# ON ENTIRE SOLUTIONS OF THE $p$-LAPLACIAN

J.CHABROWSKI

The University of Queensland, Department of Mathematics
St. Lucia 4072, Qld, Australia

## 1. Introduction.

The main purpose of this paper is to establish the existence of positive solutions of the equation

$$(1)\qquad -\sum_{i=1}^{n} D_i\left(|\nabla u(x)|^{p-2}D_i u(x)\right) + c|u(x)|^{p-2}u(x) = \sum_{i=1}^{N} r_i(x)|u(x)|^{q_i-2}u(x),$$

in $\mathbb{R}_n$, where $1 < p < \infty$ and $c > 0$ is a constant, $p < q_i < \frac{np}{n-p}$ if $n > p$ and $p < q_i < \infty$ if $p \geqq n$. The functions $r_i : \mathbb{R}_n \to [0,\infty)$ $(i = 1, ..., N)$ satisfy the hypotheses $(a_i)$ and $(b_i)$ of Section 4, which depend on whether $p < n$, $p = n$ or $p > n$. We are also interested in the behaviour of solutions at infinity and in the question of the existence of multiple solutions.

The paper is organized as follows. In Sections 2 and 3 we describe an abstract setting for the equation (1). Namely, we consider the equation

$$(2)\qquad Au = \nabla\Phi(u)$$

in a reflexive Banach space $X$, where $A$ is a potential operator and $\Phi$ is a real-valued functional on $X$. We follow here the ideas developed by Stuart in [18] and [20] for the equation

$$u = \nabla\Phi(u)$$

in a Hilbert space $H$, which is an abstract version of the equation (1) with $p = 2$. Since the equation (2) has a variational structure, solutions can be obtained by minimizing the potential $J$ given by

$$J(u) = pa(u) - p\Phi(u),$$

where $a$ is a potential of $A$. In Sections 2 and 3 we describe three methods of obtaining solutions of the equation (2). The first method is based on the mountain pass theorem (see Ambrosetti–Rabinowitz [**1**] or Rabinowitz [**13** - **14**]). In the second and third method, solutions are obtained as minimizers of $J$ subject to some constraints. As an application in Section 4 we obtain the existence of solutions of (1). In both cases we need some results on imbedding of $W^{1,p}(\mathbb{R}_n)$ into a weighted space $L^p_r(\mathbb{R}_n)$. These results are discussed in Section 4 and we distinguish three cases: $p < n$, $p = n$ and $p > n$. We also obtain some existence results when $c = 0$ if $p < n$. In particular, we show that in this case solutions of (1) converge, when $c \to 0$, to solutions of (1) with $c = 0$. In Section 5 we briefly discuss the exponential decay of positive solutions of (1). Finally, in Section 6 we establish the existence of infinitely many solutions of the equation (1). The method used in this section is based on the Lusternik–Schnirelman theory of critical points.

If $p = 2$ there are some known existence results for infinitely many solutions. In particular, Berestycki and P.L. Lions [**5**] established the existence of infinitely radial solutions. The right hand side of (1), with $p = 2$, in [**5**] is replaced by a nonlinearity $f$ depending only $u$ and having a subcritical growth at infinity. Their results were extended by Stuart [**19**] and Ruppen [**17**] to (1), with $p = 2$, in a nonradial case. We additionally prove that a sequence of level sets of the functional $J$ corresponding to the infinite sequence of solutions converges to $\infty$. The existence results obtained in this paper, even in case $p = 2$, are more general than some existence results in [**17** - **20**]. However, we are not concerned in this paper with bifurcation, which has been extensively discussed in [**19**].

We point out here that for case $p = 2$, Ruppen [**17**] proved the existence of infinitely many branches of solutions all bifurcating from $c = 0$. However, this requires some assumptions on $r_i$, which control the behaviour of these functions from below at infinity.

## 2. Application of the mountain pass theorem.

We commence with some basic definitions and notations. We consider equation (2) in a real reflexive Banach space $X$. The dual $X$ is denoted by $X^*$. We denote the duality pairing between $X$ and $X^*$ by $\langle \cdot, \cdot \rangle$. The weak convergence in $X$ and $X^*$ is denoted by $\rightharpoonup$ and the strong convergence by $\to$. For a functional $\Phi : X \to \mathbb{R}$, we use both symbols $\nabla\Phi$ and $\Phi'$ to denote the Fréchet or Gâteaux derivatives.

The following terminology is standard and can be found in the monograph of Vainberg [**22**].

A mapping $A : X \to X^*$ is said to be a potential operator with a potential $a : X \to \mathbb{R}$, if $a$ is Gâteaux differentiable and

$$\lim_{t\to 0} t^{-1}\big(a(u+tv) - a(u)\big) = \langle A(u), v\rangle$$

for all $u$ and $v$ in $X$. For a potential $a$ we always assume that $a(0) = 0$. It is known that the potential operator with the potential $a$ can be expressed by the formula

$$a(u) = \int_0^1 \langle A(tu), u\rangle \, dt.$$

We say that the mapping $A : X \to X^*$ is homogeneous of degree $\beta > 0$, if for every $u \in X$ and $t > 0$

$$A(tu) = t^\beta A(u).$$

Consequently, for the homogeneous potential operator $A$ of degree $\beta$ with the potential $a$ we have

$$a(u) = \frac{1}{\beta + 1} \langle A(u), u \rangle.$$

We make the following assumption on $A$ and $\Phi$.

$(A_1)$ The mapping $A : X \to X^*$ is a homogenous continuous potential operator of degree $p - 1$ with a potential $a$, where $p > 1$. Moreover, we assume that

(3) $$k_1 \leqq a(u) \leqq k_2$$

for some constants $k_1 > 0$ and $k_1 > 0$ and for all $\|u\| = 1$.

Since the potential $a$ is homogeneous of degree $p$, we see that (3) implies that

(4) $$k_1 \|u\|^p \leqq a(u) \leqq k_2 \|u\|^p$$

for all $u \in X$. Also, the continuity of the potential operator $A$ implies the Fréchet differentiability of its potential $a$.

$(A_2)$ $\Phi \in C^1(X, \mathbb{R})$ and

$$\phi(u) \geqq \alpha \Phi(u) > 0 \text{ on } X - \{0\},$$

for some constant $\alpha > p$, where

$$\phi(u) = \langle \nabla \Phi(u), u \rangle = \Phi'(u)u.$$

Furthermore, we assume that

$(A_3)$ $\Phi(u) \leqq C(\|u\|^\alpha + \|u\|^\gamma)$ on $X$ for some constants $\gamma > \alpha$ and $C > 0$.

Let us now set for a fixed $u \in X - \{0\}$

$$h(t) = \Phi(tu)t^{-\alpha} \text{ for all } t > 0.$$

Since

$$h'(t) = \frac{\phi(tu) - \alpha\Phi(tu)}{t^{\alpha+1}} \geqq 0 \text{ for all } t > 0,$$

we see that

$$\Phi(tu) \leqq t^\alpha \Phi(u) \text{ for } 0 < t \leqq 1 \tag{5}$$

and

$$\Phi(tu) \geqq \Phi(u)t^\alpha \text{ for } t \geqq 1. \tag{6}$$

Also, we have $\Phi(0) = \nabla\Phi(0) = 0$.

To obtain the first existence result we minimize the functional

$$J(u) = \tilde{a}(u) - p\Phi(u),$$

where $\tilde{a}(u) = pa(u)$. Since

$$J'(u)v = p\big(\langle A(u), v\rangle - \langle\nabla\Phi(u), v\rangle\big)$$

for all $u$ and $v$ in $X$, any critical point of $J$ is a solution of (2). The first existence result is based on the Ambrosetti–Rabinowitz mountain pass theorem [**1**]. To apply this theorem we need an additional asumption on $A$.

A mapping $A : X \to X^*$ is strongly monotone if there exists a continuous function $\kappa : [0, \infty) \to [0, \infty)$, which is positive on $(0, \infty)$ and $\lim_{t\to\infty} \kappa(t) = \infty$, such that

$$\langle A(u) - A(v), u - v\rangle \geqq \kappa(\|u - v\|)\|u - v\|$$

for all $u$ and $v$ in $X$.

A mapping $A : X \to X^*$ is said to satisfy the condition $S_1$ if for every sequence $\{u_j\}$ in $X$ with $u_j \rightharpoonup u$ and $A(u_j) \to v$ in $X^*$ we have $u_j \to u$. Evidently, every strongly monotone operator satisfies the condition $S_1$.

THEOREM 1. *Suppose that $A$ is a strongly monotone potential operator and that $\nabla\Phi$ is strongly sequentially continuous ( that is, $u_n \rightharpoonup u$ in $X$ implies $\nabla\Phi(u_n) \to \nabla\Phi(u)$ in $X^*$ ). Then the equation (2) has a nontrivial solution.*

PROOF: We check that the assumptions, of the mountain pass theorem are satisfied. It follows from (4) and $(A_3)$ that

$$J(u) \geqq k_1 p\|u\|^p - pC(\|u\|^\alpha + \|u\|^\gamma).$$

Since $\gamma > \alpha > p$, there exists $\rho > 0$ and $\delta > 0$ such that

$$J(u) \geqq \rho \text{ for } \|u\| = \delta$$

and also

$$J(u) > 0 \text{ for } 0 < \|u\| \leqq \delta.$$

Let $\|\bar{u}\| > \delta$, by (4) and (6) we have

$$J(t\bar{u}) \leqq k_2 p t^p \|\bar{u}\|^p - p\Phi(\bar{u})t^\alpha < 0$$

for $t > 0$ sufficiently large. We now show that $J$ satisfies the Palais-Smale condition (PS). That is, if $\{u_n\}$ is a sequence in $X$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ in $X^*$, then $\{u_n\}$ possesses a convergent subsequence in $X$. It is easy to show that, under our assumptions, these two conditions on $\{u_n\}$ imply that $\{u_n\}$ is bounded. Hence we may assume that $u_n \rightharpoonup u$ in $X$. Since

$$J'(u_n) = p\big(A(u_n) - \nabla\Phi(u_n)\big) \to 0 \text{ and } \nabla\Phi(u_n) \to \nabla\Phi(u)$$

in $X^*$, we see that $A(u_n)$ is convergent in $X^*$. According to the condition $S_1$, $u_n \to u$ in $X$. By the mountain pass theorem there exists $\tilde{u} \in X$ such that

$$J(\tilde{u}) = \inf_{g \in \Gamma} \max_{0 \leqq s \leqq 1} J(g(s)),$$

where $\Gamma = \{g \in C([0,1], X); g(0) = 0, g(1) = t\tilde{u}\}$. Since $J(\tilde{u}) \geqq \rho$, $\tilde{u} \neq 0$ and it is obvious that $\tilde{u}$ is a solution of (2).

