VARIATIONAL PROPERTIES
OF SOME SIMPLE BOUNDARY INTEGRAL EQUATIONS

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1. INTRODUCTION

In the classical method of I. Fredholm and C. Neumann, layer potentials are used to reformulate the Dirichlet and Neumann Problems for the Laplace equation as Fredholm integral equations of the second kind over the bounding curve (in two dimensions) or surface (in three dimensions). However, there are alternative reformulations which lead instead to Fredholm integral equations of the *first* kind over the boundary. My aim here is to give a fairly self-contained and non-technical account of these first-kind integral equations, emphasising the relationship between, on the one hand, the bilinear forms associated with the boundary integral operators, and on the other hand, the Dirichlet bilinear form associated with the Laplace operator. The existence and uniqueness of solutions in the energy spaces is established by showing that the first-kind integral operators are symmetric and positive-definite.

This variational approach to boundary integral equations has important consequences for numerical methods. In particular, for *any* Galerkin method, the linear system that arises has a symmetric positive-definite coefficient matrix, and Céa's lemma implies an optimal error estimate in the energy norm. Just this analysis was used by Hsiao and Wendland [11] to treat the Dirichlet problem in two dimensions, and by Giroire and Nedelec [8] to treat the Neumann problem in three dimensions. These are two of the earliest papers on the convergence of boundary element methods involving first-kind integral equations. More recently, Costabel [1], [2] and Costabel and Stephan [4] have studied variational approaches to the coupling of finite element and boundary element methods.

The theory described below has been generalized by Costabel and Wendland [6] to deal with a large class of elliptic boundary value problems, and Costabel and Stephan [5] use similar techniques to treat a transmission problem. In another extension of the theory, Costabel [3] has treated Lipschitz domains, thereby allowing application of the results to problems on regions with corners and edges.

The paper is arranged as follows. Section 2 is a rapid summary (with some proofs) of the main properties of the single and double layer potentials. The bilinear forms associated with the first-kind boundary integral operators are studied in Section 3, and the results are then applied in Section 4 to establish some mapping properties of the operators. Finally, in Section 5, there is a brief discussion of the Dirichlet and Neumann problems.

2. LAYER POTENTIALS

It will be useful to begin by reviewing some standard facts concerning the single and double layer potentials. More detailed treatments of most of this material may be found in many texts that treat potential theory, e.g., Günter [9], Kellogg [12], Mikhlin [15] and Smirnov [17].

Let Ω^+ be a bounded, open set in $\mathbf{R}^n$. For simplicity, I will assume that Ω^+ is simply-connected and has a C^∞ boundary Γ, and that the dimension n is either 2 or 3. Let Ω^- be the complement of $\Omega^+ \dot\cup \Gamma$ in $\mathbf{R}^n$, so that

$$\mathbf{R}^n = \Omega^+ \dot\cup \Gamma \dot\cup \Omega^- \quad \text{and} \quad \partial\Omega^+ = \Gamma = \partial\Omega^-;$$

here, the dot over $\cup$ indicates a disjoint union. Denote by ν the unit normal to Γ directed into Ω^+, let ds denote the element of arc length or surface area on Γ, and define the bilinear

form
$$\langle \phi, \psi \rangle \equiv \int_\Gamma \phi(x)\psi(x)\,ds_x.$$

The function $E : \mathbf{R}^n \setminus \{0\} \to \mathbf{R}$ defined by

$$E(z) \equiv \begin{cases} \dfrac{1}{2\pi} \log \dfrac{r}{|z|}, & \text{if } n = 2; \\[2mm] \dfrac{1}{4\pi |z|}, & \text{if } n = 3; \end{cases} \tag{2.1}$$

is a *fundamental solution* to Laplace's equation in n dimensions, i.e.,

$$-\nabla^2 E = \delta \quad \text{as distributions on } \mathbf{R}^n,$$

where δ is the Dirac delta functional. The full significance of the parameter $r > 0$ in the definition of E when $n = 2$ will be seen in the sequel — for the moment, it is worth noting that $E(z) > 0$ for $|z| < r$.

