

# SINGULARITY FORMATION IN GEOMETRIC EVOLUTION EQUATIONS

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In recent years major progress was made on the global behaviour of geometric evolution equations, in particular the harmonic map heatflow, the Ricci flow and the mean curvature flow. Longtime existence and regularity could be shown in a number of important cases. On the other hand, it became clear that in general singularities do occur in finite time, and an understanding of their structure should be crucial both for further development in the theory of these equations and for possible applications. In this article we will point out some of the strong analogies in the equations mentioned above and show in the case of the mean curvature flow how rescaling techniques can be used to understand the asymptotic behaviour of many singularities. We emphasize techniques applicable in all the equations under consideration and mention some open problems.

## 1. The Equations

A first major success for the heat equation method was established by Eells and Sampson [ES] in 1964. They considered a smooth map  $u_0 : (M^m, g) \rightarrow (N^n, h)$  between two compact Riemannian manifolds and solved the evolution equation

$$(1) \quad \begin{aligned} \frac{d}{dt}u(p, t) &= -\nabla E(u)(p, t) \\ u(p, 0) &= u_0(p), \end{aligned}$$

where

$$E(u) = \int_M |Du|_{g,h}^2 d\mu$$

is the Dirichlet energy of the mapping  $u$ . Stationary points of  $E$  are harmonic maps and Eells and Sampson were able to show that the *harmonic map heatflow* (1) has a smooth solution for all times which converges to a harmonic map in the same homotopy class as  $u_0$  provided  $(N^n, h)$  has nonpositive sectional curvature. This result has since been extended in various directions, see e.g. [J], but more recently it became clear that in general singularities will occur in finite time.

Technically, this loss of regularity is due to the evolution equation for  $|Du|^2$

$$(2) \quad \begin{aligned} \frac{d}{dt} |Du|^2 = & \Delta^g |Du|^2 - 2|D^2u|^2 - \text{Ric}_i^j D_i u^\alpha D_j u^\alpha \\ & + \text{Riem}_{\alpha\beta\gamma\delta}^h D_i u^\alpha D_j u^\beta D_i u^\gamma D_j u^\delta, \end{aligned}$$

where the fourth order gradient term can cause finite time blowup if the sectional curvature of  $M$  is positive.

More recently Struwe [St2,St3] obtained an estimate on the size of the singular set and proved homogeneous behaviour of solutions near singularities under very general assumptions. A crucial ingredient in this result was Moser's Harnack inequality and a monotonicity formula for the weighted energy function

$$\int_M |Du|^2 k d\mu.$$

Here  $k > 0$  is a suitably chosen backward heatkernel on  $N^n$ . We will demonstrate the use of a monotonicity formula for the case of mean curvature flow in section 2.

Given a compact Riemannian manifold  $M^n$  with metric  $g_0$ , Hamilton proposed in 1982 [Ha1] to study the equation

$$(3) \quad \frac{d}{dt}g_{ij} = -2R_{ij}^g + \frac{2}{n}rg_{ij},$$

where  $R_{ij}$  is the Ricci curvature of the evolving metric and  $r = \int Rd\mu$  is the average of its scalar curvature. Hamilton proved that on a three-manifold with positive Ricci curvature equation (3) has a smooth solution for all time which converges to a constant curvature metric on  $M$  as  $t \rightarrow \infty$ , thus classifying all compact three-manifolds of positive Ricci curvature. Since then many other global regularity and existence results were proved, compare [Ha2,Ha3,Hu4,Ch], but in general solutions of (3) may develop singularities in finite time. A typical example consists of two  $S^3$ 's which are connected by a long thin tube of type  $S^2 \times \mathbf{R}$ . It is a major open problem to understand the structure of such singularities with the ultimate goal of extending the evolution in a weak form for all time. Analytically, the singular behaviour is reflected in the evolution equation derived from (3) for the scalar curvature

$$(4) \quad \frac{d}{dt}R = \Delta R + 2|\text{Ric}|^2 - \frac{2}{n}rR,$$

where the quadratic term in the curvature is responsible for finite time blowup. The full Riemann curvature tensor satisfies a similar, but more complicated evolution equation, compare [Ha1]. Since the evolution equation in this case is not derived from an energy functional, no monotonicity result analogous to the harmonic map case was established so far. However, progress on local regularity properties of the flow was made by Hamilton [Ha2] in establishing a Harnack inequality and by Shi [Sh] who derived interior estimates for the curvature and its derivatives.