## 3. Constrained minimization.

In this section we additionally assume that

$(A_4)$ $\phi \in C^1(X, \mathbb{R})$ and the mappings $\Phi' : X \to X^*$ and $\phi' : X \to X^*$ are bounded mappings.

We now set $\tilde{\phi}(u) = \phi(u) - p\Phi(u)$ for all $u$ and assume that

$(A_5)$ $\tilde{\phi}'(u)u \geqq \alpha\tilde{\phi}(u)$ for all $u \in X - \{0\}$.

Instead of the assumption $(A_3)$ we suppose that

$(A_3')$ $\tilde{\phi}(u) \leqq K\left(\|u\|^\alpha + \|u\|^\gamma\right)$ for all $u \in X$ and some constants $\gamma > \alpha > p$ and $K > 0$.

As in [**19**] we list the consequences of these hypotheses that will be needed later.

It follows from $(A_2)$ that

$$\tilde{\phi}(u) = \phi(u) - p\Phi(u) \geqq (\alpha - p)\Phi(u) > 0 \tag{7}$$

for all $u \in X - \{0\}$. Hence by $(A_3')$ we have

$$\Phi(u) \leqq \frac{K}{\alpha - p}\left(\|u\|^\alpha + \|u\|^\gamma\right) \tag{8}$$

for all $u \in X$, that is $(A_3')$ implies $(A_3)$. Now combining (8) and $(A_3')$ we get

(9) $$0 < \phi(u) = \tilde{\phi}(u) + p\Phi(u) \leqq \left(\frac{pK}{\alpha - p} + K\right)(\|u\|^\alpha + \|u\|^\gamma) = \frac{\alpha K}{\alpha - p}(\|u\|^\alpha + \|u\|^\gamma)$$

for all $u \in X - \{0\}$. Moreover, it follows from $(A_5)$ and $(A_2)$ that

(10) $$\langle \phi'(u), u\rangle = \langle \tilde{\phi}'(u), u\rangle + p\langle \nabla\Phi(u), u\rangle \geqq \alpha\tilde{\phi}(u) + \alpha p\Phi(u) = \alpha\phi(u).$$

Let us now set for a fixed $u \in X - \{0\}$

$$k(t) = \phi(tu)t^{-\alpha} \text{ for all } t > 0.$$

Then

$$k'(t) = \frac{\phi'(tu)tu - \alpha\phi(tu)}{t^{\alpha+1}} > 0$$

and consequently $t \to \phi(tu)t^{-\alpha}$ is strictly increasing on $(0, \infty)$ and we obtain that

(11) $$\lim_{t\to 0} \phi(tu)t^{-p} = 0 \text{ and } \lim_{t\to\infty} \phi(tu)t^{-p} = \infty.$$

A similar statement can be derived for $\tilde{\phi}(u)$.

To describe the first method of the constrained minimization, we observe that if $u \in X$ is a solution of (1), then

$$g(u) = \langle A(u), u\rangle - \langle \nabla\Phi(u), u\rangle = \tilde{a}(u) - \phi(u) = 0.$$

Consequently, we minimize the functional $J$ subject to the constraint $g(u) = 0$. We set

$$V = \{u \in X - \{0\};\ g(u) = 0\}.$$

It will be shown in Lemma 2 that $V \neq \emptyset$. We now observe that

(12) $$J(u) = g(u) + \tilde{\phi}(u) \text{ for all } u \in X,$$

and

$$(13)\quad J'(u)u = \tilde{a}'(u)u - p\langle\nabla\Phi(u), u\rangle = p\big(\langle A(u), u\rangle - \langle\nabla\Phi(u), u\rangle\big)$$
$$= pg(u) = g'(u)u + \tilde{\phi}'(u)u.$$

Therefore for $u \in V$ we have

$$(14)\qquad J(u) = \tilde{\phi}(u), \quad g'(u)u = -\tilde{\phi}'(u)u$$

and moreover

$$m = \inf\{J(u);\, u \in V\} = \inf\{\tilde{\phi}(u);\, u \in V\}.$$

LEMMA 1. *(i) There exists $\delta > 0$ such that $\phi(u) \geqq pk_1\|u\|^p \geqq \delta > 0$ for $u \in V$,*

*(ii)* $\inf_{u\in V} J(u) \geqq \delta\frac{\alpha-p}{\alpha}$.

*(iii)* $\frac{\alpha}{\alpha-p}J(u) \geqq pk_1\|u\|^p$ *for* $u \in V$,

*(iv) If $u \in V$ and $J(u) = m$, then $u$ satisfies (2) and*

$$\frac{\delta(\alpha-p)}{\alpha} \leqq \frac{pk_1(\alpha-p)}{\alpha}\|u\|^p \leqq m.$$

PROOF: (i) If $u \in V$, then

$$pk_1\|u\|^p \leqq \langle A(u), u\rangle = \phi(u) \leqq \frac{K}{\alpha-p}\big(\|u\|^\alpha + \|u\|^\gamma\big).$$

Since $u \neq 0$ and $p < \alpha < \gamma$, $\|u\| \geqq$ Const $> 0$ for $u \in V$ and (i) follows.

(ii) If $u \in V$, then by (14)

$$J(u) = \tilde{\phi}(u) \text{ and } \phi(u) = \tilde{\phi}(u) + p\Phi(u) \leqq \tilde{\phi}(u) + \frac{p}{\alpha-p}\tilde{\phi}(u) = \frac{\alpha}{\alpha-p}\tilde{\phi}(u).$$

Consequently, according to (i) we have

$$J(u) = \tilde{\phi}(u) \geqq \frac{\alpha-p}{\alpha}\phi(u) \geqq \delta\frac{\alpha-p}{\alpha}$$

and this implies (ii). The estimate (iii) follows from the step (i) and the preceding inequality.

(iv) If $u \in V$ and $J(u) = m$, then there exists $\lambda \in \mathbb{R}$ such that

$$J'(u)v = \lambda g'(u)v \text{ for all } v \in X.$$

It follows from $(A_5)$, (14) and (7) that $g'(u)u < 0$. Since $J'(u)u = 0$, we must have $\lambda = 0$, that is, $J'(u)v = 0$ for all $v \in X$ and $u$ satisfies (1).

LEMMA 2. *There exists a unique function $s : X - \{0\} \to (0, \infty)$ in $C^1(X - \{0\}, \mathbb{R})$ with the following property: if $u \in X - \{0\}$ and $t > 0$, then $tu \in V$ if and only if $t = s(u)$. The gradient of $s$ satisfies the estimate*

$$|\nabla s(u)| \leqq \frac{(p\|A(s(u)u)\| + \|\nabla\phi(s(u)u)\|)s(u)^2}{(\alpha - p)\phi(s(u)u)}.$$

PROOF: The proof is similar to that of Lemma 3.2 in [**19**]. We give only an outline. We introduce a function $\psi : (0, \infty) \times X \to \mathbb{R}$ defined by

$$\psi(t, u) = \tilde{a}(u) - \phi(tu)t^{-p}.$$

According to (10) we have

$$\psi_t(t, u) = -\frac{\phi'(tu)tu - p\phi(tu)}{t^{p+1}} \leqq -\frac{(\alpha - p)\phi(tu)}{t^{p+1}};$$

also

$$\psi_u(t, u)v = p\langle A(u), v\rangle - \frac{\langle\nabla\phi(tu), v\rangle}{t^{p-1}}.$$

Since $\tilde{a}(u)$ is homogeneous of degree $p$, it is clear that for $u \in X - \{0\}$, $tu \in V$ if and only if $\psi(t, u) = 0$. Since $\phi(tu)t^{-p}$ is strictly increasing function, there exists a unique value $t = s(u)$ such that $\psi(s(u), u) = 0$. We also have

$$\psi_t(s(u), u) \leqq -\frac{(\alpha - p)\phi(s(u)u)}{s(u)^{p+1}} < 0$$

and the result follows from the implicit function theorem.

To show that the constrained minimization leads to a solution of (2) we must construct a suitable minimizing sequence.

LEMMA 3. *There exists a sequence* $\{u_n\} \subset V$ *such that* $J(u_n) \to m$, $u_n \rightharpoonup u$ *in* $X$ *and* $\nabla J(u_n) \to 0$ *in* $X^*$.

