# DISCRETE NORMS FOR THE CONVERGENCE OF BOUNDARY ELEMENT METHODS

G.A. CHANDLER

## 1. INTRODUCTION.

The simplest of elliptic problems is the interior Dirichlet problem for Laplace's equation. We are given a bounded region $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\Gamma$, and want to find the unknown function $U$ satisfying

$$-\Delta U(x) = 0\,, \quad x \in \Omega\,, \qquad \text{and}$$
$$U(x) = g(x)\,, \quad x \in \Gamma;$$

for some given function $g$.

As the fundamental solution for $-\Delta$ is

$$G(x,\xi) := \frac{1}{2\pi} \log \frac{1}{|x-\xi|}\,, \quad x, \xi \in \mathbb{R}^2\,,$$

we have Green's formula

$$\int_\Gamma G(x,.)U_\nu - \int_\Gamma G(x,.)U = U(x)\,, \quad x \in \operatorname{int}(\Omega) \tag{1.1}$$
$$= \frac{1}{2}U(x)\,, \quad x \in \Gamma\,. \tag{1.2}$$

(For any $x \in \Gamma$, $\nu(x)$ is the outward normal at $x$, $U_\nu(x) = \nu(x).\nabla U(x)$, and $G_\nu(x,\xi) = \nu(\xi).\nabla_\xi G(x,\xi)$ is the normal derivative of $G$ with respect to the second variable.) The

This work was supported by the Australian Research Council through the program grant *Numerical analysis for integrals, integral equations and boundary value problems.*

first case of Green's formula, (1.1), shows $U$ may be computed throughout $\Omega$ once the unknown $U_\nu$ has been found on $\Gamma$. The second case, (1.2), may be rewritten as

$$(1.3) \qquad \int_\Gamma G(x,.)U_\nu = \frac{1}{2}U(x) - \int_\Gamma G_\nu(x,.)U = \frac{1}{2}g(x) - \int_\Gamma G(x,.)g$$

This is a boundary integral equation for the unknown $U_\nu$. Solving such equations numerically by finite elements is known as the boundary element method (see [**1**] or [**12**] and the references there). This has been developed extensively for a number of applications:- elasticity ([**8**]), moving boundary problems in potential flow ([**9**]), and electrostatics ([**7**]).

Although there has been extensive work on proving convergence of numerical methods for boundary integral equations ([**12**]), there remain embarassing gaps, and the theory is not as complete as the corresponding theory for the standard finite element method. Convergence of the very basic method of midpoint collocation and piecewise constant approximations which we consider here, could only be shown previously under unrealistic assumptions on the grid ([**6**]). So the theory provides an imperfect reflection of the method's practical success.

Here we show how discrete norms and a coercivity result proved directly the discretised equations can be used to prove convergence of the midpoint collocation method for equation (1.3). More detail may be found in [**3**] and [**4**].

It suffices to prove the convergence of numerical methods when $\Gamma$ is the unit circle $\Gamma = \{(\text{cis}(s)) : s \in [0, 2\pi]\}$. The extension to the case of a general smooth curve is quite standard (see [**6**] for instance), and so only the circle is considered subsequently . In this special case write $v(s) = U_\nu(\text{cis}(s))$ for the unknown. As

$$G(\text{cis}(s), \text{cis}(\sigma)) = \frac{1}{2\pi} \log \left( \frac{1}{|2\sin((s-\sigma)/2)|} \right),$$

equation (1.3) becomes

$$\mathcal{V}v(s) := \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{1}{|2\sin((s-\sigma)/2)|}\right)v(\sigma)d\sigma = f(s) \tag{1.4}$$

($f(s)$ denotes the right hand side of (1.3) at $x = \mathrm{cis}(s)$). As $v = U_\nu$ and $\int_\Gamma U_\nu = 0$, $v$ should also satisfy the side condition

$$(v, 1) = \int_0^{2\pi} v = 0. \tag{1.5}$$

## 2. NUMERICAL SOLUTIONS AND STABILITY.

The method we consider here is the simplest of the boundary element methods:- midpoint collocation and piecewise constant approximations.