Given a function ϕ defined on Γ, the *single layer potential* $V\phi$ and the *double layer potential* $W\phi$ are defined by

$$(V\phi)(z) \equiv \int_\Gamma E(z - y)\phi(y)\,ds_y,$$

$$(W\phi)(z) \equiv \int_\Gamma \left\{ \frac{\partial}{\partial \nu_y} E(z - y) \right\} \phi(y)\,ds_y,$$

for $z \in \Omega^+ \,\dot\cup\, \Omega^-$. Since E is harmonic on $\mathbf{R}^n \setminus \{0\}$, it is clear that

$$\nabla^2(V\phi) = 0 = \nabla^2(W\phi) \quad \text{on } \Omega^+ \,\dot\cup\, \Omega^-, \tag{2.2}$$

and one can easily verify that as $|z| \to \infty$,

$$(V\phi)(z) = \frac{\langle \phi, 1 \rangle}{2\pi} \log \frac{r}{|z|} + O(|z|^{-1}) \quad \text{and} \quad (W\phi)(z) = O(|z|^{-1}) \quad \text{if } n = 2, \tag{2.3}$$

and

$$(V\phi)(z) = O(|z|^{-1}) \quad \text{and} \quad (W\phi)(z) = O(|z|^{-2}) \quad \text{if } n = 3. \tag{2.4}$$

Suppose u is a function defined on $\Omega^+ \,\dot\cup\, \Omega^-$. Denote the boundary values of u on Γ by $u^\pm$, i.e.,

$$u^\pm(x) \equiv \lim_{z \to x, z \in \Omega^\pm} u(z), \quad x \in \Gamma,$$

whenever these limits exist, and write

$$\tilde{u}^\pm(z) \equiv \begin{cases} u(z), & \text{if } z \in \Omega^\pm; \\ u^\pm(z), & \text{if } z \in \Gamma. \end{cases}$$

If $\tilde{u}^+$ and $\tilde{u}^-$ are C^k on $\Omega^+ \,\dot\cup\, \Gamma$ and $\Omega^- \,\dot\cup\, \Gamma$, respectively, then u is said to be *sectionally* C^k. For brevity, I will denote the normal derivatives of a sectionally C^1 function u by

$$u_\nu^\pm \equiv \nu \cdot (\nabla u)^\pm,$$

and write

$$[u] \equiv u^+ - u^- \quad \text{and} \quad [u]_\nu \equiv u_\nu^+ - u_\nu^-$$

for the jumps in u and $\partial u / \partial \nu$ across Γ.

THEOREM 2.1. *If $\phi \in C^2(\Gamma)$, then $V\phi$ and $W\phi$ are sectionally C^1 and satisfy*

$$[V\phi] = 0 = [W\phi]_\nu^-; \tag{2.5}$$

$$[W\phi] = \phi = -[V\phi]_\nu. \tag{2.6}$$

Furthermore,

$$\langle (W\phi)^\pm, \psi \rangle = \langle \phi, (V\psi)_\nu^\mp \rangle \tag{2.7}$$

for all ϕ, $\psi \in C^2(\Gamma)$.

Proof (This approach is similar to that used in Courant and Hilbert [7].) Choose any function $\zeta \in C^2(\Omega^+ \dot\cup \Gamma)$ satisfying

$$\zeta^+ = 0 \quad \text{and} \quad \zeta_\nu^+ = \phi \quad \text{on } \Gamma;$$

e.g., define ζ near Γ by $\zeta(x + t\nu_x) \equiv t\phi(x)$ for $x \in \Gamma$ and $0 < t < \epsilon$. Write $E_z(y) \equiv E(y - z)$, then $\nabla^2 E_z(y) = \delta(y - z)$ and so Green's theorem,

$$\int_\Gamma \{ E_z \zeta_\nu^+ - \zeta^+ (E_z)_\nu^+ \} \, ds = \int_{\Omega^+} \{ \zeta \nabla^2 E_z - E_z \nabla^2 \zeta \} \, dy, \tag{2.8}$$

implies

$$(V\phi)(z) = \begin{cases} -\zeta(z) - \displaystyle\int_{\Omega^+} E_z \nabla^2 \zeta \, dy, & \text{for } z \in \Omega^+; \\ -\displaystyle\int_{\Omega^+} E_z \nabla^2 \zeta \, dy, & \text{for } z \in \Omega^-. \end{cases}$$

