

## LOCAL TECHNIQUES FOR MEAN CURVATURE FLOW

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Consider immersions

$$x : M^n \rightarrow \mathbf{R}^{n+1}$$

of an  $n$ -dimensional manifold without boundary into Euclidean space. We say  $M = x(M^n)$  moves by mean curvature if there exists a one-parameter family  $x_t = x(\cdot, t)$  of immersions with corresponding images  $M_t = x_t(M^n)$  satisfying

$$\begin{aligned} \frac{d}{dt}x(p, t) &= -H(p, t)\nu(p, t) \\ x(p, 0) &= x_0(p) \end{aligned} \quad p \in M^n \quad (1)$$

for some initial data  $x_0$ . Here  $H(p, t)$  and  $\nu(p, t)$  denote mean curvature and outer unit normal of the hypersurface  $M_t$  at  $x(p, t)$ . Using the well-known formula  $\Delta x = -H\nu$  for hypersurfaces  $M$  in  $\mathbf{R}^{n+1}$  we obtain the parabolic system of differential equations

$$\left( \frac{d}{dt} - \Delta \right) x = 0 \quad \text{on } M_t \quad (2)$$

where  $\Delta$  denotes the Laplace-Beltrami operator on  $M_t$ .

Let us first mention a number of global results about mean curvature flow: In 1984, G.Huisken [11] proved that compact, convex initial surfaces converge asymptotically to round spheres. A corresponding result for convex curves in the plane was established by Gage and Hamilton [7]. Grayson [8] extended this to arbitrary embedded planar curves.

In the noncompact case it was shown in [5] that entire graphs over  $\mathbf{R}^n$  of linear growth "flatten out" and asymptotically converge to "expanding" selfsimilar solutions of (1).

Without the convexity assumption an initial surface  $M_0$  will in general develop singularities in finite time before shrinking to a point. M. Grayson [9] proved this for initial surfaces which are in part contained inside a long thin cylinder and admit two large enough internal spherical barriers near either end of this cylinder. Dziuk and Kawohl [4] showed that under certain additional conditions rotationally symmetric surfaces form an isolated singularity. Isolated singularities near which the mean curvature of the surfaces  $M_t$  "blows up" at a certain rate in time have been classified by G.Huisken [13].

In this talk we would like to concentrate on localization techniques for mean curvature flow. In particular, we want to obtain some control on the rate at which "far away bad behaviour" enters a certain confined region of space. An important tool for determining the local behaviour of geometric quantities during the evolution is the parabolic maximum principle:

**Proposition 1.** *Suppose the function  $f = f(x, t)$  satisfies the inequality*

$$\left( \frac{d}{dt} - \Delta \right) f \leq 0 \quad (\geq 0)$$

*on compact hypersurfaces  $M_t$  moving by mean curvature. Then*

$$\sup_{M_t} f \leq \sup_{M_0} f \quad (\inf_{M_t} f \geq \inf_{M_0} f)$$

*for all  $t \geq 0$ .*

Note that for some applications it is important to allow an additional linear term in the tangential gradient of  $f$ . We refer to [6] for a noncompact version of the maximum principle.

Let us now consider the evolution equation

$$\left(\frac{d}{dt} - \Delta\right)(|x|^2 + 2nt) = 0 \quad (4)$$

derived in [5]. This identity corresponds to the fact that the radius of a sphere shrinking concentrically under (1) is given by  $r(t) = \sqrt{r^2(0) - 2nt}$ . From Proposition 1 which in this case also applies to noncompact hypersurfaces we immediately obtain the following well-known sphere comparison result (e.g. [2])

**Proposition 2.** *Let  $M_t$  be a family of hypersurfaces moving by mean curvature. Then the following statements hold:*

- (i) *If  $M_0 \subset B_R(0)$  then  $M_t \subset B_{\sqrt{R^2 - 2nt}}(0)$ .*
- (ii) *If  $M_0 \subset \mathbf{R}^{n+1} \sim B_R(0)$  then  $M_t \subset \mathbf{R}^{n+1} \sim B_{\sqrt{R^2 - 2nt}}(0)$ .*