PROOF: The proof is identical to that of Lemma 3.4 in [**19**]. Therefore we only sketch the main ideas of the proof. In this proof we use assumption $(A_4)$. It follows from Ekeland's variational principle [**2**], [**9**] that there exists $\{u_n\} \subset V$ such that

$$J(u_n) \leqq m + n^{-1}$$

and

$$J(w) \geqq J(u_n) - n^{-1}\|w - u_n\|$$

for all $w \in V$. Since by Lemma 1(iii), $\{u_n\}$ is bounded, it follows from assumption $(A_4)$ that $\{\nabla\phi(u_n)\}$ is also bounded. We now write for each $v \in X - \{0\}$

$$\text{(15)} \quad J(v) - J(u_n) = J(v) - J(s(v)v) + J(s(v)v) - J(u_n) \geqq J(v) - J(s(v)v) - n^{-1}\|s(v)v - u_n\|.$$

Applying Lemma 2 we get

$$\text{(16)} \quad \|s(v)v - u_n\| \leqq C_1\|v - u_n\| \text{ and } |s(v) - 1| \leqq C_1\|v - u_n\|$$

for some $C_1 > 0$, provided $\|v - u_n\|$ is sufficiently small. To estimate $J(v) - J(s(v)v)$ we observe that

$$|J(v) - J(s(v)v)| \leqq |s(v) - 1||J'(\theta(v)v)v|,$$

where $\theta(v)$ lies between 1 and $s(v)$. On the other hand $J'(u_n)u_n = 0$, since $u_n \in V$ and consequently

$$|J(v) - J(s(v)v)| \leqq n^{-1} D\|v - u_n\| \tag{17}$$

for some constant $D > 0$, provided $\|v - u_n\|$ is sufficiently small. Combining (15), (16) and (17) we obtain

$$J(v) - J(u_n) \geqq -C_2 \frac{\|v - u_n\|}{n}$$

for some constant $C_2 > 0$. Taking $\|z\| = 1$ and $t > 0$ as small as $\|v - u_n\|$ we get

$$\frac{J(u_n + tz) - J(u_n)}{t} \geqq -\frac{C_2}{n},$$

which implies that $J'(u_n)z \geqq -\frac{C_2}{n}$ and replacing $z$ by $-z$ we derive

$$|J'(u_n)z| \leqq \frac{C_2}{n}$$

and the result follows.

We are now in a position to establish the following existence result.

THEOREM 2. *Suppose that $A$ is a strongly monotone potential operator and that $\nabla\Phi$ is strongly sequentially continuous. Let $\{u_n\}$ be sequence constructed in Lemma 3. Then $u_n \to u$ in $X$ and $u$ is a nontrivial solution of (2).*

PROOF: Since $A$ satisfies the condition $S_1$, we show as in Theorem 1 that $u_n \to u$ in $X$ and therefore $J(u) = m$, with $u \neq 0$, and the result follows from Lemma 1.

If both functionals $a$ and $\Phi$ are homogeneous, then a constrained minimization problem can be solved under different set of assumptions. In particular, we need only the Gâteaux differentiability for $a$ and $\Phi$.

To illustrate this situation we consider the following minimization problem

$$I = \min\{a(u);\ \Phi(u) = 1\}. \tag{I}$$

We follow here the approach from [7]. We assume that $A : X \to X^*$ is homogeneous of degree $p-1$, $p > 1$, with a potential $a$ satisfying (3). We stress here that we do not assume the continuity of $A$. The functional $\Phi$ is defined on a linear subspace $D(\Phi) \subset X$. Moreover, we assume that $\Phi$ is homogeneous of degree $\alpha > p$. Further assumptions on $\Phi$ will be formulated in Theorem 3 below.

THEOREM 3. *(i) Suppose that the problem (1) has a solution $u$, that is, there exists $u \in D(\Phi)$ such that $a(u) = I$ and $\Phi(u) = 1$, and that $\Phi$ has a linear continuous Gâteaux derivative $\langle \Phi'(u), v\rangle$ at all directions $v \in D(\Phi)$. Then*

$$\langle A(u), v\rangle = I\frac{p}{\alpha}\langle \Phi'(u), v\rangle \tag{18}$$

*for all $v \in D(\Phi)$. If $I > 0$ then the "scaled minimizer" $\bar{u} = \bar{\sigma}u$, where $\bar{\sigma} = \left(\frac{p}{\alpha}I\right)^{\frac{1}{\alpha-p}}$, satisfies the equation*

$$\langle A(\bar{u}), v\rangle = \langle \Phi'(\bar{u}), v\rangle \tag{19}$$

*for all $v \in D(\Phi)$.*

*(ii) Suppose that all weak limit points of every bounded subset of the level set $\Phi(u) = 1$ belong to $D(\Phi)$. If $\Phi_+$ is weakly sequentially continuous, $a$ and $\Phi_-$ are weakly sequentially lower semicontinuous, then the constrained minimization problem (1) has a nontrivial solution.*

PROOF: The proof is similar to those of Proposition 2.1 and Theorem 3.2 in [7]. Therefore we only sketch the main steps. To prove (i) we set $d = \Phi'(u)v$ and for $\sigma > 0$ we have by the definition of the Gâteaux derivative that

$$\Phi(\sigma u + \epsilon\sigma v) = \sigma^{\alpha}(1 + \epsilon d + o(\epsilon)).$$

There exists $\epsilon_\circ >$ such that $1 + \epsilon d + o(\epsilon) > 0$ for all $|\epsilon| < \epsilon_\circ$. Also, for such $\epsilon$ there exists $\sigma = \sigma(\epsilon)$ such that $\Phi(\sigma u + \epsilon\sigma v) = 1$. Consequently, if we set $c = \langle A(u), v\rangle$, we get

$$I \leqq a(\sigma u + \sigma\epsilon v) = \sigma^p a(u + \epsilon v) = a(u) + \epsilon(c - \frac{p}{\alpha} I d) + o(\epsilon).$$

Since this inequality holds for all $|\epsilon| < \epsilon_\circ$, we must have $c - \frac{p}{\alpha} I d = 0$ and this proves (18). Since for $\sigma > 0$ we have $a(\sigma u + \epsilon v) = \sigma^p a(u + \epsilon\sigma^{-1} v)$ and $\Phi(\sigma u + \epsilon v) = \sigma^\alpha \Phi(u + \epsilon\sigma^{-1} v)$, it is easy to see that

$$\langle A(\sigma u), v\rangle = \frac{p}{\alpha} I \sigma^{p-\alpha} \langle \nabla\Phi(\sigma u), v\rangle.$$

Therefore, if $I > 0$, then $\bar{u} = \bar{\sigma} u$, with $\bar{\sigma} = \left(\frac{p}{\alpha} I\right)^{\frac{1}{\alpha - p}}$ satisfies (19).

(ii) Let $\{u_j\}$ be a minimizing sequence. Since $\{u_j\}$ is bounded, we may assume that $u_j \rightharpoonup u$ in $X$. According to our assumptions $u \in D(\Phi)$ and $a(u) \leqq I$. We now have

$$\Phi_-(u_j) = \Phi_+(u_j) - 1 \to \Phi_+(u) - 1,$$

as $j \to \infty$, and consequently

$$\Phi_-(u) \leqq \lim_{j\to\infty} \Phi_-(u_j) = \Phi_+(u) - 1.$$

Therefore $\Phi(u) \geqq 1$ and $u \neq 0$. Assuming that $\Phi(u) > 1$, we have $\sigma^\alpha \Phi(u) = 1$, with $0 < \sigma^\alpha < 1$, hence $a(u) \leqq I \leqq a(\sigma u) = \sigma^p a(u)$, which implies that $1 \leqq \sigma^p$ and we arrive at the contradiction.

**Remark.** We now compare the constrained minimizations of Theorems 3 and 2. First, under the assumptions of Theorem 3 $\nabla\Phi$ is a potential operator with a potential $\Phi$ of degree $\alpha$. Therefore we have

$$\Phi(u) = \frac{1}{\alpha} \langle \nabla\Phi(u), u\rangle.$$

It is easy to check that a solution $\bar{u}$ of the constrained minimization (I) satisfies the equation

$$\frac{p}{\alpha}a(\bar{u}) = \Phi(\bar{u}).$$

We now set $H(u) = \frac{1}{p}J(u)$, that is, $H(u) = a(u) - \Phi(u)$. We show that

$$H(\bar{u}) = \inf\{H(v);\ v \in V\}.$$

If $g(v) = 0$, then $\langle A(v), v\rangle - \langle\nabla\Phi(v), v\rangle = 0$, which is equivalent to $\frac{p}{\alpha}a(v) = \Phi(v)$. Let $v \neq 0$ satisfy $g(v) = 0$. Then in view of (3) and the last identity $\Phi(v) > 0$. Hence there exists $\sigma > 0$ such that $\sigma^\alpha = \Phi(v)$. Let $v_\circ = \sigma^{-1}v$. Then the equation $g(v) = 0$ can be written as $\sigma^\alpha = \frac{p}{\alpha}\sigma^p a(v_\circ)$, that is $\sigma = \left(\frac{p}{\alpha}a(v_\circ)\right)^{\frac{1}{\alpha-p}}$. Hence

$$H(v) = (1-\frac{p}{\alpha})a(v) = (1-\frac{p}{\alpha})\sigma^p a(v_\circ) = (1-\frac{p}{\alpha})\left(\frac{p}{\alpha}a(v_\circ)\right)^{\frac{p}{\alpha-p}}a(v_\circ).$$

On the other hand, for $\bar{u}$ we have

$$H(\bar{u}) = (1-\frac{p}{\alpha})\bar{\sigma}^p a(u) = (1-\frac{p}{\alpha})\left(\frac{p}{\alpha}I\right)^{\frac{p}{\alpha-p}}I.$$

Since $\Phi(v_\circ) = 1$, we have $I \leqq a(v_\circ)$ and consequently

$$H(\bar{u}) \leqq (1-\frac{p}{\alpha})\left(\frac{p}{\alpha}a(v_\circ)\right)^{\frac{p}{\alpha-p}}a(v_\circ) = H(v)$$

and this justifies our claim.