(Strictly speaking, one should excise a small disk or ball of radius ρ about $z \in \Omega^+$, apply Green's theorem, and then send $\rho \to 0$.) It is not difficult to verify that $\int_{\Omega^+} E_z \nabla^2 \zeta \, dy$ is continuously differentiable across Γ as a function of z, so $V\phi$ is sectionally C^1, with

$$(V\phi)^+ - (V\phi)^- = -\zeta^+ = 0 \quad \text{and} \quad (V\phi)_\nu^+ - (V\phi)_\nu^- = -\zeta_\nu^+ = -\phi \quad \text{on } \Gamma,$$

as claimed in (2.5) and (2.6).

To handle the double layer potential, one extends ϕ to a C^2 function on $\Omega^+ \dot\cup \Gamma$ in such a way that $\phi_\nu^+ = 0$ on Γ; e.g., near Γ let $\phi(x + t\nu_x) \equiv \phi(x) + t^2$. Replacing ζ by ϕ in (2.8) gives

$$(W\phi)(z) = \begin{cases} \phi(z) + \displaystyle\int_{\Omega^+} E_z \nabla^2 \phi \, dy, & \text{for } z \in \Omega^+; \\ \displaystyle\int_{\Omega^+} E_z \nabla^2 \phi \, dy, & \text{for } z \in \Omega^-; \end{cases}$$

thus, $W\phi$ is sectionally C^1 with

$$(W\phi)^+ - (W\phi)^- = \phi \quad \text{and} \quad (W\phi)_\nu^+ - (W\phi)_\nu^- = \phi_\nu^+ = 0 \quad \text{on } \Gamma,$$

which completes the proof of (2.5) and (2.6).

By applying the divergence theorem to the vector field $u\nabla v$, one finds that

$$\int_\Gamma u^\pm v_\nu^\pm \, ds = \mp \int_{\Omega^\pm} (\nabla u \cdot \nabla v + u\nabla^2 v) \, ds, \tag{2.9}$$

provided u and v decay suitably at infinity. Extend ϕ to a C^2 function with compact support in $\mathbf{R}^n$, then put $u = \phi$ and $v = E_z$ in (2.9) to obtain

$$(W\phi)(z) = \pm \int_{\Omega^{\mp}} \nabla \phi(y) \cdot \nabla E(y - z)\, dy, \quad z \in \Omega^{\pm}.$$

The right hand sides are continuous functions of z, so

$$(W\phi)^{\pm}(x) = \pm \int_{\Omega^{\mp}} \nabla \phi(y) \cdot \nabla E(y - x)\, dy, \quad x \in \Gamma.$$

Thus,

$$
\begin{aligned}
\langle (W\phi)^{\pm}, \psi \rangle &= \int_{\Gamma} \left\{ \pm \int_{\Omega^{\mp}} \nabla \phi(y) \cdot \nabla E(y - x)\, dy \right\} \psi(x)\, ds_x \\
&= \pm \int_{\Omega^{\mp}} \nabla \phi(y) \cdot \left\{ \int_{\Gamma} \nabla E(y - x) \psi(x)\, ds_x \right\} dy \\
&= \pm \int_{\Omega^{\mp}} \nabla \phi(y) \cdot \nabla (V\psi)(y)\, dy = \int_{\Gamma} \phi^{\mp}(V\psi)^{\mp}_{\nu}\, ds,
\end{aligned}
$$

which proves (2.7). The last step follows by taking $u = \phi$ and $v = V\psi$ in (2.9). $\square$