In [3] U. Dierkes showed that the function  $|x|^2 - nx_{n+1}^2$  is subharmonic on minimal hypersurfaces in  $\mathbf{R}^{n+1}$  and used this to establish certain nonexistence results for connected minimal hypersurfaces having more than one boundary component. Let us apply his idea to mean curvature flow. Using the identity

$$\left(\frac{d}{dt} - \Delta\right)x_{n+1}^2 = -2|\nabla x_{n+1}|^2$$

from [5] as well as the fact that  $|\nabla x_{n+1}| \leq 1$  ( here  $\nabla$  denotes the tangential gradient on  $M_t$  ) we obtain

$$\left( \frac{d}{dt} - \Delta \right) (|x|^2 - (n - \beta)x_{n+1}^2 + 2\beta t) \leq 0$$

for  $0 \leq \beta \leq n$ . Proposition 1 then yields

**Proposition 3.** *Let  $M_t$  be a family of compact hypersurfaces moving by mean curvature.*

*Then for  $0 \leq \beta \leq n$  and  $t \leq \frac{\epsilon}{2\beta}$*

$$M_0 \subset \{x \in \mathbf{R}^{n+1} / (n - 1 - \beta)x_{n+1}^2 \geq x_1^2 + \dots + x_n^2 - \epsilon\}$$

*implies*

$$M_t \subset \{x \in \mathbf{R}^{n+1} / (n - 1 - \beta)x_{n+1}^2 \geq x_1^2 + \dots + x_n^2 - \epsilon + 2\beta t\}.$$

*Remark.* For  $\beta = n - 1$  this yields that  $M_t$  will remain inside a shrinking cylinder of radius  $\sqrt{\epsilon - 2(n - 1)t}$ . For  $0 \leq \beta < n - 1$  we infer that  $M_{\frac{\epsilon}{2\beta}}$  will lie inside a cone with vertex  $x = 0$ . Using two appropriately positioned internal spherical barriers of radius  $R$  for  $M_0$  as in Proposition 2 (ii) and assuming that  $M_0$  is contained inside a hyperboloid as in Proposition 3 where  $\epsilon$  is small enough compared with  $R$  we obtain a "neck-pinching" result for a wide class of initial surfaces.

In [2] Brakke studied mean curvature flow for varifolds using a weak version of the evolution equation for the surface measure  $\mu_t$  of  $M_t$ ,

$$\frac{d}{dt}\mu_t = -H^2\mu_t. \tag{5}$$

An important local result he obtained is the "clearing out lemma" which we state here in the case of smooth hypersurfaces:

**Proposition 4.** ([2]) *Let  $M_t$  be a family of hypersurfaces moving by mean curvature:*

*Suppose  $M_0$  satisfies*

$$|M_0 \cap B_R(x_0)| \leq \epsilon R^n$$

*for some  $x_0 \in \mathbb{R}^{n+1}$ ,  $R > 0$  and sufficiently small  $\epsilon = \epsilon(n) > 0$  where  $|\cdot|$  denotes  $n$ -dimensional Hausdorff measure. Then there exist constants  $c > 0$  and  $0 < \alpha < 1$  depending on  $n$  such that*

$$|M_t \cap B_{\frac{R}{4}}(x_0)| = 0$$

*for some  $t \leq c\epsilon^\alpha R^2$ .*

*Remark.* In case  $n = 2$  we can choose  $\alpha = 1$ .

**Proof.** By scaling and translating we may assume w.l.o.g. that  $x_0 = 0$  and  $R = 1$ . We give separate proofs for the cases  $n = 2$  and  $n \geq 3$ . The use of the p-Sobolev inequality in the case  $n \geq 3$  is due to J. Hutchinson.