### 4. Existence results for the equation (1).

As an application of the results of Sections 2 and 3, we prove the existence of positive solutions of (1). We recall some basic properties of the $p$-Laplacian

$$\Delta_p u = \sum_{i=1}^{n} D_i\left(|\nabla u|^{p-2}D_i u\right).$$

All these properties can be found in J.L. Lions [11]. The $p$-Laplacian $\Delta_p$ maps $W^{1,p}(\mathbb{R}_n)$ onto $W^{-1,p'}(\mathbb{R}_n) = \left(W^{1,p}(\mathbb{R}_n)\right)^*$, with $\frac{1}{p} + \frac{1}{p'} = 1$. If we now set

$$A(u) = -\Delta_p u + c|u|^{p-2}u,$$

where $c > 0$ is a constant, then for every $h \in W^{-1,p'}$ there exists a unique $u \in W^{1,p}(\mathbb{R}_n)$ such that $A(u) = h$. Furthermore, $A$ is a strictly monotone potential operator with a potential $a : W^{1,p}(\mathbb{R}_n) \to \mathbb{R}$ given by

$$a(u) = \frac{1}{p}\left(\int_{\mathbb{R}_n} |\nabla u(x)|^p \, dx + c\int_{\mathbb{R}_n} |u(x)|^p \, dx\right).$$

It is known that $A : W^{1,p}(\mathbb{R}_n) \to W^{-1,p'}(\mathbb{R}_n)$ is uniformly continuous on bounded sets.

To formulate the assumptions on $r_i : \mathbb{R}_n \to [0,\infty)$ $(i = 1,...,N)$, guaranteeing the existence of a compact imbedding of $W^{1,p}(\mathbb{R}_n)$ in $L^{q_i}_{r_i}(\mathbb{R}_n)$, we distinguish three cases. In what follows we denote by $Q(x,l)$ the cube of the form

$$Q(x,l) = \{y \in \mathbb{R}_n,\ |y_j - x_j| < \frac{l}{2},\ j = 1,...,n\}.$$

Case $p < n$.

$(a_1)$ $r_i \in L_{\text{loc}}^{\frac{q_i+\epsilon_i}{\epsilon_i}}(\mathbb{R}_n)$, where $p < q_i < q_i + \epsilon_i < \frac{np}{n-p}$ $(i = 1,...,N)$ and

$(b_1)$ $\lim_{|x|\to\infty} \int_{Q(x,l)} r_i(y)^{\frac{q_i+\epsilon_i}{\epsilon_i}} \, dy = 0$ $(i = 1,...,N)$ for some $l > 0$.

Case $p = n$.

$(a_2)$ $r_i \in L^{s_i}_{\text{loc}}(\mathbb{R}_n)$ for some $s_i > 1$ $(i = 1,...,N)$ and

$(b_2)$ $\lim_{|x|\to\infty} \int_{Q(x,l)} r_i(x)^{s_i} \, dx = 0$ $(i = 1,...,N)$ for some $l > 0$.

Case $p > n$.

$(a_3)$ $r_i \in L^1_{\text{loc}}(\mathbb{R}_n)$ $(i = 1,...,N)$ and

$(b_3)$ $\lim_{|x|\to\infty} \int_{Q(x,l)} r_i(x) \, dx = 0$ $(i = 1,...,N)$ some $l > 0$.

We assume that $q_1 < \dots < q_N$ and that at least one of the functions $r_i$ is not identically equal to 0.

The functionals $\Phi : W^{1,p}(\mathbb{R}_n) \to \mathbb{R}$, $J : W^{1,p}(\mathbb{R}_n) \to \mathbb{R}$, $\phi : W^{1,p}(\mathbb{R}_n) \to \mathbb{R}$ and $\tilde{\phi} : W^{1,p}(\mathbb{R}_n) \to \mathbb{R}$ are given by

$$\Phi(u) = \sum_{i=1}^{N} \frac{1}{q_i} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx,$$

$$J(u) = p\Big(a(u) - \sum_{i=1}^{N} \frac{1}{q_i} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx\Big) = \int_{\mathbb{R}_n} |\nabla u(x)|^p\, dx$$
$$+ c \int_{\mathbb{R}_n} |u(x)|^p\, dx - p\sum_{i=1}^{N} \frac{1}{q_i} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx,$$

$$\phi(u) = \sum_{i=1}^{N} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx$$

and

$$\tilde{\phi}(u) = \sum_{i=1}^{N}(1 - \frac{p}{q_i}) \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx = \sum_{i=1}^{N} \frac{q_i - p}{q_i} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx.$$

To show that all these functionals are well defined we need some results on compact imbedding of $W^{1,p}(\mathbb{R}_n)$ into each space $L^{q_i}_{r_i}(\mathbb{R}_n)$ $(i = 1, \dots, N)$ defined by

$$L^{q_i}_{r_i}(\mathbb{R}_n) = \{u : \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx < \infty\}$$

and equipped with the norm

$$\|u\|_{r_i,q_i} = \Big(\int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i}\, dx\Big)^{\frac{1}{q_i}}.$$

LEMMA 4. *(i) Case $n > p$. Suppose that $r_i$ $(i = 1, \dots, N)$ satisfy $(a_1)$ and $(b_1)$. Then $W^{1,p}(\mathbb{R}_n)$ is compactly imbedded in each space $L^{q_i}_{r_i}(\mathbb{R}_n)$.*

*(ii) Case $n = p$. Suppose that $r_i$ $(i = 1, ...., N)$ satisfy $(a_2)$ and $(b_2)$. Then $W^{1,n}(\mathbb{R}_n)$ is compactly imbededd in each space $L^q_{r_i}(\mathbb{R}_n)$ for every $q \geqq n$.*

*(iii) Case $n < p$. Suppose that $r_i$ $(i = 1, ..., N)$ satisfy $(a_3)$ and $(b_3)$. Then $W^{1,p}(\mathbb{R}_n)$ is compactly imbedded in each space $L^q_{r_i}(\mathbb{R}_n)$ for each $q \geqq p$.*

PROOF: Case (i), is a special case of a slightly more general version of a result due to Berger–Schechter [6]. An independent proof of the compact imbedding of $W^{1,2}(\mathbb{R}_n)$ into $L^2_r(\mathbb{R}_n)$ with $r$ bounded and satisfying $(b_3)$ can be found in [**12**]. We follow the method from [**12**] to show (ii) and (iii). Obviously, the proof of (ii) and (iii), presented here, can also be used to prove (i). For simplicity we set $r = r_i$. In both cases it suffices to show that for every $\delta > 0$ there exists $R > 0$ such that

$$\|f - \chi_{Q(0,R)} f\|_{r,q} < \delta \tag{20}$$

for all $f$ such that $\|f\|_{W^{1,p}(\mathbb{R}_n)} \leqq 1$, where $\chi_Q$ is the characteristic function of the cube. Indeed, let $\{f_m\}$ be a bounded sequence in $W^{1,p}(\mathbb{R}_n)$. We assume that $\|f_m\|_{W^{1,p}} \leqq 1$ for all $m \geqq 1$. Consequently, we may assume that $f_m \rightharpoonup f$ in $W^{1,p}(\mathbb{R}_n)$ and in view of standard Sobolev compactness theorems there exists a subsequence of $\{f_m\}$, denoted again by $\{f_m\}$, such that $f_m \to f$ in $L^q(Q(0,R))$ in case (ii) and $f_m \to f$ uniformly on $Q(0,R)$ in case (iii). On the other hand by (20) we have

$$\left(\int_{\mathbb{R}_n - Q(0,R)} |f_m - f|^q r \, dx\right)^{\frac{1}{q}} \leqq 2\delta.$$

Combining this with the previous observation, we easily conclude that $f_m \to f$ in $L^q_r(\mathbb{R}_n)$.

To prove (20) we cover $\mathbb{R}_n$ with cubes $Q(z,1)$, $z \in \mathbb{Z}^n$, where $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. We may assume that $(a_i)$ and $(b_i)$ $(i = 2,3)$ hold with $l = 1$. For $\eta > 0$, in case (ii), we use $(b_2)$ to find $N > 0$ such that

$$\int_Q r(x)^s \, dx < \eta$$

for each $Q = Q(z,1)$ outside $Q(0,N)$. We now use the fact that if $f \in W^{1,n}(Q)$ then $f \in L^q(Q)$ for each $n \leqq q < \infty$ and

$$\|f\|_{L^q(Q)} \leqq C(q,n)\|f\|_{W^{1,n}(Q)}$$

for some constant $C(q,n) > 0$. Hence by the Hölder inequality we have

$$\int_Q |f|^q r \, dx \leqq \left(\int_Q r^s \, dx\right)^{\frac{1}{s}} \left(\int_Q f^{qs'} \, dx\right)^{\frac{1}{s'}} \leqq \eta^{\frac{1}{s}} C(q,n)\|f\|^q_{W^{1,n}(Q)}.$$

We now choose $C(q,n)\eta^{\frac{1}{s}} < \delta$ and add these inequalities over all $Q(z,1)$ outside $Q(0,N)$ to obtain (20) with $R = N$. In case (iii) we use $(b_3)$ to get $\int_Q r \, dx < \eta$ for each $Q = Q(z,1)$ outside $Q(0,N)$. We now repeat the previous argument using the inequality $\|f\|_{L^\infty(Q)} \leqq C\|f\|_{W^{1,p}(Q)}$, which holds for all $f \in W^{1,p}(Q)$, with $p > n$, for some constant $C > 0$.

