing with ψ on $\partial\Omega$ in some way, with $u(x) \geq \theta(x)$ for $x \in F$ (excluding the exceptional values) and such that

$$\sum_{i=1}^{n} \int_\Omega a_i(x, u(x), b(x)) \frac{\partial \phi}{\partial x_i}(x)dx + \int_\Omega b(x, u(x), Du(x))\phi(x)dx \geq 0 \tag{14}$$

for all $\phi \in W_0^{1,\alpha}(\Omega)$ with

$$\phi(x) \geq \theta(x) - u(x) \tag{15}$$

for all $x \in F$ (excluding the exceptional values).

If the inequality (14) comes from a variational problem, then the existence of u ; can be proved by variational methods as described earlier. If it does not, then under certain assumptions on θ, ψ and $\partial\Omega$, monotone or pseudo-monotone operators may be used.

First of all, one must decide in what sense the solution is required to be $\geq \theta$. The solution will be a Sobolev function and these are defined almost everywhere, so one is tempted to say $u(x) \geq \theta(x)$ except for a set of measure zero. This approach would rule out an important case, discussed by H. Lewy in 1968. In Lewy's case, $\Omega \subset R^3$, the set F is a straight line segment and θ is a continuous function on F. Since F has measure zero, then for *every* Sobolev function v, one has $v(x) \geq \theta(x)$ for almost all $x \in F$.

In some work done jointly with W.P. Ziemer on obstacle problems, capacity was used. In this work, Bessel capacity was used, but it will be easier to explain here, if I use Riesz capacity. The Riesz capacity $R(E)$ of a bounded subset E of R^n is defined to be

$$\inf \int_{R^n} f(x)^\alpha dx, \tag{16}$$

where the infimum is taken over all non-negative $f \in L^\alpha(R^n)$ such that

$$\int_{R^n} |y - x|^{1-n} f(x)dx \geq 1 \tag{17}$$

for all $y \in E$. Capacity is subadditive but not additive. A set of capacity zero is much smaller than a set of measure zero.

For each Sobolev function $g \in W^{1,\alpha}(\Omega)$, it can be shown that

$$\lim_{\rho \to 0+} \fint_{B_\rho(x)} g(\xi) d\xi \tag{18}$$

exists, except for a set of capacity zero ($\fint$ denotes the integral average). (18) is used to extend the domain of definition of each Sobolev function.

We then require that the solution u of (14) should satisfy $u(x) \geq \theta(x)$ for quasi-all $x \in F$; i.e. for all $x \in F$ except for a set of capacity zero and that (15) should hold for quasi-all $x \in F$. It is not difficult to show that a theory based on capacity will include the Lewy example. When applying the theory of pseudo-monotone operators to (14), one needs the following theorem of Egeroff type, which is proved in [Mi].

2.1. THEOREM

Let Γ be an open set in R^n and let $\{v_r\}$ converge strongly to v in $W^{1,\alpha}(\Gamma)$. Then there is a subsequence $\{v_{r_s}\}$ which converges pointwise to v, quasi-everywhere on Γ.

In [MZ] it is assumed that Ω satisfies a Wiener criterion at each boundary point and that (i), (ii) and (iii) hold. A restriction has to be placed on the obstacle function θ, particularly when θ is unbounded. By using pseudo-monotone operators, it is shown that there exists a function $u \in W^{1,\alpha}_{loc}(\Omega)$, such that

$$u(x) \geq \theta(x) - u(x)$$

for quasi-all $x \in F$, (14) holds for every $\phi \in W^{1,\alpha}(\Omega)$ which vanishes outside a compact subset of Ω and has

$$\phi(x) \geq \theta(x) - u(x)$$

for quasi-all $x \in F$ and

$$\lim_{\xi \to x} u(x) = \psi(x)$$

for all $x \in \partial\Omega$.

In [Mi] obstacle problems involving higher order operators are discussed, general bounded domains are considered, but the solutions only satisfy the boundary conditions in a weak sense. Again, the theory of pseudo-monotone operators is used.

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