Choose a grid of $n$ points

$$0 = x_0 < x_1 < \ldots x_{n-1} < x_n = 2\pi\,,$$

and let $h_j = x_j - x_{j-1}$ and $h = \max\{h_j\}$. The midpoints of the intervals are $t_{i-\frac{1}{2}} := (x_i + x_{i-1})/2$. The basis functions are defined by

$$\chi_i(s) := \begin{cases} 1, & s \in (x_{i-1}, x_i] \\ 0, & \text{otherwise}\ . \end{cases}$$

The numerical solution to the boundary integral equation (1.4)–(1.5) is written as a linear combination of these basis functions

$$v^n(s) = \sum_{j=1}^{n} v_{j-\frac{1}{2}}\chi_j(s)$$

(and so $v_{j-\frac{1}{2}} = v^n(t_{j-\frac{1}{2}})$ ). We aim to choose the unknowns $\{v_{j-\frac{1}{2}}\}$ so that $v^n$ satisfies the integral equation (1.4) at the midpoints. That is

$$\mathcal{V}v^n(t_{i-\frac{1}{2}}) = f(t_{i-\frac{1}{2}})\,, \tag{2.1}$$

or in matrix vector notation

$$\left[\mathcal{V}\chi_j(t_{i-\frac{1}{2}})\right][v_{j-\frac{1}{2}}] = [f(t_{i-\frac{1}{2}})], \quad i,j = 1,2,\ldots,n. \tag{2.2}$$

We would also like $v^n$ to satisfy the side condition (1.5),

$$(v^n, 1) = \sum_{j=1}^{n} h_j v_{j-\frac{1}{2}} = 0. \tag{2.3}$$

The difficulty in proving convergence of the method is to show that these equations determine $v^n$ uniquely, that is that they are nonsingular. To show this we transform the coefficient matrix in (2.2) by taking differences between neighbouring rows and columns, which gives a matrix which can be proved to be diagonally dominant. Differencing was originally introduced in [5] as a way of improving the convergence of iterative methods.

Firstly taking differences amongst neighbouring rows, (2.2) becomes

$$[\delta_i \mathcal{V}\chi_j][v_{j-\frac{1}{2}}] = [\delta_i f], \quad i = 1,2,\ldots,n-1, \quad j = 1,2,\ldots,n, \tag{2.4}$$

where

$$\delta_i \mathcal{V}\chi_j := \mathcal{V}\chi_j(t_{i+\frac{1}{2}}) - \mathcal{V}\chi_j(t_{i-\frac{1}{2}}) \quad \text{and} \quad \delta_i f = f(t_{i+\frac{1}{2}}) - f(t_{i-\frac{1}{2}}). \tag{2.5}$$

Now take divided differences between columns to get an $n-1 \times n-1$ system with a new solution $[u_1, u_2, \ldots u_{n-1}]$. The $\{v_{j-\frac{1}{2}}\}$ are then computed from the $\{u_j\}$. That is

$$A[u_j] = \left[\delta_i \mathcal{V}\left(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}\right)\right][u_j] = [\delta_i f], \quad i,j = 1,2,\ldots,n-1 \tag{2.6}$$

and then

$$v_{j-\frac{1}{2}} = \frac{1}{h_j}(u_{j-1} - u_j) \quad ; \tag{2.7}$$

where we have set

$$u_0 = u_n = 0. \tag{2.8}$$

Conveniently, any $\{v_{j-\frac{1}{2}}\}$ calculated by (2.7) and (2.8) satisfies the side condition (2.3), because

$$(v^n, 1) = \sum_{j=1}^{n} h_j v_{j-\frac{1}{2}} = \sum_{j=1}^{n} (u_{j-1} - u_j) = u_0 - u_n = 0\,.$$

In fact (2.7) gives a 1–1 correspondence between the $\{v_{j-\frac{1}{2}}\}$ satisfying (2.3) and the $\{u_j\}$ satisfying (2.8). Thus the $\{v_{j-\frac{1}{2}}\}$ calculated by (2.6) and (2.7) satisfies (2.4). However the original collocation equations, (2.1), will not be satisfied exactly, but only to within a small constant that may be proved to be $O(h^2)$ ([**3**]). In spite of this, the solution of the differenced collocation equations, (2.6), will continue to be termed the collocation solution.

The basis of the convergence proof is a careful evaluation of the signs of the elements of $A$. This needs additional restrictions on the mesh. It is assumed subsequently that for some parameters $\mu_1$ and $\mu_2$ which are independent of n, the mesh satisfies

(2.9) $$\frac{1}{\mu_1} \le \frac{h_{j+1}}{h_j} \le \mu_1 \quad \text{and}$$

(2.10) $$h_j \le \frac{\mu_2}{n}$$

for all $j$. We call $\mu_1$ and $\mu_2$ the mesh parameters. Although it is not known whether these assumptions are necessary for the convergence of the method, they are harmless in practice. They do not prevent graded meshes being used to approximate singularities in the solution ([**2**]) or the geometric meshes used in ([**11**]), and might be readily included in routines for generating meshes automatically.