It is evident that under the assumptions of Lemma 4, $\Phi$ is Fréchet differentiable and

$$\langle \nabla\Phi(u), v\rangle = \sum_{i=1}^{N} \int_{\mathbb{R}_n} r_i(x)|u(x)|^{q_i-2}u(x)v(x) \, dx$$

for all $u$ and $v$ in $W^{1,p}(\mathbb{R}_n)$.

It is now readily seen that all assumptions concerning regularity of $\phi$, $\Phi$ and $\tilde{\phi}$ are satisfied. Therefore each of Theorems 1 or 2 yield the existence result for (1).

THEOREM 4. *Suppose that the assumptions of Lemma 4 hold. Then the equation (1) has at least one positive solution $u$ in $W^{1,p}(\mathbb{R}_n)$.*

Since a solution is obtained as a critical point of $J$, which has the property $J(u) = J(|u|)$, we may assume that $u \geqq 0$ on $R_n$. The strict positivity of $u$ follows from the Harnack inequality.

If $c = 0$ a suitable Sobolev space for the equation (1) is a space $E^{1,p}(\mathbb{R}_n) = \{$ completion of the space $C_\circ^\infty(\mathbb{R}_n)$ with respect to the norm $\|Du\|_p\}$.
Here $C_\circ^\infty(\mathbb{R}_n)$ is the space of $C^\infty$-functions with compact supports. We only establish the existence result in case $p < n$. By the Sobolev inequality

$$\|u\|_{\frac{np}{n-p}} \leqq S\|Du\|_p$$

for all $u \in C_\circ^\infty(\mathbb{R}_n)$ with a constant $S = S(n,p)$. Consequently, $E^{1,p}(\mathbb{R}_n)$ can be regarded as a subspace of $L^{\frac{np}{n-p}}(\mathbb{R}_n)$. Also, $W^{1,p}(\mathbb{R}_n) \subset E^{1,p}(\mathbb{R}_n)$ with continuous injection. We now extend Lemma 4 to the space $E^{1,p}(\mathbb{R}_n)$.

LEMMA 5. *Let $p < n$ and suppose that $r_i$ $(i = 1, ..., N)$ satisfy $(a_1)$ and that $r_i \in L^1(\mathbb{R}_n)$. Then for each $i = 1, ..., N$ the space $E^{1,p}(\mathbb{R}_n)$ is compactly imbedded in $L^{q_i}_{r_i}(\mathbb{R}_n)$.*

PROOF: The proof is similar to that of Lemma 4. For simplicity we set $q = q_i$, $r = r_i$ and $\epsilon = \epsilon_i$. It suffices to show that for every $\delta > 0$ there exists $R > 0$ such that the inequality (20) holds for all $f$ such that $\|Df\|_p \leqq 1$. Indeed, let $\{f_m\}$ be a bounded sequence in $E^{1,p}(\mathbb{R}_n)$. We may assume that $\|Df_m\|_p \leqq 1$ for all $m \geqq 1$. Since $p < q + \epsilon < \frac{np}{n-p}$, we may assume that there exists $f \in E^{1,p}(\mathbb{R}_n)$ such that $f_m \to f$ in $L^{q+\epsilon}(Q(0,R))$ and $Df_m \rightharpoonup Df$ in $L^p(\mathbb{R}_n)$. Thus

$$\int_{Q(0,R)} |f_m - f|^q r\, dx \leqq \left(\int_{Q(0,R)} |f_m - f|^{q+\epsilon}\, dx\right)^{\frac{q}{q+\epsilon}} \left(\int_{Q(0,R)} r^{\frac{q+\epsilon}{\epsilon}}\, dx\right)^{\frac{\epsilon}{q+\epsilon}}.$$

On the other hand by (20) we have

$$\left(\int_{\mathbb{R}_n - Q(0,R)} |f_m - f|^q r\, dx\right)^{\frac{1}{q}} \leqq 2\delta.$$

Since $f_m \to f$ in $L^{q+\epsilon}(Q(0,R))$, the last two inequalities give the convergence of $f_m$ in $L^q_r(\mathbb{R}_n)$.

To prove (20) we cover $\mathbb{R}_n$ with cubes $Q(z,1)$, $z \in Z^n$. For $\eta > 0$ we use $(a_1)$ to find $N > 0$ such that

$$\int_Q r(y)^{\frac{q+\epsilon}{\epsilon}}\, dy \leqq \eta$$

for every $Q = Q(z,1)$ outside $Q(0,N)$ and so that

$$\int_{\mathbb{R}_n - Q(0,N)} r(y)\, dy < \eta.$$

If $Q$ is any such cube we have by the Sobolev and Hölder inequalities that

$$\int_Q |f|^{q+\epsilon}\, dx \leqq \left(\int_Q |f|^{\frac{np}{n-p}}\, dx\right)^{\frac{(n-p)(q+\epsilon)}{np}} \leqq S^{q+\epsilon} \|Df\|_p^{q+\epsilon}.$$

and consequently, setting $f_Q = \frac{1}{|Q|}\int_Q f(x)\, dx$, we obtain by Poincaré's inequality

$$\begin{aligned}\int_Q |f|^q r\, dx &\leqq 2^{q-1}\left[\int_Q |f - f_Q|^q r\, dx + + \int_Q \left(\int_Q |f(x)|\, dx\right)^q r(y)\, dy\right] \\ &\leqq 2^{q-1}\Bigg[\left(\int_Q |f - f_Q|^{q+\epsilon}\, dx\right)^{\frac{q}{q+\epsilon}} \left(\int_Q r^{\frac{q+\epsilon}{\epsilon}}\, dx\right)^{\frac{\epsilon}{q+\epsilon}} \\ &\quad + \left(\int_Q |f|^{\frac{np}{n-p}}\, dx\right)^{\frac{q(n-p)}{n-p}} \int_Q r(y)\, dy\Bigg] \\ &\leqq K\left(\int_Q |Df|^p\, dx\right)^{\frac{q}{p}} \left(\int_Q r^{\frac{q+\epsilon}{\epsilon}}\, dx\right)^{\frac{\epsilon}{q+\epsilon}} + 2^{q-1} S^q \left(\int_{\mathbb{R}_n} |Df|^p\, dx\right)^{\frac{q}{p}} \int_Q r(x)\, dx,\end{aligned}$$

where $K > 0$ is a constant. We now add all these inequalities over all $Q(z,1)$ outside $Q(0,N)$ to get

$$\int_{\mathbb{R}_n - Q(0,N)} |f|^q r\, dx \leqq K\eta^{\frac{\epsilon}{q+\epsilon}} + 2^{q-1} S^q \int_{\mathbb{R}_n - Q(0,N)} r\, dx \leqq K\eta^{\frac{\epsilon}{q+\epsilon}} + 2^{q-1} S^q \eta.$$

We now choose $\eta$ so that $K\eta^{\frac{\epsilon}{q+\epsilon}} + 2^{q-1}S^q\eta < \delta$ and add these inequalities over all $Q(z,1)$ outside $Q(0,N)$ to obtain (20) with $R = N$.

It is clear that, in case $p < n$, Theorem 4 continues to hold for the equation (1) with $c = 0$, provided $r_i$ $(i = 1,...,N)$ satisfy the assumptions of Lemma 5. Obviously in this case a solution belongs to $E^{1,p}(\mathbb{R}_n)$. As an application of Theorem 3 we consider the equation (1) with $c \geqq 0$ and $N = 1$, that is,

$$(21) \qquad -\Delta_p u + c|u|^{p-2}u = r(x)|u|^{q-2}u \text{ in } \mathbb{R}_n,$$

with the function $r(x)$ varying in sign.

THEOREM 5. *Let $p < n$. If $c > 0$ we suppose that $r_+$ satisfies $(a_1)$ and $(b_1)$ for some $\epsilon > 0$ and that $r_- \in L^1_{loc}(\mathbb{R}_n)$. If $c = 0$ we suppose that $r_+$ satisfies $(a_1)$ and that $r_+ \in L^1(\mathbb{R}_n)$ and $r_- \in L^1_{loc}(\mathbb{R}_n)$. Then the equation (21) admits at least one nontrivial solution in $E^{1,p}(\mathbb{R}_n)$ if $c = 0$ and belonging to $W^{1,p}(\mathbb{R}_n)$ if $c > 0$.*

PROOF: We only consider the case $c > 0$. For a functional $\Phi(u) = \frac{1}{q}\int_{\mathbb{R}_n} r(x)|u(x)|^q\,dx$ we set $D(\Phi) = \{u;\ \int_{\mathbb{R}_n}|r(x)||u(x)|^q\,dx < \infty\}$. It is obvious that $D(\Phi)$ is a linear subspace of $E^{1,p}(\mathbb{R}_n)$ containing $C_o^\infty(\mathbb{R}_n)$. In view of Theorem 5 it suffices to show that weak limit points of every bounded subset of $\{\Phi(u) = 1\}$ belong to $D(\Phi)$. Indeed, let $u_m$ be bounded sequence in $\{\Phi(u) = 1\}$. We may assume that $u_m \rightharpoonup u$ in $W^{1,p}(\mathbb{R}_n)$ and $u_m \to u$ in $L^p_{r+}(\mathbb{R}_n)$ and hence

$$\lim_{m\to\infty}\int_{\mathbb{R}_n}|u_m|^q r_+\,dx = \int_{\mathbb{R}_n}|u|^q r_+\,dx$$

and

$$\lim_{m\to\infty}\int_{\mathbb{R}_n}|u_m|^q r_-\,dx = \lim_{m\to\infty}\int_{\mathbb{R}_n}|u_m|^q r_+\,dx - q = \int_{\mathbb{R}_n}|u|^q r_+\,dx - q\,.$$

By Fatou's lemma we see that $r_-|u|^q \in L^1(\mathbb{R}_n)$ and

$$\int_{\mathbb{R}_n} |u|^q r_- \, dx \leqq \lim_{m\to\infty} \int_{\mathbb{R}_n} |u_m|^q r_- \, dx = \int_{\mathbb{R}_n} |u|^q r_+ \, dx - q,$$

therefore $u \in D(\Phi)$. Since $\Phi$ is only Gâteaux differentiable at $u$ in every direction $v \in C_o^\infty(\mathbb{R}_n)$, the equation (21) is satisfied in the distributional sense, that is,

$$\int_{\mathbb{R}_n} (|Du|^{p-2} DuDv + c|u|^{p-2}uv) \, dx = \int_{\mathbb{R}_n} r|u|^{q-2}uv \, dx$$

for each $v \in C_o^\infty(\mathbb{R}_n)$.