$n = 2$  : In this case the argument can be based on the standard result that in two dimensions a local  $L^2$ -bound on the mean curvature implies a lower density bound for the surface ([1]). For the convenience of the reader we include a short outline of the argument: Let  $B_\rho = B_\rho(0)$ ,  $\rho > 0$  where  $0 \in M$ . Then by the Sobolev inequality ([14], [1])

$$|M \cap B_\rho|^{\frac{1}{2}} \leq c_0 \left( |M \cap \partial B_\rho| + \int_{M \cap B_\rho} |H| d\mu \right)$$

holds for a.e.  $\rho$  where  $c_0$  is the two dimensional isoperimetric constant. Using the coarea formula and Hölder's inequality we obtain

$$c_0^{-1} |M \cap B_\rho|^{\frac{1}{2}} \left( 1 - c_0 \left( \int_{M \cap B_\rho} H^2 d\mu \right)^{\frac{1}{2}} \right) \leq \frac{d}{d\rho} |M \cap B_\rho|.$$

Let  $\rho_0 > 0$  be such that  $\left(\int_{M \cap B_{\rho_0}} H^2 d\mu\right)^{\frac{1}{2}} < c_0^{-1}$  and  $|M \cap B_{\frac{\rho_0}{2}}| > 0$ . Integrating yields

$$|M \cap B_{\rho_0}| \geq \theta \rho_0^2 \quad (6)$$

where  $\theta = \frac{1}{16c_0^2} \left(1 - c_0 \left(\int_{M \cap B_{\rho_0}} H^2 d\mu\right)^{\frac{1}{2}}\right)^2$ .

We now proceed in the following way: Let  $\psi(r) = (1 - r)_+^2$  where the subscript denotes the positive part of a function and set  $r = |x|^2 + 2nt$ . Using (4) and (5) one readily checks the inequality

$$\frac{d}{dt} \int_{M_t} \psi d\mu_t \leq - \int_{M_t} H^2 \psi d\mu_t \quad (7)$$

which implies, in particular,

$$\int_{M_t} \psi d\mu_t \leq \int_{M_0} \psi d\mu_0 \leq |M_0 \cap B_1| \leq \epsilon.$$

This yields

$$|M_t \cap B_{\frac{1}{2}}| \leq 4\epsilon \quad (8)$$

for all  $t \leq \frac{1}{16}$  in view of the fact that  $\psi \geq \frac{1}{4}$  in  $B_{\frac{1}{2}}$  for those  $t$ . We want to show that for  $\delta > 0$  small enough the inequality

$$\frac{d}{dt} \int_{M_t} \psi \leq -\delta \quad (9)$$

holds for all  $t \leq \frac{1}{16}$  unless the integral already vanishes at an earlier time. For  $\epsilon \leq \frac{\delta}{16}$  inequality (9) implies  $|M_t \cap B_{\frac{1}{2}}| = 0$  when  $t = \frac{\epsilon}{\delta}$ , hence the result.

Suppose (9) does not hold. Then for any  $\delta > 0$ , by the above properties of  $\psi$ , there exists a time  $t \leq \frac{1}{16}$  for which

$$\int_{M_t \cap B_{\frac{1}{2}}} H^2 d\mu_t \leq 4 \int_{M_t} H^2 \psi d\mu_t \leq 4\delta.$$

If  $|M_t \cap B_{\frac{1}{2}}| = 0$  for one of these  $t$  we are done. Otherwise we use (6) with  $\rho_0 = \frac{1}{2}$  to obtain

$$|M_t \cap B_{\frac{1}{2}}| \geq \frac{1}{64c_0^2} \left(1 - 2c_0\delta^{\frac{1}{2}}\right)^2$$

which obviously contradicts (8) for  $\epsilon \leq \frac{\delta}{16}$  and small enough  $\delta$ .

$n \geq 3$ : Consider  $\varphi = (1 - r)_+$  as above where now  $r = |x|^2 + (2n + m - 1)t$  for  $m > 1$ .