LEMMA 1. *Let*

$$a_{ij} = \delta_i \mathcal{V}\left(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}\right)$$

*be the elements of the coefficient matrix $A$ in the differenced equations* (2.6). *Then*

(2.11) $$a_{jj} > 0 \quad \textit{and} \quad a_{ij} < 0\ , \textit{ for } i \ne j\,.$$

*Moreover there is a constant $\gamma$ depending only on the mesh parameters such that for all $h$ sufficiently small*

$$a_{jj} \geq 2\gamma \quad \text{and} \quad a_{j\pm 1j} \leq -\gamma\,. \tag{2.12}$$

PROOF: This is most easily done by using

$$\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j} = -D\phi_j\,,$$

where $\phi_j$ is the hat function centred at $x_j$; i.e.

$$\phi_j(s) = \begin{cases} (s - x_{j-1})/h_j\,, & s \in [x_{j-1}, x_j] \\ (x_{j+1} - s)/h_{j+1}\,, & s \in [x_j, x_{j+1}] \\ 0\,, & \text{otherwise.} \end{cases}$$

Then integration by parts gives

$$\begin{aligned} \mathcal{V}(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j})(x_j + s) &= -\frac{1}{2\pi}\int_{-h_j}^{h_{j+1}} \log\frac{1}{|2\sin((s-\sigma)/2)|} D\phi_j(\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{-h_j}^{h_{j+1}} \frac{1}{2\tan((s-\sigma)/2)}\phi_j(\sigma)d\sigma =: \Phi_0(s) \end{aligned}$$

(the final integral is a Cauchy principal value integral). Now as $\phi_j > 0$, it is easily seen that $D\Phi_0(s)$ is strictly decreasing for $s \notin [-h_j, h_{j+1}]$, and so

$$a_{ij} = \Phi_0(t_{i+\frac{1}{2}} - x_j) - \Phi_0(t_{i-\frac{1}{2}} - x_j) < 0$$

for $i \neq \{j \pm 1, j\}$. The remaining three cases will follow from the second inequality (2.12).

Write

$$\begin{aligned} \mathcal{V}(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j})(x_j + s) &= -\frac{1}{2\pi}\int_{-h_j}^{h_{j+1}} \left(\log\frac{1}{|s-\sigma|} + \log\frac{1}{|\mathrm{sinc}((s-\sigma)/2)|}\right) D\phi_j(\sigma)d\sigma \\ &= \frac{1}{2\pi}\int_{-h_j}^{h_{j+1}} \frac{1}{s-\sigma}\phi_j(\sigma)d\sigma \\ &\quad + \frac{1}{2\pi}\int_{-h_j}^{h_{j+1}} \log|\mathrm{sinc}((s-\sigma)/2)| D\phi_j(\sigma)d\sigma \\ &=: T_1(s) + T_2(s) \end{aligned} \tag{2.13}$$

using integration by parts.

Firstly note that $\kappa(s) := \log(\mathrm{sinc}(s/2))$ is smooth on $[-\pi, \pi]$ and

$$T_2(s) = -(D^{-1}\kappa)[s + h_{j+1}, s, s - h_j](h_{j+1} - h_j)$$

( $D^{-1}\kappa$ is an indefinite integral of $\kappa$, and $(D^{-1}\kappa)[.,.,.]$ is a second order divided difference of $D^{-1}\kappa$.) Thus

$$T_2(s) = -\frac{1}{2}(D\kappa)(s + \eta)(h_{j+1} - h_j)$$

for some $\eta \in [-h_j, h_{j+1}]$. Hence $T_2(s) = O(h_j)$ and similarly

$$T_2(s_2) - T_2(s_1) = O(s_2 - s_1)O(h_j)\,. \tag{2.14}$$

These show that $T_2$ makes an insignificant contribution to the $a_{ij}$.

To look at the more important first term in (2.13), define

$$\Phi(s) := \frac{1}{2\pi}\int_0^1 \frac{1}{s - \sigma}(1 - \sigma)d\sigma\,.$$

A change of variable shows

$$T_1(s) = \Phi(\frac{s}{h_{j+1}}) - \Phi(-\frac{s}{h_j})\,.$$

Now remembering the definition of $a_{ij}$, (2.13) and (2.14) give

$$\begin{aligned} a_{ij} &= \delta_i \mathcal{V}(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}) \\ &= \left(\Phi(\frac{t_{i+\frac{1}{2}} - x_j}{h_{j+1}}) - \Phi(\frac{t_{i-\frac{1}{2}} - x_j}{h_{j+1}})\right) \\ &\quad + \left(\Phi(\frac{t_{i-\frac{1}{2}} - x_j}{h_j} - \Phi(\frac{t_{i+\frac{1}{2}} - x_j}{h_j})\right) + O(h_i h_j) \end{aligned} \tag{2.15}$$

