Functional Calculus for Non-Commuting Operators

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1. Introduction

Let $\mathscr{B}$ be a unital Banach algebra and $a = (a_1, \ldots, a_m) \in \mathscr{B}^m$. We construct a functional calculus $\Phi_a : \mathscr{F} \to \mathscr{B}$ with a joint spectrum $\gamma(a)$. The space $\mathscr{F}$ is a Banach algebra of functions $f : \mathbb{R}^m \to \mathbb{C}$ and Φ_a is a bounded linear transformation with compact support supp (Φ_a) in $\mathbb{R}^m$. If the a_j commute then Φ_a is a homomorphism and if also f is a polynomial in a neighbourhood of supp (Φ_a) then $\Phi_a(f) = f(a)$. In the non-commuting case weaker properties are retained.

Our primary interest is in the case $\mathscr{B} = \mathscr{B}(X)$, the space of bounded linear operators on the Banach space X. However, it is convenient to formulate the results in the more general setting. This work extends that of Taylor [9], Anderson [1], McIntosh and Pryde [5] and Pryde [6].

1980 Mathematics subject classification (1985 version): 47A10, 47A60.

2. Vector-valued distribution

The construction of Φ_a is via $\mathscr{B}$-valued distributions and the compactness of supp (Φ_a) follows from the Paley-Wiener theorem.

For this section we require only that $\mathscr{B}$ be a Banach space. Let $\mathscr{S}(\mathbb{R}^m)$ denote the Schwartz space of rapidly decreasing functions with its natural Frechet topology. Let $L(\mathscr{S}(\mathbb{R}^m), \mathscr{B})$ denote the space of continuous linear functions from $\mathscr{S}(\mathbb{R}^m)$ to $\mathscr{B}$, that is the space of $\mathscr{B}$-valued tempered distributions.

A function $e : \mathbb{C}^m \to \mathscr{B}$ is called *entire* if it is norm differentiable in each variable ζ_j at each $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$. Such a function is of *Paley-Wiener type (s, r),* where $s, r \geq 0$, if

$$\| e(\zeta) \| \leq c(1 + |\zeta|)^s e^{r|\mathrm{Im}\, \zeta|}$$

for all $\zeta \in \mathbb{C}^m$ and some $c > 0$.

If e is entire of Paley-Wiener type then it *generates* a distribution $E : \mathscr{S}(\mathbb{R}^m) \to \mathscr{B}$ where $E(f) = (2\pi)^{-m} \int_{\mathbb{R}^m} e(\xi)\, f(\xi)\, d\xi$. This integral is the Bochner integral of the $\mathscr{B}$-valued integrand.

Each tempered distribution $E : \mathscr{S}(\mathbb{R}^m) \to \mathscr{B}$ has a *Fourier transform* $\hat{E} : \mathscr{S}(\mathbb{R}^m) \to \mathscr{B}$ defined by $\hat{E}(f) = E(\hat{f})$ where $\hat{f}(\lambda) = \int_{\mathbb{R}^m} e^{-i<\lambda,\, \xi>} f(\xi)\, d\xi$ and $<\lambda, \xi> = \lambda_1\xi_1 + \ldots + \lambda_m\xi_m$. So, if E is generated by e then its Fourier transform W is given by $W(f) = (2\pi)^{-m} \int_{\mathbb{R}^m} e(\xi)\, \hat{f}(\xi)\, d\xi$.

The *support*, supp (W), of a distribution W is the smallest closed set K in $\mathbb{R}^m$ such that W(f) = 0 whenever f has compact support disjoint from K.

Theorem 2.1 (Paley-Wiener theorem) Let $W \in L(\mathscr{S}(\mathbb{R}^m), \mathscr{B})$. Then W has compact support if and only if W is the Fourier transform of a distribution E generated by an entire function e of Paley-Wiener type (s, r) for some s, $r \geq 0$. In that case, $\text{supp}(W) \subseteq \{\lambda \in \mathbb{R}^m : |\lambda| \leq r\}$.

The proof of this theorem follows readily from the corresponding theorem for scalar-valued distributions. For the latter, see for example Reed and Simon [8]. The entire function e is obviously unique and we shall call it the *symbol* of W.