Now define the linear operators

$$R, T^t : C^2(\Gamma) \to C(\Gamma) \quad \text{and} \quad S, T : C^2(\Gamma) \to C^1(\Gamma)$$

by

$$
\begin{aligned}
R\phi &\equiv -(W\phi)^+_{\nu} = -(W\phi)^-_{\nu}, & T^t\phi &\equiv (V\phi)^+_{\nu} + (V\phi)^-_{\nu}, \\
S\phi &\equiv (V\phi)^+ = (V\phi)^-, & T\phi &\equiv (W\phi)^+ + (W\phi)^-,
\end{aligned}
\tag{2.10}
$$

then (2.6) implies

$$(W\phi)^{\pm} = \tfrac{1}{2}(\pm \phi + T\phi) \quad \text{and} \quad (V\phi)^{\pm}_{\nu} = \tfrac{1}{2}(\mp \phi + T^t \phi), \tag{2.11}$$

and (2.7) implies

$$\langle T\phi, \psi \rangle = \langle \phi, T^t \psi \rangle.$$

Thus, T^t is the transpose of T. It is not difficult to show that S, T and T^t can be written as integral operators,

$$(S\phi)(x) = \int_{\Gamma} E(x - y) \phi(y)\, ds_y,$$

$$(T\phi)(x) = 2 \int_{\Gamma} \left\{ \frac{\partial}{\partial \nu_y} E(x - y) \right\} \phi(y)\, ds_y,$$

$$(T^t\phi)(x) = 2 \int_{\Gamma} \left\{ \frac{\partial}{\partial \nu_x} E(y - x) \right\} \phi(y)\, ds_y,$$

for $x \in \Gamma$. Notice that

$$\frac{\partial}{\partial \nu_y} E(x - y) = \frac{1}{\omega_n} \frac{\nu(y) \cdot (x - y)}{|x - y|^n},$$

where $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$. Since Γ is smooth,

$$\nu(y) \cdot (x - y) = O(|x - y|^2) \quad \text{for } x, y \in \Gamma,$$

which implies that T and T^t have smooth kernels if $n = 2$, and only weakly singular kernels if $n = 3$. The operator S is also only weakly singular, however R has formally the hyper-singular kernel

$$-\frac{\partial}{\partial \nu_x} \frac{\partial}{\partial \nu_y} E(x - y) = \frac{1}{\omega_n} \left\{ \frac{\nu_x \cdot \nu_y}{|x - y|^n} - n \frac{\nu_x \cdot (x - y)\, \nu_y \cdot (x - y)}{|x - y|^{n+2}} \right\}.$$

Some further relationships between the four boundary operators in (2.10) are consequences of the following properties of Harmonic functions.

THEOREM 2.2. *If $u \in C^2(\Omega^+ \dot{\cup} \Gamma)$ and $\nabla^2 u = 0$ on Ω^+, then*

$$u = Wu^+ - Vu_\nu^+ \quad on \ \Omega^+, \tag{2.12}$$

with

$$Su_\nu^+ = -\tfrac{1}{2}(u^+ - Tu^+) \quad and \quad Ru^+ = -\tfrac{1}{2}(u_\nu^+ + T^t u_\nu^+) \quad on \ \Gamma. \tag{2.13}$$

Proof Replacing ζ by u in (2.8) yields

$$(Vu_\nu^+)(z) - (Wu^+)(z) = -u(z) \quad \text{for } z \in \Omega^+,$$

which proves (2.12), and then (2.13) follows at once from (2.11). □

COROLLARY 2.3. *For all $\phi \in C^2(\Gamma)$,*

$$SR\phi = \tfrac{1}{4}(\phi - T^2\phi) \quad and \quad RS\phi = \tfrac{1}{4}(\phi - (T^t)^2\phi).$$

Proof In (2.13), take $u = W\phi$ and $V\phi$, respectively, and then make use of (2.11). □

3. BILINEAR FORMS

Recall that, given any open set $\Omega \subseteq \mathbf{R}^n$, the *Dirichlet bilinear form* associated with the Laplace operator on Ω is defined by

$$D_\Omega(u, v) \equiv \int_\Omega \nabla u \cdot \nabla v \, dy.$$

The next two theorems set out the relationship between $D_{\Omega^+ \dot{\cup} \Omega^-}$ and the bilinear forms associated with the boundary operators R and S defined in (2.10).