According to Theorem 5 the equation (21), with $c > 0$, has a solution in $W^{1,p}(\mathbb{R}_n)$. Let us denote this solution by $u_c$. In Corollaries 1 and 2, below, we examine the behaviour of $u_c$, as $c \to 0$.

COROLLARY 1. *Let $p < n$ and suppose that $r \geqq 0$ on $\mathbb{R}_n$, $r \in L^1(\mathbb{R}_n)$ and that the assumption $(a_1)$ holds. Then $u_c \rightharpoonup u$ in $E^{1,p}(\mathbb{R}_n)$, as $c \to 0$, where $u$ is a nontrivial solution of (22) with $c = 0$.*

PROOF: The function $u_c$ is a solution of the constrained minimization

$$a(u) = \inf\{a(v); \frac{1}{q}\int_{\mathbb{R}_n} |v(x)|^q r(x) \, dx = 1\}.$$

Repeating the approximation argument used in the proof of Lemma 4 in [ 16 ] we conclude that

$$\int_{\mathbb{R}_n} (|Du_c|^p + c|u_c|^p) \, dx = q \int_{\mathbb{R}_n} r|u_c|^q \, dx.$$

Consequently $\{u_c\}$ is bounded in $E^{1,p}(\mathbb{R}_n)$. Therefore we may assume that $Du_c \rightharpoonup Du$ in $L^p(\mathbb{R}_n)$, $u_c \to u$ in $L^p(K)$ on every bounded subset $K \subset \mathbb{R}_n$ and a.e. on $\mathbb{R}_n$. On the other hand in view of Lemma 5 $u_c \to u$ in $L_r^p(\mathbb{R}_n)$ as $c \to 0$. To complete the proof it is sufficient to show that $Du_c \to Du$ in $L^p(\mathbb{R}_n)$. We first show that $Du_c \to Du$ a.e. on $\mathbb{R}_n$. To show this we consider

$$0 \leqq \int_{\mathbb{R}_n} \sum_{i=1}^{n} (|\nabla u_c|^{p-2} D_i u_c - |\nabla u|^{p-2} D_i u) D_i(u_c - u)\, dx$$
$$= \int_{\mathbb{R}_n} r|u_c|^{q-2} u_c(u_c - u)\, dx - \int_{\mathbb{R}_n} \sum_{i=1}^{n} |\nabla u|^{p-2} D_i u D_i(u_c - u)\, dx = J_1 + J_2.$$

By the weak convergence of $Du_c$ we have $\lim J_2 = 0$. The following inequality

$$|J_2| \leqq \left( \int_{\mathbb{R}_n} r|u_c|^q\, dx \right)^{\frac{q-1}{q}} \left( \int_{\mathbb{R}_n} r|u_c - u|^q\, dx \right)^{\frac{1}{q}}$$

yields that $\lim_{c \to 0} J_2 = 0$. Therefore

$$\lim_{c \to 0} \int_{\mathbb{R}_n} \sum_{i=1}^{n} (|\nabla u_c|^{p-2} D_i u_c - |\nabla u|^{p-2} D_i u) D_i(u_c - u)\, dx = 0 \tag{22}$$

and consequently repeating the argument from [**3**] we see that $\lim_{c \to 0} Du_c = Du$ a.e. on $\mathbb{R}_n$. Applying the Hölder inequality we check that $\sum_{i=1}^{n} |\nabla u_c|^{p-2} D_i u_c D_i u$ is equiintegrable and consequently

$$\lim_{c \to 0} \int_{\mathbb{R}_n} \sum_{i=1}^{n} |\nabla u_c|^{p-2} D_i u_c D_i u\, dx = \int_{\mathbb{R}_n} |\nabla u|^p\, dx.$$

It now follows from (22) that

$$\lim_{c \to} \int_{\mathbb{R}_n} |\nabla u_c|^p\, dx = \int_{\mathbb{R}_n} |\nabla u|^p\, dx$$

and the result follows from the uniform convexity of the space $L^r(\mathbb{R}_n)$.

If $p = 2$ $(2 < n)$ this result can be slightly improved by allowing $r$ to vary in sign.

COROLLARY 2. *Let $2 < n$ and suppose that $r$ satisfies the assumptions of Theorem 5 with $p = 2$. Then $u_c \rightharpoonup u$ in $E^{1,2}(\mathbb{R}_n)$, as $c \to 0$, where $u$ is a nontrivial solution of (21) with $c = 0$ and $p = 2$.*

PROOF: The function $u_c$ is a solution of the constrained minimization

$$a(u) = \inf\{a(v);\ \frac{1}{q}\int_{\mathbb{R}_n} |v|^q r\,dx = 1\},$$

where $a(u) = \frac{1}{2}\int_{\mathbb{R}_n}(|Du|^2 + cu^2)\,dx$ and moreover

$$\int_{\mathbb{R}_n}(Du_c Dv + cu_c v)\,dx = \int_{\mathbb{R}_n} r|u_c|^{q-2}u_c v\,dx.$$

for each $v \in C_o^\infty(\mathbb{R}_n)$. Inspection of the proof of Theorem 5 shows that $|u_c|^q r \in L^1(\mathbb{R}_n)$. Repeating the approximation argument used in the proof of Lemma 4 in **[16]** we conclude that

$$\int_{\mathbb{R}_n}(|Du_c|^2 + cu_c^2)\,dx = \int_{\mathbb{R}_n} ru_c^q\,dx = q.$$

Consequently $\{u_c\}$ is bounded in $E^{1,2}(\mathbb{R}_n)$. Therefore we may assume that $Du_c \rightharpoonup Du$ in $L^2(\mathbb{R}_n)$, $u_c \to u$ in $L^2(K)$ on every bounded subset $K \subset \mathbb{R}_n$ and a. e. on $\mathbb{R}_n$.

On the other hand in view of Lemma 5 $u_c \to u$ in $L^2_{r^+}(\mathbb{R}_n)$ as $c \to 0$, hence by Fatou's lemma

$$\int_{\mathbb{R}_n} |u|^q r_-\,dx \leqq \lim_{c\to 0}\int_{\mathbb{R}_n} |u_c|^q r_+\,dx - q = \int_{\mathbb{R}_n} |u|^q r_+\,dx - q.$$

Consequently, $|u|^q r \in L^1(\mathbb{R}_n)$ and $1 \leqq \int_{\mathbb{R}_n} |u|^q r\,dx$, so $u \neq 0$. It is easy to check that $u$ satisfies (22), with $c = 0$, in the distributional sense.

**Remark.** If $p = 2$ the existence result obtained in Theorem 5 is more general than in [7] ( see Proposition 4.1 there ). For related existence results we refer to papers [**14- 15**].

The above discussion shows that the solution $u$ is obtained either by applying the mountain pass theorem or by a constrained minimization. However, we were unable to show whether these to methods give rise to the same or different solutions.

### 5. Exponential decay at infinity.

Inspection of the proof of Theorem 1.1 in [10] show that if $r_i$ $(i = 1, ..., N)$ are bounded for $|x| > R$ for some $R > 0$, then any solution of (1) converges to 0 as $|x| \to \infty$. If $p > n$ this follows from the fact that $u \in W^{1,p}(\mathbb{R}_n)$. To establish the exponential decay at infinity of a positive solution of (1) we apply the maximum principle (see Lemma 3.1 in [21]). First we observe that for $u \in C^2(\mathbb{R}_n)$ we have

$$\Delta_p u = (p-2)|\nabla u|^{p-4} \sum_{i,k=1}^{n} D_{ki}uD_k uD_i u + |\nabla u|^{p-2} \sum_{i=1}^{n} D_{ii}u.$$

Let $H(x, \delta) = \prod_{i=1}^{n} \cosh \delta x_i$ for $x \in \mathbb{R}_n$ and $\delta > 0$. Then by a direct computation we check that

$$\Delta_p(H^{-1}) = H^{-(p-1)}O(\delta^p).$$

Therefore there exists $\delta_\circ > 0$ such that for $0 < \delta < \delta_\circ$ and all $x \in \mathbb{R}_n$ we have

$$-\Delta_p(H^{-1}) + cH^{-(p-1)} = H^{-(p-1)}\big(O(\delta^p) + c\big) > 0.$$

THEOREM 6. *Suppose that the functions $r_i$ $(i = 1, ..., N)$ satisfy the hypotheses of Lemma 4 and that they are bounded for $|x| \geq R$ for some $R > 0$. Then if $u \in W^{1,p}(\mathbb{R}_n)$ is a positive solution of (1), we have*

$$u(x) \leq Ce^{-\delta \sum_{i=1}^{n} |x_i|} \quad \text{in } \mathbb{R}_n \tag{23}$$

*for some constants $C > 0$ and $\delta > 0$.*

PROOF: The proof is a modification of the proof of Proposition 4.4 in [**19**]. Let $M = \max_{\mathbb{R}_n} u(x)$ and we choose $R_1 \geqq R$ and $0 < \delta_1 \leqq \delta_o$ so that

$$-c + \sum_{i=1}^{N} r_i(x) u^{q_i - p} \leqq -\frac{c}{2}$$

for $|x| \geqq R_1$ and

$$-\Delta_p(H_1) + \left(c - \sum_{i=1}^{N} r_i u^{q_i - p}\right) H_1^{p-1} > 0$$

on $\mathbb{R}_n$ for $0 < \delta \leqq \delta_1$, where $H_1(x,\delta) = M^{-1} H(x - \bar{R}, \delta)^{-1}$ and $\bar{R} = (R, ..., R)$. We now define an open set

$$D(L) = \{x \in \mathbb{R}_n;\ R_1 < |x| < L,\ u(x) > H_1(x,\delta)\}$$

for $R_1 < L$. It is clear that

$$-\Delta_p(u - H_1) < -\left(c - \sum_{i=1}^{N} r_i u^{q_i - p}\right)\left(u^{p-1} - H_1^{p-1}\right) < 0$$

on $D(L)$ in the distributional sense. We now deduce from Lemma 3.1 in [**21**] that

$$u(x) - H_1(x,\delta) \leqq \max\left(0, \max_{|x|=L}\left(u(x) - H_1(x,\delta)\right)\right)$$

on $D(L)$ and letting $L \to \infty$ the result follows.