Let $C_c^\infty(\mathbb{R}^m)$ denote the space of infinitely differentiable functions on $\mathbb{R}^m$ with compact support. If $p : \mathbb{R}^m \to \mathbb{C}$ is a polynomial, say

$$p(\lambda) = \sum_{|\alpha| \leq m} a_\alpha \lambda^\alpha ,$$

then

$$p(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

where $D = (D_1, \ldots, D_m)$ and $D_j = \frac{1}{i}\frac{\partial}{\partial\lambda_j}$. The following result was proved in [1].

Theorem 2.2 Let $W : \mathbb{R}^m \to \mathscr{B}$ be a compactly supported distribution with symbol e. Let $\theta \in C_c^\infty(\mathbb{R}^m)$ be identically 1 on a neighbourhood of supp (W). Then for all polynomials $p : \mathbb{R}^m \to \mathbb{C}$, $W(\theta p) = p(D)e(0)$.

3. Examples

In this section we exhibit some entire functions e of Paley-Wiener type. These symbols give rise to functional calculi constructed in the following section. Again let $\mathscr{B}$ denote a unital Banach algebra.

We shall say that an m-tuple $a = (a_1, \ldots, a_m) \in \mathscr{B}^m$ is of Paley-Wiener type (s, r), where $s, r \geq 0$, if

$$\| e^{i<a,\, \zeta>} \| \leq c(1 + |\zeta|)^s \, e^{r|\mathrm{Im}\, \zeta|}$$

for all $\zeta \in \mathbb{C}^m$ and some $c > 0$. As elsewhere, $<a, \zeta> = a_1\zeta_1 + \ldots + a_m\zeta_m$.

So an m-tuple a is of Paley-Wiener type (s, r) if and only if the function $e_a : \zeta \mapsto e^{i<a,\, \zeta>}$ is of Paley-Wiener type (s, r).

<u>Example 3.1</u> Let $a_1, \ldots, a_m$ be bounded self-adjoint operators on a Hilbert space H. Taylor [9] proved that $a = (a_1, \ldots, a_m)$ is of Paley-Wiener type $(0, r)$ where $r = (\| a_1 \|^2 + \ldots + \| a_m \|^2)^{1/2}$.

<u>Example 3.2</u> Let $b \in \mathscr{B}(X)$ where X is a Banach space. It is proved in Colojoara and Foias [3] that b is a generalized scalar operator with real spectrum if and only if $\| e^{ib\xi} \| \leq c(1 + |\xi|)^s$ for all $\xi \in \mathbb{R}$ and some $s, c \geq 0$. Hence b is generalized scalar with real spectrum if and only if it is of Paley-Wiener type.

It follows that for commuting operators $a_1, \ldots, a_m$ in $\mathscr{B}(X)$, the function e_a is of Paley-Wiener type if and only if each a_j is generalized scalar with real spectrum.

<u>**Example 3.3**</u> Let $a_j \in \mathscr{B}$ be of Paley-Wiener type (s_j, r_j) for $1 \leq j \leq m$. Let $\tau \in S_m$ the group of permutations on $(1, \ldots, m)$. The function

$$e_{a,\tau} : \zeta \mapsto e^{i<a_{\tau(1)}, \zeta_1>} \ldots . e^{i<a_{\tau(m)}, \zeta_m>}$$

is of Paley-Wiener type (s, r) where $s = s_1 + \ldots + s_m$ and $r = (r_1^2 + \ldots + r_m^2)^{1/2}$.

<u>**Example 3.4**</u> Let $\mathscr{B} = M_n$ the algebra of n by n complex matrices with a suitable norm. Suppose $a_1, \ldots, a_m$ are simultaneously triangularizable matrices in $\mathscr{B}$ with real spectra. It is proved in Pryde [7] that $a = (a_1, \ldots, a_m)$ is of Paley-Wiener type $(n - 1, r(a))$ where $r(a) = \sup \{|\lambda| : \lambda \in \gamma(a)\}$ and $\gamma(a) = \{\lambda \in \mathbb{R}^m : \sum_1^m (a_j - \lambda_j)^2$ is not invertible$\}$. Also proved is an extension of this result to the case of certain triangularizable m-tuples in $\mathscr{B}(H)$ for a separable Hilbert space H.