In particular if $i = j$ then (2.15) becomes

$$a_{jj} = \left(\Phi(\frac{1}{2}) - \Phi(-\frac{1}{2}\frac{h_j}{h_{j+1}})\right) + \left(\Phi(\frac{1}{2}) - \Phi(-\frac{1}{2}\frac{h_{j+1}}{h_j})\right) + O(h_j^2)$$

It is simple to prove that $\Phi$ is decreasing on $[-\infty, 0)$ and on $[1, \infty)$, and that $\Phi(\frac{1}{2}) = \Phi(1)$. So using (2.9),

$$a_{jj} \geq 2\left(\Phi(1) - \Phi(-\frac{1}{2\mu_1})\right) + O(h_j^2) \tag{2.16}$$

which is positive for $h_j$ sufficiently small. Similarly if $i = j+1$, (2.15) becomes

$$\begin{aligned} a_{j+1j} &= \left(\Phi(1 + \frac{h_{j+2}}{2h_{j+1}}) - \Phi(\frac{1}{2})\right) + \left(\Phi(-\frac{h_{j+1}}{2h_j}) - \Phi(-\frac{h_{j+1} + \frac{1}{2}h_{j+2}}{h_j})\right) + O(h_j^2) \\ &\leq \left(\Phi(1 + \frac{1}{2\mu_1}) - \Phi(1)\right) + O(h_j^2), \end{aligned} \tag{2.17}$$

which is negative for $h_j$ sufficiently small. The case $a_{j-1j}$ is similar to (2.17). Thus looking at the right sides of (2.16) and (2.17), we may select a constant $\gamma$ sufficiently small so that (2.12) holds.

□

Lemma 1 states that the $n-1 \times n-1$ matrix $A$ has the sign pattern

$$A = \begin{bmatrix} + & - & \cdots & & - \\ - & + & - & & \vdots \\ \vdots & & \ddots & & \\ & & & \ddots & - \\ - & & \cdots & - & + \end{bmatrix}.$$

This leads to a quick proof that $A$ is strictly diagonally dominant and therefore nonsingular. To prove row diagonal dominance it must be shown that

$$a_{ii} - \sum_{\{j:j\neq i\}} |a_{ij}| > 0.$$

But by the first part of lemma 1 and the definition of $a_{ij}$

$$\begin{aligned} a_{ii} - \sum_{\{j:j\neq i\}} |a_{ij}| &= a_{ii} - \sum_{\{j:j\neq i\}} -a_{ij} = \sum_{j=1}^{n} a_{ij} \\ &= \sum_{j=1}^{n} \delta_i \mathcal{V}(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}) = \delta_i \mathcal{V}(\frac{\chi_n}{h_n} - \frac{\chi_1}{h_1}) \\ &= \delta_i \mathcal{V}(\frac{\chi_0}{h_0} - \frac{\chi_1}{h_1}) = -a_{i0} > 0, \end{aligned} \tag{2.18}$$

where the last line comes from remembering the problem is periodic and applying lemma (2.11) for $j = 0$.

Although it is encouraging to know that the equations which must be solved numerically are in fact nonsingular, to prove convergence of the numerical solution it must be shown that $A$ cannot become too close to being singular as $n$ increases. This is done in the next theorem by showing $\gamma T \le A$ for the tridiagonal matrix

$$T = [\,t_{ij}\,] = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Again the statement of this theorem uses the convention $u_0 = u_n = 0$.

THEOREM 2. (Coercivity) *For all $h$ sufficiently small and any vector $u \in \mathbb{R}^{n-1}$,*

$$\gamma u^T T u = \gamma \sum_{j=1}^{n} (u_j - u_{j-1})^2 \le u^T A u\,, \tag{2.19}$$

*where $\gamma$ is the constant of lemma 1.*

PROOF: Let $A = \gamma T + E$, and let $e_{ij}$ be the elements of $E$. We show $E$ is diagonally dominant by rows and columns.