THEOREM 3.1. *For every ϕ, $\psi \in C^2(\Gamma)$,*

$$\langle R\phi, \psi \rangle = D_{\Omega^+ \dot{\cup} \Omega^-}(W\phi, W\psi), \tag{3.1}$$

and

$$\langle R\phi, \phi \rangle = 0 \quad implies \quad \phi = constant. \tag{3.2}$$

Proof Using the first of the jump relations (2.6), and putting $u = W\psi$ and $v = W\phi$ in (2.9), one obtains

$$\langle R\phi, \psi \rangle = \langle R\phi, [W\psi] \rangle = -\langle (W\phi)_\nu^+, (W\psi)^+ \rangle + \langle (W\phi)_\nu^-, (W\psi)^- \rangle$$

$$= \int_{\Omega^+} \nabla(W\phi) \cdot \nabla(W\psi) \, dy + \int_{\Omega^-} \nabla(W\phi) \cdot \nabla(W\psi) \, dy,$$

which proves (3.1). (Application of the divergence theorem to $u\nabla v$ on Ω^- is justified by the behaviour of the double layer potential at infinity; see (2.3) and (2.4).)

Now suppose that $\langle R\phi, \phi \rangle = 0$, then (3.1) implies $\nabla W\phi = 0$ on $\Omega^+ \dot{\cup} \Omega^-$, so there are constants c^+ and c^- such that $W\phi = c^\pm$ on $\Omega^\pm$. Hence, $\phi = [W\phi] = c^+ - c^- = $ constant on Γ. □

By applying the divergence theorem to the vector field $\nabla E(\cdot - z)$, it is easy to see that the double layer potential of the constant function $\phi = 1$ is just

$$W1(z) = \begin{cases} 1 & \text{if } z \in \Omega^+; \\ 0 & \text{if } z \in \Omega^-. \end{cases} \tag{3.3}$$

Hence, $R1 = 0$ and so, for any $\phi \in C^2(\Gamma)$,

$$R\phi = 0 \quad \text{if and only if} \quad \phi = constant, \tag{3.4}$$

which shows that the converse of (3.2) holds.

For the operator S, there are differences between the two- and three-dimensional cases.

THEOREM 3.2. *Suppose* $\phi, \psi \in C^2(\Gamma)$. *If* $n = 3$, *then*

$$\langle S\phi, \psi \rangle = D_{\Omega^+ \dot{\cup} \Omega^-}(V\phi, V\psi), \qquad \qquad (3.5)$$

and

$$\langle S\phi, \phi \rangle = 0 \quad implies \quad \phi = 0. \qquad (3.6)$$

If $n = 2$, *then* (3.5) *and* (3.6) *hold provided* $\langle \phi, 1 \rangle = 0$.
Proof This time, use the second of the jump relations (2.6), and put $u = V\phi$ and $v = V\psi$ in (2.9), to obtain

$$\langle S\phi, \psi \rangle = \langle S\phi, -[V\psi]_\nu \rangle = -\langle (V\phi)^+, (V\psi)_\nu^+ \rangle + \langle (V\phi)^-, (V\psi)_\nu^- \rangle$$
$$= \int_{\Omega^+} \nabla(V\phi) \cdot \nabla(V\psi) \, dy + \int_{\Omega^-} \nabla(V\phi) \cdot \nabla(V\psi) \, dy,$$

which proves (3.5). When $n = 2$, the condition $\langle \phi, 1 \rangle = 0$ is needed to ensure that $(V\phi)(z)$ is $O(|z|^{-1})$ rather than $O(\log |z|)$ as $|z| \to \infty$; see (2.3).