4. <u>Functional calculus</u>

For $a = (a_1, \ldots, a_m) \in \mathscr{B}^m$ and $\tau \in S_m$, consider the entire functions e_a and $e_{a,\tau}$ defined in section 3. If e_a (resp. $e_{a,\tau}$) is of Paley-Wiener type then it is the symbol of a compactly supported $\mathscr{B}$-valued distribution which we denote by W_a (resp. $W_{a,\tau}$). In such a case, let $\theta \in C_c^\infty(\mathbb{R}^m)$ be identically 1 in a neighbourhood of the support.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ let $p_\alpha : \mathbb{R}^m \to \mathbb{C}$ denote the monomial $p_\alpha(\lambda) = \lambda^\alpha$ and set $|\alpha| = \alpha_1 + \ldots + \alpha_m$ and $\alpha! = \alpha_1! \ldots \alpha_m!$.

<u>**Theorem 4.1**</u> Let $(a_1, \ldots, a_m)$ be of Paley-Wiener type. For each multi-index α, $W_a(\theta p_\alpha) = \frac{\alpha!}{|\alpha|!} \sum_\sigma a_{\sigma(1)} \cdots a_{\sigma(|\alpha|)}$ where the summation is over all maps $\sigma : \{1, \ldots, |\alpha|\} \to \{1, \ldots, m\}$ which assume the value j exactly α_j times for $1 \le j \le m$.

<u>**Theorem 4.2**</u> Let each a_j be of Paley-Wiener type. For each multi-index α and each permutation $\tau \in S_m$, $W_{a,\tau}(\theta p_\alpha) = a_{\tau(1)}^{\alpha_1} \cdots\cdot a_{\tau(m)}^{\alpha_m}$.

These two theorems follow readily from theorem 2.2. For the first, see Anderson [1].

Following McIntosh and Pryde [5], we extend W_a (resp. $W_{a,\tau}$) from $\mathscr{S}(\mathbb{R}^m)$ to a large function space $\mathscr{F}^s$. Indeed, for $s \ge 0$ let $L_1^s = L_1(d\mu)$ where $d\mu = (1 + |\xi|)^s d\xi$ and $d\xi$ is Lebesgue measure on $\mathbb{R}^m$. Then $\mathscr{F}^s$ is the space of inverse Fourier transforms of elements of L_1^s. With the norm

$$\| f \| = (2\pi)^{-m} \int_{\mathbb{R}^m} (1 + |\xi|)^s \, |\hat{f}(\xi)| \, d\xi ,$$

$\mathscr{F}^s$ becomes a Banach algebra under pointwise operations. Moreover, $\mathscr{S}(\mathbb{R}^m)$ is dense in $\mathscr{F}^s$.

If e_a is of Paley-Wiener type (s, r) then $\| W_a(f) \| \le c \| f \|$ for all $f \in \mathscr{S}(\mathbb{R}^m)$ and some $c \ge 0$. Hence W_a extends uniquely to a bounded linear operator $\Phi_a : \mathscr{F}^s \to \mathscr{B}$. Moreover,

$$\text{supp } (\Phi_a) = \text{supp } (W_a) \subseteq \{\lambda \in \mathbb{R}^m : |\lambda| \le r\} .$$

Similarly, if $e_{a,\tau}$ is of Paley-Wiener type (s, r) then $W_{a,\tau}$ extends to a bounded linear operator $\Phi_{\alpha,\tau} : \mathscr{F}^s \to \mathscr{B}$ with

$$\text{supp}(\Phi_{\alpha,\tau}) = \text{supp } (W_{a,\tau}) \subseteq \{\lambda \in \mathbb{R}^m : |\lambda| \le r\} .$$

5. Joint spectrum

Much use was made in [5] and [6] of spectral sets of the following form. Let $a = (a_1, \ldots, a_m) \in \mathscr{B}^m$ and for $\lambda \in \mathbb{R}^m$ define $p(\lambda, a) = \sum_1^m (a_j - \lambda_j)^2$. By $\mathscr{C}$ we will denote a closed unital subalgebra of $\mathscr{B}$ containing each a_j, and by $\mathscr{A}$ the intersection of all such $\mathscr{C}$. So $\mathscr{A} \subseteq \mathscr{C} \subseteq \mathscr{B}$. If $x \in \mathscr{C}$ then $\sigma_{\mathscr{C}}(x)$ denotes its spectrum as an element of $\mathscr{C}$ and $\rho_{\mathscr{C}}(x)$ its resolvent. The spectral sets are defined by

$$\gamma_{\mathscr{C}}(a) = \{\lambda \in \mathbb{R}^m : 0 \in \sigma_{\mathscr{C}}(p(\lambda, a))\}$$

and $\gamma(a) = \gamma_{\mathscr{A}}(a)$.