By lemma (2.11)

$$e_{ii} \ge 0\,, \qquad \text{and} \quad e_{ij} \le 0\,, \qquad \text{for } i \ne j\,.$$

In the same way that (2.18) was derived, for $i \ne 1, n-1$

$$e_{ii} - \sum_{\{j:j\ne i\}} |e_{ij}| = \sum_{j=1}^{n} e_{ij} = \sum_{j=1}^{n} a_{ij} - \gamma \sum_{j=1}^{n} t_{ij} = \sum_{j=1}^{n} a_{ij} = -a_{i0} > 0\,,$$

as $T$ has zero row sums. For the extreme case $i = 1$,

$$e_{11} - \sum_{\{j:j\ne 1\}} |e_{1j}| = \sum_{j=1}^{n} a_{1j} - \gamma \sum_{j=1}^{n} t_{1j} = \sum_{j=1}^{n} a_{ij} + \gamma = -a_{i0} + \gamma > 0 \tag{2.20}$$

by (2.12). The case $i = n$ is similar, so $E$ is row diagonally dominant.

The proof that $E$ is column diagonally dominant is analogous. For $j \neq 1$,

$$e_{jj} - \sum_{\{i:i\neq j\}} |e_{ij}| = \sum_{i=1}^{n} a_{ij} = -a_{0j} > 0\,,$$

and then a similar modification to (2.20), deals with $i = 1, n$.

Since $E$ is row and column diagonally dominant, it is positive definite; i.e.

$$u^T A u \geq 0\,. \tag{2.21}$$

This follows because the minimum of $u^T E u$ on the unit circle $u^T u = 1$ is an eigenvalue of $E + E^T$. As $E$ is diagonally dominant by rows and columns, $E + E^T$ is diagonally dominant, and the Gershgorin circle theorem shows its eigenvalues must be non–negative.

So now by (2.21), for any $u \in \mathbb{R}^2$,

$$u^T A u = u^T(\gamma T + E)u = \gamma u^T T u + u E u \geq \gamma u^T T u = \gamma \sum_{j=1}^{n} (u_j - u_{j-1})^2\,.$$

□

The coercivity theorem is expressed in terms of the the vector $u^n$, but using (2.7) it translates to a result about piecewise constant approximations. The left side of (2.19) becomes just $\sum h_i^2 (v_{i-\frac{1}{2}}^n)^2$, while some manipulation shows

$$\begin{aligned}
u^T A u &= \sum_{ij} u_i \delta_i \mathcal{V}\big(\big(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}\big)u_j\big) \\
&= -\sum_{ij} (u_i - u_{i-1}) \mathcal{V}\big(\big(\frac{\chi_{j+1}}{h_{j+1}} - \frac{\chi_j}{h_j}\big)u_j\big)(t_{i-\frac{1}{2}}) \\
&= -\sum_{ij} (u_i - u_{i-1}) \mathcal{V}\big(\frac{\chi_j}{h_j}(u_{j-1} - u_j)\big)(t_{i-\frac{1}{2}}) \\
&= \sum_{ij} h_i v_{i-\frac{1}{2}} (\mathcal{V} v^n)(t_{i-\frac{1}{2}})
\end{aligned}$$

Thus in terms of $v^n$ lemma 2 becomes

$$\gamma \sum_{i=1}^{n} h_i^2 (v_{i-\frac{1}{2}})^2 \leq \sum_{i=1}^{n} h_i v_{i-\frac{1}{2}} (\mathcal{V} v^n)(t_{i-\frac{1}{2}}) \tag{2.22}$$

The right hand side is an obvious approximation to the $-\frac{1}{2}$ norm $\|v^n\|^2_{-\frac{1}{2}} := (\mathcal{V}v^n, v^n)$, but the left hand side does not have a continuous analogue due to the $h_i^2$ term. It is clearly smaller than the approximation $\sum_i h_i (v_{i-\frac{1}{2}})^2$ to the $L^2$ inner product $(v^n, v^n)$.

## 3. CONSISTENCY AND CONVERGENCE.

To use the coercivity result we have to make the additional assumption that the mesh is capable of a good approximation to the solution. On the interval $[x_{j-1}, x_j]$ let

$$\bar{v}_{j-\frac{1}{2}} := \frac{1}{h_j} \int_{x_{j-1}}^{x_j} v \,, \tag{3.1}$$

so that

$$\bar{v}^n := \sum_{j=1}^{n} \bar{v}_{j-\frac{1}{2}} \frac{\chi_j}{h_j}$$

is the approximation to the solution $v$ that would be computed if $v$ were actually known. It is assumed subsequently

$$\|\bar{v}^n - v\|_\infty := \max\{|\bar{v}^n(s) - v(s)| : s \in [0, 2\pi]\} \leq d(v)\frac{1}{n} \tag{3.2}$$

for some unknown constant $d(v)$ depending on $v$ but not on $n$. If $v$ is smooth, then any mesh will staisfy (3.2). Alternatively if $v$ has a weak singularity, such as $v(s) = |s|^{\frac{1}{2}}$, the mesh must be graded in the neighbourhood of the singularity. Equation (3.2) will become true for $|s|^{\frac{1}{2}}$ if the mesh $\{x_i\} = \{\pi(2i/n)^2 : -n/2 \leq i \leq n/2\}$ is used (see **[10]**).