We now proceed to estimate the gradient of $u$.

THEOREM 7. *Let $u$ be a positive solution in $W^{1,p}(\mathbb{R}_n)$ of the equation (1). Suppose that $r_i$ $(i = 1, ..., N)$ satisfy the assumptions of Theorem 6. Then there exists a constant $\lambda > 0$ such that*

$$\int_{\mathbb{R}_n} |\nabla u(x)|^p e^{\lambda \sum_{i=1}^n |x_i|} \, dx < \infty.$$

PROOF: Let $\psi : \mathbb{R}_n \to [0,1]$ be a $C^1$–function with properties: $\psi(x) = 1$ for $|x| \leqq k$, $\psi(x) = 0$ for $|x| \geqq k+1$ with $|\nabla\psi|$ bounded independently of $k$. Taking $v = uG^p\psi^p$, with $G(x,p) = \prod_{i=1}^n \cosh \lambda x_i$, $0 < \lambda$, as a test function we get

$$(24) \quad \int_{\mathbb{R}_n} |\nabla u|^{p-2} \sum_{i=1}^n D_i u D_i u G^p \psi^p \, dx + p \int_{\mathbb{R}_n} |\nabla u|^{p-2} \sum_{i=1}^n D_i u u D_i G G^{p-1} \psi^p \, dx$$
$$+ p \int_{\mathbb{R}_n} |\nabla u|^{p-2} \sum_{i=1}^n D_i u u G^p D_i \psi \psi^{p-1} \, dx + \int_{\mathbb{R}_n} c u^p G^p \psi^p \, dx$$
$$= \int_{\mathbb{R}_n} \sum_{i=1}^N r_i |u|^{q_i} G^p \psi^p \, dx.$$

Let us denote the second and third integral on the left side by $J_1$ and $J_2$, respectively. A straightforward application of the Hölder inequality gives for $\epsilon > 0$

$$|J_1| \leqq (p-1)\epsilon^{\frac{p}{p-1}} \int_{\mathbb{R}_n} |\nabla u|^p G^p \psi^p \, dx + \epsilon^{-p} \int_{\mathbb{R}_n} u^p |\nabla G|^p \psi^p \, dx$$

and

$$|J_2| \leqq (p-1)\epsilon^{\frac{p}{p-1}} \int_{\mathbb{R}_n} |\nabla u|^p G^p \psi^p \, dx + \epsilon^{-p} \int_{\mathbb{R}_n} u^p G^p |\nabla \psi|^p \, dx.$$

Inserting these two estimates into (24) and choosing $(p-1)\epsilon^{\frac{p}{p-1}} = \frac{1}{4}$, we get

$$(25) \quad \frac{1}{2} \int_{\mathbb{R}_n} |\nabla u|^p G^p \psi \, dx + \int_{\mathbb{R}_n} (cG^p - (4(p-1))^{p-1} |\nabla G|^p) \psi^p u^p \, dx$$
$$\leqq (4(p-1))^{p-1} \int_{\mathbb{R}_n} u^p G^p |\nabla \psi|^p \, dx + \int_{\mathbb{R}_n} \sum_{i=1}^N r_i u^{q_i} G^p \psi^p \, dx.$$

Let $\delta > 0$ be the constant from the estimate (23). Since $|\nabla G|^p = O(\lambda^p)G^p$, we see that there exists $0 < \lambda_\circ < \min_{1 \leqq i \leqq N}(q_i\delta)$ such that

$$cG^p - (4(p-1))^{p-1}|\nabla G|^p \geqq \frac{c}{2}G^p$$

for all $\in \mathbb{R}_n$ and $0 < \lambda < \lambda_\circ$. Letting $k \to \infty$ and using Theorem 7 we derive from (25) that

$$\frac{1}{2}\int_{\mathbb{R}_n} |\nabla u|^p G^p\,dx + \frac{c}{2}\int_{\mathbb{R}_n} u^p G^p\,dx \leqq \int_{\mathbb{R}_n} \sum_{i=1}^N r_i u^{q_i} G^p\,dx < \infty$$

and the result follows.

## 6. Existence of infinitely many solutions.

To obtain the existence of infinitely many solutions we apply the Lusternik-Schnirelman theory of critical points. We assume that the functions $r_i$ $(i = 1, ..., N)$ are nonnegative and satisfy the assumptions of Lemma 4.

LEMMA 6. *Suppose that* $\sum_{i=1}^N r_i(x) > 0$ *on* $\mathbb{R}_n$. *Then for any* $m$*-dimensional subspace* $E_m$ *of* $W^{1,p}(\mathbb{R}_n)$ *the set*

$$F_m = \{u \in W^{1,p}(\mathbb{R}_n);\ J(u) \geqq 0\} \cap E_m$$

*is bounded.*

PROOF: Let $E_m =$ span $\{\Psi_1, ..., \Psi_m\}$, where $\Psi_i \in W^{1,p}(\mathbb{R}_n)$ $(i = 1, ..., m)$ are linearly independent. Suppose that $F_m$ is unbounded. Therefore, there exists a sequence $\{u_j\}$ in $F_m$ such that $\|u_j\|_{W^{1,p}} \to \infty$ as $j \to \infty$. Since $u_j = \sum_{i=1}^m t_i^j \Psi_i$, this implies that $|t^j| \to \infty$, where $t^j = (t_1^j, ..., t_m^j)$ and we may assume that $\tau_i = \lim_{j\to\infty} \frac{t_i^j}{|t^j|}$ $(i = 1, ..., m)$ and $\sum_{i=1}^m \tau_i = 1$. For every $u_j$ we have

$$K|t^j|^p \geqq \int_{\mathbb{R}_n} \sum_{s=1}^n r_s(x)|u_j(x)|^{q_s}\, dx, \tag{26}$$

where $K > 0$ is a constant independent of $j$. Since $\{\Psi_i\}$ $(i = 1, ..., m)$ are linearly independent $\lim_{j\to\infty} | \sum_{i=1}^m \frac{t_i^j}{|t^j|}\Psi_i| > 0$ on a set of positive measure. According to our assumption on $\{r_i\}$ there exists $r_k$ such that

$$r_k(x) \lim_{j\to\infty} |\sum_{i=1}^m \frac{t_i^j}{|t^j|}\Psi_i| > 0 \tag{27}$$

on a set of positive measure. On the other hand the inequality (26) yields that

$$K|t^j|^{p-q_k} \geqq \int_{\mathbb{R}_n} r_k(x)|\sum_{i=1}^m \frac{t_i^j}{|t^j|}\Psi_i(x)|^{q_k}\, dx.$$

Letting $j \to \infty$ we derive from this inequality that

$$r_k(x) \lim_{j\to\infty} |\sum_{i=1}^m \frac{t_i^j}{|t^j|}\Psi_i(x)| = 0$$

a.e. on $\mathbb{R}_n$ contradicting (27) and this completes the proof.

**Remark.** If there is an open set $\mathcal{O} \subset \mathbb{R}_n$ such that $r_i(x) = 0$ on $\mathcal{O}$ $(i = 1, ..., N)$, then Lemma is not true. In fact, if supp $\Psi_i \subset \mathcal{O}$ for $(i = 1, ..., m)$, then the inequality (26) takes the form $\sum_{i=1}^n |t_i|^p \geqq 0$, which shows that the set $P_m$ is unbounded.

We now introduce a sequence of subspaces $E_m \in W^{1,p}(\mathbb{R}_n)$ with dim $E_m = m$ and $E_m \subset E_{m+1}$ for each $m$. We assume that linear manifold generated by $\bigcup_{m\geqq 1} E_m$ is dense in $W^{1,p}(\mathbb{R}_n)$. By $E_m^c$ we denote a topological and algebraic complement of $E_m$. We set $\Gamma^* = \{h; h : W^{1,p}(\mathbb{R}_n) \to W^{1,p}(\mathbb{R}_n), h(0) =$

0, $h$ is an odd homeomorphism, and $h(B) \subset \{u;\ J(u) \geqq 0\}\}$, where $B$ denotes the unit ball in $W^{1,p}(\mathbb{R}_n)$ centered at 0.

The following theorem is an immediate consequence of Theorem 2.13 in [1].

THEOREM 8. *Suppose that* $\sum_{i=1}^{N} r_i(x) > 0$ *on* $\mathbb{R}_n$. *Then*

$$c_m = \sup_{h \in \Gamma^*} \inf_{u \in \partial B \cap E^c_{m-1}} J(h(u))$$

*is a critical value of* $J$. *Moreover,* $c_m \leqq c_{m+1}$ *and* $\alpha_\circ \leqq c_m$ *for all* $m$.