In general $\gamma_{\mathscr{B}}(a) \subseteq \gamma_{\mathscr{C}}(a) \subseteq \gamma(a)$. However, if for all $\lambda \in \mathbb{R}^m$ the resolvent set $\rho_{\mathscr{B}}(p(\lambda, a))$ has no bounded connected components then $\gamma_{\mathscr{B}}(a) = \gamma_{\mathscr{C}}(a) = \lambda(a)$. This is the case for example if $\mathscr{B}$ is finite dimensional or if $\sigma_{\mathscr{B}}(p(\lambda, a)) \subseteq \mathbb{R}$ for all $\lambda \in \mathbb{R}^m$.

The following theorem was proved in [5]. There it was stated for the case $\mathscr{B} = \mathscr{B}(X)$, X a Banach space, but the same proof is valid in the more general setting. Part (b) for $m = 1$ is due to Foias [4]. Part (c) is a spectral mapping theorem.

<u>Theorem 5.1</u> Let $a = (a_1, \ldots, a_m)$ be a commuting m-tuple in $\mathscr{B}$ of Paley-Wiener type (s, r).

(a) $\Phi_a : \mathscr{F}^S \to \mathscr{A}$ is a homomorphism of Banach algebras.

(b) $\operatorname{supp}(\Phi_a) = \gamma(a)$.

(c) $\sigma(\Phi_a(f)) = f(\gamma(a))$ for all $f \in \mathscr{F}^S$.

Again let $\mathscr{C}$ be a closed unital subalgebra of $\mathscr{B}$ containing $a_1, \ldots, a_m$. Let rad $\mathscr{C}$ be the Jacobson radical of $\mathscr{C}$. So rad $\mathscr{C}$ is the intersection of all maximal left ideals of $\mathscr{C}$ and is a closed two-sided ideal. (See Bonsall and Duncan [2].) Let $\pi : \mathscr{C} \to$ /rad $\mathscr{C}$ be the natural homomorphism and set $\pi(a) = (\pi(a_1), \ldots, \pi(a_m))$. We shall say that $a = (a_1, \ldots, a_m)$ commutes modulo rad $\mathscr{C}$ if $\pi(a)$ is a commutative m-tuple. The integrands in the expresions used to define Φ_a and $\Phi_{a,\tau}$ are elements of $\mathscr{C}$. Hence these operators have range in $\mathscr{C}$. A theorem similar to the following was announced in [6].

<u>Theorem 5.2</u> Let $a = (a_1, \ldots, a_m)$ be an m-tuple in $\mathscr{B}$ which commutes modulo rad $\mathscr{C}$ and for which e_a is of Paley-Wiener type (s, r).

(a) $\pi(a)$ is a commuting m-tuple of Paley-Wiener type (s, r) and $\pi \circ \Phi_a = \Phi_{\pi(a)}$.

(b) supp $(\Phi_a) \supseteq \gamma_{\mathscr{C}}(a)$.

(c) $\sigma_{\mathscr{C}}(\Phi_a(f)) = f(\gamma_{\mathscr{C}}(a))$ for all $f \in \mathscr{F}^s$.

<u>Theorem 5.3</u> The previous theorem remains valid with e_a replaced by $e_{a,\tau}$ and Φ_a by $\Phi_{a,\tau}$ for any permutation $\tau \in S_m$.

<u>Corollary 5.4</u> Let $a = (a_1, \ldots, a_m)$ be an m-tuple in $\mathscr{B}$ which commutes modulo rad $\mathscr{C}$ and for which each a_j is of Paley-Wiener type. Then $\gamma_{\mathscr{B}}(a) = \gamma(a)$.

(Complete proofs of these results will appear elsewhere.)

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