Now suppose that $\langle S\phi, \phi \rangle = 0$, and assume $\langle \phi, 1 \rangle = 0$ if $n = 2$, then (3.5) implies $\nabla V\phi = 0$ on $\Omega^+ \dot{\cup} \Omega^-$ and hence $\phi = -[V\phi]_\nu = 0$. □

The remainder of this section is devoted to a more detailed study of the operator S in the two-dimensional case; cf. Sloan and Spence [16].
THEOREM 3.3. *Suppose* $n = 2$. *There exists a unique function* $\theta \in C^\infty(\Gamma)$ *such that* $S\theta$ *is constant on* Γ *and* $\langle \theta, 1 \rangle = 1$. *Moreover, this function satisfies* $\theta \geq 0$ *on* Γ.
Proof By (2.10) and (3.3), the constant function $\phi = 1$ satisfies $T1 = 1$, and is therefore a non-trivial solution of the homogeneous equation $\phi - T\phi = 0$. Since T is a compact linear operator on $L_2(\Gamma)$, the transposed equation $\phi - T^t\phi = 0$ has a non-trivial solution, say $\phi = \theta \in L_2(\Gamma)$, by the Fredholm alternative. Furthermore, since T^t has a C^∞ kernel, the function $\theta = T^t\theta$ is C^∞ on Γ. By taking $u = v = V\theta$ in (2.9), and recalling (2.11), one sees that

$$D_{\Omega^+}(V\theta, V\theta) = -\langle (V\theta)^+, (V\theta)_\nu^+ \rangle = \langle S\theta, \tfrac{1}{2}(\theta - T^t\theta) \rangle = 0,$$

so $V\theta$ is constant on Ω^+, and hence $S\theta = (V\theta)^+$ is constant on Γ. Next, observe that if $\langle \theta, 1 \rangle = 0$, then $\langle S\theta, \theta \rangle = (\text{constant})\langle 1, \theta \rangle = 0$ and so $\theta = 0$ by Theorem 3.2. This contradiction shows that $\langle \theta, 1 \rangle \neq 0$, and hence θ can be normalized so that $\langle \theta, 1 \rangle = 1$. To see that θ is unique, suppose $S\theta_j = c_j$ and $\langle \theta_j, 1 \rangle = 1$ for $j = 1$ and 2, then the difference $\phi = \theta_1 - \theta_2$ satisfies $\langle S\phi, \phi \rangle = \langle c_1 - c_2, \theta_1 - \theta_2 \rangle = 0$ and $\langle \phi, 1 \rangle = 0$, so $\phi = 0$ by Theorem 3.2. Finally, (2.3) implies that $(V\theta)(z) \to -\infty$ as $|z| \to \infty$, and therefore, since $(V\theta)^- = S\theta$ is constant on Γ, the maximum principle implies that $(V\theta)_\nu^- \geq 0$ on Γ. Also, $(V\theta)_\nu^+ = \tfrac{1}{2}(-\theta + T^t\theta) = 0$, and hence $\theta = -[V\theta]_\nu = (V\phi)_\nu^- \geq 0$. □

I will write $S = S_r$ and $\theta = \theta_r$ whenever it is necessary to indicate the dependence on the parameter r appearing in (2.1). Let $\kappa_r = 2\pi S_r \theta_r$, and, given any $\phi \in C^2(\Gamma)$, write

$$\phi_{0,r} \equiv \phi - \langle \phi, 1 \rangle \theta_r$$

so that $\langle \phi_{0,r}, 1 \rangle = 0$, then a simple calculation shows

$$\langle S_r \phi, \psi \rangle = \langle S_r \phi_{0,r}, \psi_{0,r} \rangle + \frac{\kappa_r}{2\pi} \langle \phi, 1 \rangle \langle \psi, 1 \rangle \qquad (3.7)$$

This means that if

$$\kappa_r > 0, \qquad (3.8)$$

then $\langle S_r\phi, \phi\rangle \geq 0$, and there is no need to assume $\langle \phi, 1\rangle = 0$ in (3.6). Also, since $\theta_r \geq 0$ it is clear that

$$\kappa_r \geq \log \frac{r}{\mathrm{diam}(\Gamma)},$$

so a sufficient condition for (3.8) is that

$$r > \mathrm{diam}(\Gamma).$$

The simplest case is a circle: if $\Gamma = \{\, z \in \mathbf{R}^2 : |z - z_0| = \rho \,\}$, then

$$\theta_r = \frac{1}{2\pi\rho} \quad \text{and} \quad \kappa_r = 2\pi\big(V\theta_r\big)(z_0) = \log\frac{r}{\rho},$$

because from symmetry, θ_r must be constant on Γ.