The following technical lemma shows that if (3.2) is true, then there is only a very small error when the integral operator is applied to the approximation $\bar{v}^n$ rather than $v$ itself.

LEMMA 3. *If (3.2) is true, then*

$$\|\mathcal{V}(\bar{v}^n - v)\|_\infty \le C d(v) \frac{\log(n)}{n^2}, \tag{3.3}$$

*for some constant $C$ depending only on the mesh parameters.*

PROOF: We need to bound

$$\mathcal{V}(\bar{v}^n - v)(s) = \frac{1}{2\pi} \sum_j \int_{x_{j-1}}^{x_j} \log\left(\frac{1}{2|\sin((s-\sigma)/2)|}\right)(\bar{v}_{j-\frac{1}{2}} - v(\sigma))d\sigma \tag{3.4}$$

for arbitrary $s$. Without loss of generality, assume that

$$s \in [x_{i-1}, x_i] \quad \text{and} \quad \pi \in [x_{i-1}, x_i] \quad ;$$

that is the intervals are renumbered if necessary so that $s$ is in an interval near the middle. Let

$$\kappa(\sigma) := \frac{1}{2\pi} \log \frac{1}{|\sin((s-\sigma)/2)|}.$$

By the definition of $\bar{v}^n$,

$$\int_{x_{j-1}}^{x_j} (\bar{v}^n - v(\sigma))\kappa_j d\sigma = 0,$$

for any constant $\kappa_j$, and so from (3.4)

$$\begin{aligned} \mathcal{V}(\bar{v}^n - v)(s) &= \sum_j \int_{x_{j-1}}^{x_j} (\kappa(\sigma) - \kappa_j)(\bar{v}_{j-\frac{1}{2}} - v(\sigma))d\sigma \\ &\le \left(\sum_j \int_{x_{j-1}}^{x_j} |\kappa(\sigma) - \kappa_j| d\sigma\right) \|\bar{v}^n - v\|_\infty . \end{aligned} \tag{3.5}$$

Now we have to choose the constants $\kappa_j$ so that the sum in (3.5) is small. By assumption (2.10), we can find $j' < i$ and $j'' > i$ with

$$\frac{\mu_2}{n} \le x_{i-1} - x_{j'} \le \frac{2\mu_2}{n} \quad \text{and} \quad \frac{\mu_2}{n} \le x_{j''-1} - x_i \le \frac{2\mu_2}{n}.$$

(So that $[x_{j'}, x_{j''-1}]$ is a small interval excluding $s$.) Define

$$\bar{\kappa}_{j-\frac{1}{2}} := \begin{cases} \frac{1}{h_j}\int_{x_{j-1}}^{x_j} \kappa(\sigma)d\sigma\,, & j \le j' \text{ or } j \ge j'' \\ 0\,, & \text{, otherwise} \end{cases}$$

Elementary arguments with Taylor's theorem show for $\sigma \in [x_{j-1}, x_j]$ and $j \ge j''$, that

(3.6)

$$|\kappa(\sigma) - \bar{\kappa}_{j-\frac{1}{2}}| \le h_j \max\{|D\kappa(\tau)| : \tau \in [x_{j-1}, x_j]\} \le h_j \frac{1}{4\pi|\tan((s - x_{j-1})/2)|}\,.$$

Now using the assumption on the local mesh ratio, for $j \ge j''$,

$$\frac{1}{|\tan((s - x_{j-1})/2)|} \le C\frac{1}{|\tan((s - x_j)/2)|}\,, \tag{3.7}$$

where $C$ is a constant that depends only on $\mu_1$ and $\mu_2$. Equations (3.6) and (3.7) give

$$\begin{aligned} \sum_{j \ge j''} \int_{x_{j-1}}^{x_j} |\kappa(\sigma) - \bar{\kappa}_{j-\frac{1}{2}}| d\sigma &\le C \sum_{j\ge j''} h_j^2 \frac{1}{2|\tan((s - x_j)/2)|} \\ &\le Ch \sum_{j \ge j''} h_j \frac{1}{2|\tan((s - x_j)/2)|} \\ &\le Ch \int_{x_{j''-1}}^{2\pi} \frac{1}{2|\tan((s - \sigma)/2)|} d\sigma \le Ch\log(\frac{1}{n}) \end{aligned} \tag{3.8}$$

(using the fact that the sum is a lower Riemann sum for the second last integral). A similar argument bounds the sum over $j \le j'$.