The third example of a geometric evolution equation concerns smooth immersions  $F_t : M^n \rightarrow \mathbf{R}^{n+1}$  of hypersurfaces in Euclidean space which move in direction of their mean curvature vector. That is, the immersions  $F_t = F(\cdot, t)$  satisfy the evolution equation

$$(5) \quad \begin{aligned} \frac{d}{dt} F(p, t) &= -H(p, t)\nu(p, t) & p \in M^n, \quad t > 0 \\ F(p, 0) &= F_0(p), \end{aligned}$$

where  $H$  and  $\nu$  are the mean curvature and unit normal on  $M_t = F_t(M^n)$  respectively. As before this is a nonlinear parabolic system of equations and it is well known that a solution will at least exist for short time under reasonable assumptions on the initial data. Also, a number of global existence results have been obtained. For example, it was shown that compact convex surfaces contract smoothly to a round point in finite time [GH,Hu1], that closed embedded curves in  $\mathbf{R}^2$  become convex [Gr1] and entire graphs over  $\mathbf{R}^n$  exist for all time [EH1].

But again, it is clear that singularities, in particular pinching can occur in finite time, the standard example consisting of two large spheres connected by a long, thin tube. Another example of a singularity arises from a curve with winding number two in the plane, where a small loop pinches off forming a cusp, compare [A3]. This singular behaviour is again reflected in the evolution equation for the curvature, e.g. the square of the norm of the second fundamental form  $|A|^2$ , computed in [Hu1],

$$(6) \quad \frac{d}{dt} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

Notice that the equation for the extrinsic curvature term  $|A|^2$  here has the same scaling behaviour as the equation for  $|Du|^2$  in (2) for the harmonic map heatflow and the equation

for the intrinsic curvature in (4) for the Ricci flow. It is not surprising therefore that the behaviour of solutions near singularities exhibits strong analogies for all three equations. In particular, it appears that solutions in many situations tend to be asymptotically *selfsimilar* when approaching a singularity. We will now show for the special case of mean curvature flow how rescaling techniques can be used to prove this selfsimilar behaviour for a large number of cases.

## 2. Behaviour of Singularities for the Mean Curvature Flow

We assume in this section that  $M^n$  is compact without boundary and  $F_t$  satisfies equation (5). It is well known, see [Hu1], that the mean curvature flow can only become singular in finite time if the curvature becomes infinite. Thus, assuming that the singularity occurs at time  $T$  at the origin let us assume that there is  $p_0 \in M^n$  such that  $F(p_0, t) \rightarrow 0$  and  $|A|^2(p_0, t) \rightarrow \infty$  as  $t \rightarrow T$ . In view of equation (6) we have that the quantity  $U = \max_{M_t} |A|^2$  satisfies the inequality  $\frac{d}{dt}U \leq 2U^2$ . It is then easy to see that  $U$  must blow up at least like  $(T - t)^{-1}$ , i.e.

$$(7) \quad \max_{M_t} |A|^2 \geq \frac{2}{(T - t)}.$$

It is much more difficult to determine an upper bound for the blowup rate near a singularity. It proves to be useful to distinguish two classes of singularities, depending on whether the blowup is as in (7) or higher. We refer to a *type 1 singularity* if we have

$$(8) \quad \max_{M_t} |A|^2 \leq \frac{C_0}{(T - t)}$$

for some constant  $C_0$ , otherwise we call it a *type 2 singularity*. It is known that spheres, convex surfaces, cylinders and rotational symmetric shrinking necks in  $\mathbf{R}^3$  are of type 1,

see [Hu1,Hu2] whereas shrinking loops forming a cusp are of type 2, see [A3]. In [A3] Angenent proves that a shrinking loop approaches a *selftranslating* solution of the mean curvature flow after rescaling, i.e. a curve which moves under translation without changing its shape. Apart from this result very little is known of the general behaviour of type 2 singularities.

In the following we will concentrate on the case of type 1 singularities. In this case we rescale the flow by setting

$$(9) \quad \tilde{F}(p, s) = (2(T - t))^{-\frac{1}{2}} F(p, t), \quad s = -\frac{1}{2} \log(T - t),$$

such that the new time variable  $s$  tends to infinity as  $t \rightarrow T$ . The rescaled position vector then satisfies the equation

$$\frac{d}{dt} \tilde{F} = -\tilde{H}\nu + \tilde{F}.$$

In view of (8) the rescaled surfaces  $\tilde{M}_s = \tilde{F}(\cdot, s)(M^n)$  have bounded curvature and one can then prove estimates for all higher derivatives of the second fundamental form, see e.g. [Hu2] or [EH2]:

$$(10) \quad |\tilde{\nabla}^m \tilde{A}|^2 \leq C_m(C_0) \quad \forall m \geq 1.$$

Furthermore, using again assumption (8) one can show that  $\tilde{F}(p_0, s)$  remains bounded and it is then possible to prove that a subsequence of  $\tilde{M}_s$  converges to a smooth limiting hypersurface  $\tilde{M}_\infty$ .