Here $\alpha_\circ > 0$ is a constant such that $J(u) \geqq \alpha_\circ$ for $\|u\| = \rho_\circ$, where $\rho_\circ$ is sufficiently small.

Let $\Sigma(W^{1,p}(\mathbb{R}_n))$ denote the class of closed subsets of $W^{1,p}(\mathbb{R}_n) - \{0\}$ symmetric with respect to origin. We say that a set $A$ in $\Sigma(W^{1,p}(\mathbb{R}_n))$ has genus $m$, denoted by $\gamma(A) = m$, if $m$ is smallest integer for which there exists an odd mapping $\Phi \in C^1(A, \mathbb{R}_n - \{0\})$. $\gamma(A) = \infty$ if there exists no finite such $m$ and $\gamma(\emptyset) = 0$. For properties of genus we refer to papers [8] and [13]. By $\Gamma_m$ we denote the following subset of $\Sigma(W^{1,p}(\mathbb{R}_n))$:

$$\Gamma_m = \{K \subset W^{1,p}(\mathbb{R}_n));$$

$K$ is compact, symmetric with respect to the origin and for all $h \in \Gamma^*$,

$$\gamma(K \cap h(\partial B)) \geqq m\}.$$

We are now in a position to state the following existence result which follows from Theorem 2.8 in [1].

THEOREM 9. *Suppose that* $\sum_{i=1}^{N} r_i(x) > 0$ *on* $\mathbb{R}_n$ *and let*

$$b_m = \inf_{K \in \Gamma_m} \max_{u \in K} J(u).$$

*Then* $0 < \alpha_o \leqq b_m \leqq b_{m+1}$ *is a critical value of* $J$. *Moreover, if* $b_{m+1} = ... = b_{m+s} = b$, *then* $\gamma(K_b) \geqq s$, *where* $K_b = \{u \in W^{1,p}(\mathbb{R}_n);\ J(u) = b,\ J'(u) = 0\}$.

According to Theorems 2.8 and 2.13 in [1] we also have $c_m \leqq b_m$. In Theorem 10, below, we show that both sequences $b_m$ and $c_m$ converge to infinity. In the proof we use some ideas from the prooof Theorem 3.14 [1] ( see also the proof of Theorem 4.3 in [4]).

THEOREM 10. *Suppose that* $\sum_{i=1}^{N} r_i(x) > 0$ *on* $\mathbb{R}_n$. *Then* $\lim_{m\to\infty} c_m = \infty$.

PROOF: Let

$$M = \{u \in W^{1,p}(\mathbb{R}_n) - \{0\};\ \int_{\mathbb{R}_n} (|Du|^p + c|u|^p)\, dx \leqq \int_{\mathbb{R}_n} \sum_{i=1}^{N} \frac{p}{q_i} r_i(x)|u(x)|^{q_i}\, dx\}.$$

First we show that there exists $b > 0$ such that

$$\int_{\mathbb{R}_n} \sum_{i=1}^{N} r_i(x)|u(x)|^{q_i}\, dx \geqq b$$

for all $u \in M$. Indeed, by a compact imbedding of $W^{1,p}(\mathbb{R}_n)$ into $L^{q_i}_{r_i}(\mathbb{R}_n)$ $(i = 1, ..., N)$ (see Lemma 4) we have for each $k$

$$\left(\int_{\mathbb{R}_n} r_k|u|^{q_k}\, dx\right)^{\frac{1}{q_k}} \leqq \sum_{i=1}^{N}\left(\int_{\mathbb{R}_n} r_i|u|^{q_i}\, dx\right)^{\frac{1}{q_i}} \leqq B\left(\int_{\mathbb{R}_n} (|Du|^p + c|u|^p)\, dx\right)^{\frac{1}{p}}$$
$$\leqq B\left(\int_{\mathbb{R}_n} \sum_{i=1}^{N} \frac{p}{q_i} r_i|u|^{q_i}\, dx\right)^{\frac{1}{p}},$$

and this implies that

$$\int_{\mathbb{R}_n} \sum_{k=1}^{N} r_k |u|^{q_k}\, dx \leqq \sum_{k=1}^{N} B^{q_k} \left( \int_{\mathbb{R}_n} \sum_{i=1}^{N} \frac{p}{q_i} r_i |u|^{q_i}\, dx \right)^{\frac{q_k}{p}}.$$

Since $p < q_k$ for $k = 1, \dots, N$, our claim easily follows. We now define

$$d_m = \inf\{\|u\|_{W^{1,p}};\ u \in M \cap E_m^c\}.$$

We now show that $d_m \to \infty$ as $m \to \infty$. In the contrary case there exists $0 < d < \infty$ and $u_m \in M \cap E_m^c$ such that $\|u_m\|_{W^{1,p}} \leqq d$, so we may assume that $u_m \to 0$ weakly in $W^{1,p}(\mathbb{R}_n)$ and strongly in $L^{q_i}_{r_i}(\mathbb{R}_n)$ $(i = 1, \dots, N)$. Obviously this contradicts the previous step.

For $R > 1$ we define the mapping $h_m : E_m^c \to E_m^c$ by $h_m(u) = R^{-1} d_m u$ for $u \in E_m^c$. We check that $h_m$ can be extended to an application belonging to $\Gamma^*$. Indeed, it is easy to see that for each $u \in W^{1,p}(\mathbb{R}_n) - \{0\}$ there exists a unique $\beta(u) > 0$ such that $\beta(u)u \in M$ and $J(tu) \geqq 0$ for all $0 \leqq t \leqq \beta(u)$. This $\beta(u)$ can be found as a unique root of the equation

$$t^p \int_{\mathbb{R}_n} (|Du|^p + c|u|^p)\, dx - \sum_{i=1}^{N} \int_{\mathbb{R}_n} \frac{p}{q_i} r_i t^{q_i} |u|^{q_i}\, dx = 0. \tag{28}$$

Therefore for an arbitrary $u_\circ \in E_m^c \cap B - \{0\}$ we have

$$R^{-1} d_m \leqq d_m = \inf\{\|u\|_{W^{1,p}};\ u \in M \cap E_m^c\} \leqq \beta(u_\circ)\|u_\circ\|_{W^{1,p}}.$$

Consequently, this implies that

$$h_m(E_m^c \cap B) \subset \{u \in W^{1,p}(\mathbb{R}_n);\ J(u) \geqq 0\}.$$

For an $\epsilon > 0$ we define by $Z_\epsilon$ a set of all sums $u + v$ with $u \in R^{-1} d_m(E_m^c \cap B)$ and $v \in \epsilon(E_m \cap B)$. We now show that

$$Z_\epsilon \subset \{W^{1,p}(\mathbb{R}_n) - \{0\} - M\}. \tag{29}$$

In the contrary case there exists a sequence $R^{-1}d_m u_j + \epsilon_j v_j \in M$ with $\epsilon_j \to 0$, as $j \to \infty$. Since $u_j$ and $v_j$ are bounded in $W^{1,p}(\mathbb{R}_n)$, $\epsilon_j v_j \to 0$ in $W^{1,p}(\mathbb{R}_n)$ and we may assume that $R^{-1}d_m u_j \to R^{-1}d_m u$ weakly in $W^{1,p}(\mathbb{R}_n)$ and strongly in $L^{q_i}_{r_i}(\mathbb{R}_n)$ for $i = 1, ..., N$. By the previous step of the proof we must have

$$b \leqq \sum_{i=1}^{N} \frac{p}{q_i} \int_{\mathbb{R}_n} r_i |R^{-1}d_m u_j + \epsilon_j v_j|^{q_i}\, dx.$$

Also, $R^{-1}d_m u \in M$. On the other hand we have $0 < R^{-1}d_m < \beta(u)$, so $J(R^{-1}d_m u) > 0$, that is,

$$\int_{\mathbb{R}_n} (|Du|^p + c|u|^p)\, dx > \sum_{i=1}^{N} \int_{\mathbb{R}_n} r_i (R^{-1}d_m)^{q_i - p} |u|^{q_i}\, dx$$

and we arrive at a contradiction. We can now define the extension of $h_m$ by

$$\bar{h}_m(u) = \begin{cases} h_m(u) & \text{for } u \in E_m^c, \\ \epsilon u & \text{for } u \in E_m. \end{cases}$$

It follows from (29) that $\bar{h}_m(B) \subset \{u \in W^{1,p}(\mathbb{R}_n);\ J(u) \geqq 0\}$. To complete the proof we show that $\inf_{\partial B \cap E_m^c} J(\bar{h}_m(u)) \to \infty$, as $m \to \infty$. Let $u \in \partial B \cap E_m^c$, then

$$J(\bar{h}_m(u)) = (R^{-1}d_m)^p \left( \min(1, c) - \sum_{i=1}^{N} \frac{p}{q_i} \int_{\mathbb{R}_n} r_i (R^{-1}d_m)^{q_i - p} |u|^{q_i}\, dx \right).$$

On the other hand by (28) we have

$$\sum_{i=1}^{N} \frac{p}{q_i} \int_{\mathbb{R}_n} r_i (R^{-1}d_m)^{q_i - p} |u|^{q_i}\, dx \leqq R^{-(q_1 - p)} \sum_{i=1}^{N} \frac{p}{q_i} \int_{\mathbb{R}_n} r_i \beta^{q_i - p} |u|^{q_i}\, dx$$
$$\leqq R^{-(q_1 - p)} \max(1, c).$$

Hence

$$J(\bar{h}_m(u)) \geqq (R^{-1}d_m)^p \left(\max(1, c) - R^{-(q_i - p)} \max(1, c)\right)$$

and taking $R$ sufficiently large, our claim follows.

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