Finally it is merely to be remarked that for some $C$ depending only on the mesh parameters

$$\int_{x_{j'}}^{x_{j''-1}} \kappa(\sigma)d\sigma \le C\frac{\log(n)}{n}\,. \tag{3.9}$$

Equations (3.8) and (3.9) bound the sum in (3.5) by $C\log(n)/n$. With the assumption (3.2), this proves the lemma.

□

It is important to point out that lemma 3 can be proved under weaker assumptions than (3.2). In fact assumptions that are only slightly stronger than $\|D^{-1}(v^n - \bar{v}^n)\|_\infty \le$

$O(\log(n)/n^2)$. This enables strong singularities of the form $|s|^\beta$, $-1 < \beta$ to be included. These are the singularities occurring when the boundary integral method is applied to domains with corners. The reference [4] contains more details.

We are now able to prove the main result

THEOREM 4. (Convergence Theorem) *Under the assumptions* (2.9), (2.10), *and* (3.2)*; the midpoint collocation solution, $v^n$ defined by the differenced collocation equations* (2.6)*, satisfies*

$$\left(\sum_{j=1}^{n} h_j^2 (v_{j-\frac{1}{2}} - \bar{v}_{j-\frac{1}{2}})^2\right)^{\frac{1}{2}} \leq C \frac{\log(n)}{n^{\frac{3}{2}}} d(v)\,, \tag{3.10}$$

*where the constant $C$ depends only on the mesh parameters.*

PROOF: Write the error

$$e^n = v^n - \bar{v}^n = \sum_j (v_{j-\frac{1}{2}} - \bar{v}_{j-\frac{1}{2}}) \frac{\chi_j}{h_j}\,.$$

The coercivity theorem applied to the difference $v^n - \bar{v}^n$, (i.e. (2.22) for $e^n$) shows

$$\sum_j h_j^2 (e^n_{j-\frac{1}{2}})^2 \leq \frac{1}{\gamma} \sum_i h_i (\mathcal{V} e^n)(t_{i-\frac{1}{2}}) e^n_{i-\frac{1}{2}}\,. \tag{3.11}$$

But as

$$\delta_i \mathcal{V} e^n = \delta_i \mathcal{V}(v^n - \bar{v}^n) = \delta_i \mathcal{V}(v - \bar{v}^n) \tag{3.12}$$

by the collocation equations (2.5), we have

$$(\mathcal{V} e^n)(t_{i-\frac{1}{2}}) = \mathcal{V}(v - \bar{v}^n)(t_{i-\frac{1}{2}}) + \alpha \tag{3.13}$$

for some constant $\alpha$ independent of $i$. But also

$$\sum_j h_j \alpha e^n_{t_{j-\frac{1}{2}}} = \alpha(e^n, 1) = \alpha(v^n - \bar{v}^n, 1) = 0\,. \tag{3.14}$$

(Because $v$ satisfies (1.5), so does $\bar{v}^n$ by its construction in (3.1).) Thus (3.13) in (3.11) and then using (3.14) gives

$$\sum_j h_j^2 (e_{j-\frac{1}{2}}^n)^2 \le \frac{1}{\gamma} \sum_i h_i(\alpha + \mathcal{V}(v - \bar{v}^n)(t_{i-\frac{1}{2}}))e_{i-\frac{1}{2}}^n$$
$$= \frac{1}{\gamma} \sum_i h_i \mathcal{V}(v - \bar{v}^n)(t_{i-\frac{1}{2}})e_{i-\frac{1}{2}}^n ,$$

and then the Cauchy-Schwartz inequality shows

$$\sum_j h_j^2 (e_{j-\frac{1}{2}}^n)^2 \le \frac{1}{\gamma} \left( \sum_j h_j^2 (e_{j-\frac{1}{2}}^n)^2 \right)^{\frac{1}{2}} \left( \sum_i |\mathcal{V}(v - \bar{v}^n)(t_{i-\frac{1}{2}})|^2 \right)^{\frac{1}{2}} .$$

Then removing the common factor on the right and left sides

$$\left( \sum_j h_j^2 (e_{j-\frac{1}{2}}^n)^2 \right)^{\frac{1}{2}} \le \left( \sum_i |\mathcal{V}(v - \bar{v}^n)(t_{i-\frac{1}{2}})|^2 \right)^{\frac{1}{2}} \le C d(v) \frac{\log(n)}{n^{\frac{3}{2}}}$$

after applying (3.2) for each term in the final sum.