To understand the structure of  $\tilde{M}_\infty$  we will use a *monotonicity formula*. Let  $\rho$  be a function

on  $\mathbf{R}^{n+1}$  defined by

$$\rho(x, t) = \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} \exp\left(\frac{-|x|^2}{4(T-t)}\right), \quad t < T,$$

which is  $(T-t)^{\frac{1}{2}}$  times the backward heatkernel in  $\mathbf{R}^{n+1}$ . It then follows from equation (5) that  $\rho$  satisfies the evolution equation

$$\frac{d}{dt}\rho = -\Delta\rho + \rho\left(\frac{\langle F, \nu \rangle H}{(T-t)} - \frac{1}{4} \frac{\langle F, \nu \rangle^2}{(T-t)^2}\right)$$

on the hypersurface  $M_t$ . Since the area element on  $M_t$  changes according to the rule  $\frac{d}{dt}d\mu = -H^2 d\mu$  this implies the monotonicity formula

$$(11) \quad \frac{d}{dt} \int_{M_t} \rho d\mu_t = - \int_{M_t} \rho \left( H - \frac{\langle F, \nu \rangle^2}{2(T-t)} \right)^2 d\mu_t.$$

This equation is similar to the monotonicity formula for minimal surfaces and states that surface area can not concentrate too fast near a singular point. After rescaling and setting  $\tilde{\rho}(x) = \exp(-\frac{1}{2}|x|^2)$  equation (11) takes the form

$$\frac{d}{ds} \int_{\tilde{M}_s} \tilde{\rho} d\tilde{\mu}_s = - \int_{\tilde{M}_s} \tilde{\rho} \left( \tilde{H} - \langle \tilde{F}, \nu \rangle \right)^2 d\tilde{\mu}_s$$

and the integrand is no longer explicitly depending on time. Since this weighted area functional is always nonnegative and bounded at time  $t = 0$ , we conclude that

$$(12) \quad \int_{s_0}^{\infty} \int_{\tilde{M}_s} \tilde{\rho} d\tilde{\mu}_s \leq C.$$

Hence, in view of the uniform regularity estimates in (10) every limit surface  $\tilde{M}_{\infty}$  has to satisfy the equation

$$(13) \quad H = \langle F, \nu \rangle.$$

This is a second order elliptic equation and it is easy to see that any hypersurface  $M_0$  satisfying this equation gives rise to a selfsimilar solution of the mean curvature flow by setting  $M_t = (2(T-t))^{\frac{1}{2}} M_0$ . So we have shown

**Theorem 1.** *A type 1 singularity of the mean curvature flow is asymptotically selfsimilar, the surfaces  $M_t$  approach a homothetically shrinking solution as  $t \rightarrow T$ .*

It is an open problem to classify type 2 singularities in a similar way, it may be conjectured that type 2 singularities are asymptotically *selftranslating*. The reader should compare the above result with the paper [GK] by Giga and Kohn, where a similar method was used to understand singularities of certain semilinear heat equations. For the harmonic map heatflow the behaviour near singularities is quite well understood, compare [St2], whereas for the Ricci flow very little is known, mainly because no monotonicity formula has been established.

### 3. Selfsimilar solutions

There is a large variety of selfsimilar, contracting solutions of the mean curvature flow as described by equation (13). Apart from the obvious examples  $\mathbf{S}^n$  and  $\mathbf{S}^{n-m} \times \mathbf{R}^m$  Abresch and Langer [AL] found a discrete two-parameter family of convex immersed curves in  $\mathbf{R}^2$  which contract under homotheties. Furthermore, it is easy to numerically compute a very large variety of closed rotationally symmetric surfaces in  $\mathbf{R}^3$  satisfying equation (13), including an embedded torus, compare [A4]. It seems impossible at this stage to obtain a classification of all solutions to equation (13). However, we can prove a complete classification in the class of surfaces having nonnegative mean curvature:

**Theorem 2.** *If  $\tilde{M}_\infty$  is a smooth limiting hypersurface in  $\mathbf{R}^{n+1}$  satisfying (13), with nonnegative mean curvature  $H \geq 0$ , then  $\tilde{M}_\infty$  is one of the following:*

- (i)  $\mathbf{S}^n$
- (ii)  $\mathbf{S}^{n-m} \times \mathbf{R}^m$
- (iii)  $\Gamma \times \mathbf{R}^{n-1}$ ,

where  $\Gamma$  is one of the homothetically shrinking curves in  $\mathbf{R}^2$  found by Abresch and Langer.

A proof of Theorem 2. for the compact case can be found in [Hu2], the noncompact case will appear elsewhere. Together with Theorem 1 this yields a fairly good description of type 1 singularities with nonnegative mean curvature. It is an open problem to show that generically these are the only singularities that can occur for embedded surfaces moving by mean curvature. Also, it would be very desirable to have a higher order approximation near singularities, i.e. estimates describing the rate of convergence toward the limiting hypersurface  $\tilde{M}_\infty$ . This would open a possibility for extending the flow in a unique and controlled way beyond such singularities.

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