Then one derives using (4) and (5)

$$\frac{d}{dt} \int_{M_t} \varphi^m d\mu_t \leq -m(m-1) \left[ \int_{M_t} (\varphi^{m-1} + \varphi^{m-2} |\nabla\varphi|^2 + H^2\varphi^m) d\mu_t \right]$$

where for reasons of exposition we also require  $m(m-1) \leq 1$ . The Sobolev inequality for

$\lambda > 0$ ,  $1 \leq p < 2$  yields

$$\begin{aligned} \left( \int_{M_t} \varphi^{\lambda \frac{np}{n-p}} d\mu_t \right)^{\frac{n-p}{n}} &\leq c(n, p, \lambda) \left[ \int_{M_t} (\varphi^{p(\lambda-1)} |\nabla\varphi|^p + |H|^p \varphi^{\lambda p}) d\mu_t \right] \\ &\leq c(n, p, \lambda) \left[ \int_{M_t} (\varphi^{m-2} |\nabla\varphi|^2 + H^2\varphi^m + \varphi^{\frac{p}{2-p}(2\lambda-m)}) d\mu_t \right] \end{aligned}$$

in view of Young's inequality. Let now  $\lambda = m \frac{n-p}{np}$  and  $p = \frac{2n}{n+2m}$ . (Note that for  $n \geq 3$

this is possible with  $m > 1$ .) Since then  $\frac{n-p}{n} = 1 - \frac{2}{n+2m}$  we arrive at

$$\frac{d}{dt} \left( \int_{M_t} \varphi^m d\mu_t \right) \leq -c \left( \int_{M_t} \varphi^m d\mu_t \right)^{1 - \frac{2}{n+2m}}$$

where  $c = c(n, m) > 0$ . Integrating this inequality implies the result in view of the definition of  $\varphi$ .

Let us now turn to local results for higher order geometric quantities such as gradient and curvature which were obtained in joint work with G.Huisken. In the case of the Ricciflow on Riemannian manifolds introduced by R. Hamilton [10] local estimates have been derived by W.-X. Shi [15].

Suppose  $M_0$  can be locally written as a graph over its tangent plane at some point. Define  $v = \nu_{n+1}^{-1}$  for a choice of normal such that  $\nu_{n+1} > 0$ . If  $M_t = \text{graph } u_t$  then  $v = \sqrt{1 + |Du_t|^2}$  up to tangential diffeomorphisms. In [6] the following gradient estimate is derived:

**Theorem 1.** ([6]) *Let  $R > 0$  and  $x_0 \in \mathbf{R}^{n+1}$  be arbitrary. Then*

$$v(x, t)(R^2 - |x|^2 - 2nt)_+ \leq \sup_{M_0} (v(R^2 - |x|^2)_+)$$

as long as  $v(x, t)$  is defined everywhere in  $M_t \cap \{x \in \mathbf{R}^{n+1} / |x|^2 + 2nt \leq R^2\}$ .

The proof of this theorem is again based on the maximum principle. Using the evolution equation for  $v$

$$\left(\frac{d}{dt} - \Delta\right)v = -|A|^2v - 2v^{-1}|\nabla v|^2$$

from [5] and equation (4) one can show that the function  $f = v(R^2 - |x|^2 - 2nt)_+$  satisfies the inequality

$$\left(\frac{d}{dt} - \Delta\right)f \leq a \cdot \nabla f$$

for some vectorfield  $a$  on  $M_t$ .

*Remark.* One can also prove a gradient estimate in terms of the height of  $M_t$  over an  $n$ -dimensional ball in the tangent space of  $M_t$ . In this case the radius of the ball is time-independent as long as there is a well-defined height-function for  $M_t$  inside the cylinder over this ball (see [6], Theorem 2.3).

Having obtained local gradient estimates we are then able to establish local bounds for the second fundamental form and all its derivatives. This can be achieved using a variety of distance functions for the localization of the problem. Admissible distance functions  $r = r(x, t) \geq 0$  satisfy the conditions

$$\left| \left( \frac{d}{dt} - \Delta \right) r \right| \leq c(n) \quad \text{and} \quad |\nabla r|^2 \leq c(n)r$$

as well as compactness of  $\{x \in M_t / r(x, t) \leq R^2\}$  for  $R > 0$  and  $t \geq 0$ . We may choose  $r = |x|^2 + 2nt$  for estimates inside  $n$ -dimensional shrinking spheres or  $r = |x|^2 - x_{n+1}^2$  for estimates inside cylinders over  $n$ -dimensional balls of fixed radius. The following curvature and higher order estimates hold:

**Theorem 2.** ([6]) *Let  $R > 0$  and assume that  $\{x \in M_t / r(x, t) \leq R^2\}$  can be written as a graph over some hyperplane for  $t \in [0, T]$ . Let  $r = r(x, t) \geq 0$  be as above. Then for any  $t \in [0, T]$  the estimate*

$$\sup_{\{x \in M_t / r(x, t) \leq R^2\}} |A|^2 \leq c(n) \left( \frac{1}{t} + \frac{1}{R^2} \right) \sup_{\{x \in M_s / r(x, s) \leq R^2, s \in [0, t]\}} v^4.$$

holds.

*Remark.* Note that using  $r = |x|^2 + 2nt$  we obtain  $|A|^2 \sim t^{-1}$ .

**Theorem 3.** ([6]) *Under the assumptions of the previous theorem the estimate*

$$\sup_{\{x \in M_t / r(x,t) \leq R^2\}} |\nabla^m A|^2 \leq c_m \left( \frac{1}{t} + \frac{1}{R^2} \right)^{m+1}$$

holds for any  $m \geq 0$  where  $c_m = c_m \left( n, \sup_{\{x \in M_s / r(x,s) \leq R^2, s \in [0,t]\}} v \right)$ .

To prove Theorem 2 we establish an inequality of the type

$$\left( \frac{d}{dt} - \Delta \right) f \leq -\delta f^2 + a \cdot \nabla f + \text{lower order terms}$$

where  $f = |A|^2 \varphi(v^2)$  for some suitable choice of  $\varphi$ . The  $-\delta f^2$  - term then allows us to multiply  $f$  by a cut-off function  $\eta = \eta(r)$  such that  $g = f\eta$  satisfies an inequality of the type

$$\left( \frac{d}{dt} - \Delta \right) g \leq a \cdot \nabla g$$

to which we can apply the parabolic maximum principle.

Finally we would like to mention an application of the interior estimates to the mean curvature evolution of entire graphs.

Let  $M_0 = \text{graph } u_0$ , where  $u_0 : \mathbf{R}^n \rightarrow \mathbf{R}$ . In the class of graphs problem (1) is up to tangential diffeomorphisms equivalent to the parabolic equation

$$\begin{aligned} \frac{d}{dt} u &= \sqrt{1 + |Du|^2} \operatorname{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) \\ u(0) &= u_0. \end{aligned} \tag{10}$$

We prove the following theorem:

**Theorem 4.** ([6]) *Let  $u_0$  be locally Lipschitz continuous on  $\mathbf{R}^n$ . Then (10) (and therefore (1)) has a smooth solution for all  $t > 0$ .*

*Remark.* Note that in contrast to the above result a solution of the Cauchy problem for the ordinary heat equation becomes unbounded in finite time, unless certain growth conditions at infinity are imposed on the initial data.

To prove Theorem 4 we solve the Dirichlet problem with initial and boundary data  $u_0$  on increasing balls  $B_R \subset \mathbf{R}^n$ . The results in [2] guarantee that the solution  $u_R$  on  $B_R$  exists for all times. Using spherical barriers which depend on a fixed time  $T > 0$  and the initial height  $u_0$  over a fixed compact set  $\Omega \subset \mathbf{R}^n$  we establish a bound of the form

$$\sup_{\Omega \times [0, T]} |u_R| \leq c(\Omega, T, u_0)$$

which is independent of  $R$ . We then use the interior estimates to prove bounds for  $|D^m u_R|$  on  $\Omega \times [0, T]$  for all  $m \geq 0$  independent of  $R$ . Letting  $R \rightarrow \infty$  we may now select a subsequence of radii  $R_k \rightarrow \infty$  for which  $u_{R_k}$  converges to a solution of (10) on compact subsets of  $\mathbf{R}^n \times (0, \infty)$ .

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