□

This is a perfectly respectable convergence result, as the quantity on the left side of (3.10) is a legitimate norm, that is a sensible measure of the error between the computed solution and the average of the true solution on the intervals of the partition. It is however an unusual way to measure the error, as it has no continuous analogue. So we briefly describe ways the convergence theorem can be used to give results in more common norms.

The simplest such result is obtained under the additional assumption that the grid is quasi-uniform, that is

$$h_{\min} := \min\{h_i : 1 \le i \le n\} \ge \mu_3/n \tag{3.15}$$

for some constant $\mu_3$ independent of $n$. This is a very restrictive assumption that prevents graded meshes being used to get good approximations to weakly singular solutions.

Proceeding nevertheless, (3.15) in (3.10) gives

$$h_{\min}^{\frac{1}{2}}\|v-\bar{v}^n\|_{L^2} = h_{\min}^{\frac{1}{2}}\left(\sum_j h_j (e_{i-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}} \le \left(\sum_j h_j^2 (e_{i-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}} \le Cd(v)\frac{\log(n)}{n^{\frac{3}{2}}},$$

that is

$$\|v^n - \bar{v}\|_{L^2} \le Cd(v)\frac{\log(n)}{n}.$$

This is an error estimate in the more traditional $L^2$ norm.

If the restrictive assumption (3.15) is not made, no $L^2$ error estimates are known for this collocation method, or indeed any other collocation or Galerkin method. But it is possible to to prove an energy estimate

$$\|v^n - \bar{v}\|_{-\frac{1}{2}} \le d(v)\frac{\log(n)}{n^{\frac{3}{2}}}.$$

The key to this is to establish

$$\gamma' \sum_{i=1}^n h_i^2 (v_{i-\frac{1}{2}})^2 \le (v^n, \mathcal{V}v^n) =: \|v^n\|_{-\frac{1}{2}}^2.$$

This is a coercivity result for $(v^n, \mathcal{V}v^n)$, that is for the matrix that occurs when the Galerkin method is used with piecewise constants to solve (1.4). The proof imitates lemmas 1 and 2 above. Details may be found in [4].

## REFERENCES.

[1] C.A. Brebbia, J.C.F. Telles, and L.C. Wrobel, "Boundary Element Techniques," Springer-Verlag, Berlin Heidelberg NewYork Tokyo, 1984.

[2] G.A. Chandler, *Galerkin's method for boundary integral equations on polygonal domains*, J. Austral. Math. Soc. Ser. B. **26** (1984), 1 – 13.

[3] G.A. Chandler, *Mid-point collocation for Cauchy singular integral equations*, (submitted for publication).

[4] G.A. Chandler, *Optimal order convergence of midpoint collocation for first kind integral equations*, (submitted for publication).

[5] N. Dyn and D. Levin, *Iterative solution of systems originating from integral equations and surface interpolation*, SIAM J. Numer. Anal. **20** (1983), 377-390.

[6] F.R. de Hoog, "Product Integration Techniques for the Numerical Solution of Integral Equations," Ph.D. thesis, Australian National University, Canberra, 1973.

[7] M.A. Jaswon, and G.T. Symm, "Integral Equations in Potential Theory and Elastostatics," Academic Press, London, 1977.

[8] G. Kuhn, *Boundary element technique in elastostatics and linear fracture mechanics*, in "Finite Element and Boundary Element Techniques from a Mathematical and Engineering Point of View," (E. Stein and W.L. Wendland, Eds.), Springer – Verlag, Wien – New York, pp. 109 – 170.

[9] M.S. Longuet – Higgins and E.D. Cokelet, *The deformation of steep surface waves on water. I. A numerical method of computation*, Proc. R. Soc. London A **350** (1976), 1 – 26.

[10] J.R. Rice, *On the degree of convergence of nonlinear spline approximation*, in "Approximation with Special Emphasis on Spline Functions," (I.J. Schoenberg, ed.), Academic Press, New York, 1969.

[11] E.P. Stephan, *The h–p version of the Galerkin boundary element method for integral equations on polygons and open arcs*, in "Boundary Elements X," (C.A. Brebbia, (Ed.)), Computational Mechanics, Southhampton, 1988, pp. 239 – 245.

[12] W. Wendland, *Strongly elliptic boundary integral equations*, in "The State of the Art in Numerical Analysis," (A. Iserles, and M.J.D. Powell, Eds.), Clarendon, Oxford, 1987, pp. 511 – 561.

Department of Mathematics,
The University of Queensland, Queensland 4072
Australia

Email: gac@axiom.maths.uq.oz.au