Another consequence of (3.7) is that

$$\langle S_r\phi, \phi\rangle \geq \frac{\kappa_r}{2\pi} = \langle S_r\theta_r, \theta_r\rangle \quad \text{whenever } \langle \phi, 1\rangle = 1,$$

and therefore

$$\kappa_r = 2\pi \min\{\, \langle S_r\phi, \phi\rangle : \phi \in C^2(\Gamma) \text{ and } \langle \phi, 1\rangle = 1 \,\}.$$

In fact, since $\theta_r \geq 0$,

$$\kappa_r = 2\pi \min\{\langle S_r\phi, \phi\rangle : \phi \in C^2(\Gamma),\ \phi \geq 0 \text{ and } \langle \phi, 1\rangle = 1 \}.$$

The quantity κ_1 is called *Robin's constant* for the curve Γ, and the related quantity $e^{-\kappa_1}$ is called the *transfinite diameter* of Γ; see Hille [10, p. 280]. Notice that for any $r > 0$ and $r' > 0$,

$$S_r\phi = S_{r'}\phi + \frac{\langle \phi, 1\rangle}{2\pi}\log\frac{r}{r'}$$

and hence

$$\kappa_r = \kappa_{r'} + \log\frac{r}{r'}.$$

In particular, $\kappa_r > \kappa_{r'}$ whenever $r > r' > 0$.

4.   THE ENERGY SPACES

Throughout this section, the parameter r in (2.1) is assumed to be large enough so that the condition (3.8) is satisfied.

Introduce the linear operator

$$R_1\phi \equiv R\phi + \langle \phi, 1\rangle,$$

then

$$\langle R_1\phi, \psi\rangle = \langle R\phi, \psi\rangle + \langle \phi, 1\rangle\langle \psi, 1\rangle$$

and consequently, by (3.2),

$$\langle R_1\phi, \phi\rangle = 0 \quad \text{if and only if} \quad \phi = 0.$$

Notice that $R_1 1 = |\Gamma|$, where $|\Gamma| = \langle 1, 1\rangle = \int_\Gamma ds$ is the arc length or surface area of Γ. For $\phi, \psi \in C^2(\Gamma)$, define

$$(\phi|\psi)_R \equiv \langle R_1\phi, \psi\rangle \quad \text{and} \quad (\phi|\psi)_S \equiv \langle S\phi, \psi\rangle,$$

then Theorems 3.1 and 3.2, together with (3.7) in the case $n = 2$, imply that $(\cdot|\cdot)_R$ and $(\cdot|\cdot)_S$ are *real* inner products on $C^2(\Gamma)$. Denote the associated norms by

$$\|\phi\|_R \equiv \sqrt{(\phi|\phi)_R} \quad \text{and} \quad \|\phi\|_S \equiv \sqrt{(\phi|\phi)_S},$$

and let $\mathcal{H}_R$ and $\mathcal{H}_S$ be the respective abstract Hilbert spaces formed by completion of $C^2(\Gamma)$ in these norms. Thus, $\mathcal{H}_R$ is the energy space of the operator R, and $\mathcal{H}_S$ is the energy space of S.

Next, introduce the dual spaces $\mathcal{H}'_R$ and $\mathcal{H}'_S$, with norms

$$\|f\|'_R \equiv \sup_{\|\phi\|_R = 1} |\langle f, \phi \rangle| \quad \text{and} \quad \|f\|'_S \equiv \sup_{\|\phi\|_S = 1} |\langle f, \phi \rangle|,$$

then, for all $\phi \in C^2(\Gamma)$,

$$\|R_1\phi\|'_R = \|\phi\|_R \quad \text{and} \quad \|S\phi\|'_S = \|\phi\|_S.$$

Hence, R_1 and S have unique extensions to bounded linear operators

$$R_1 : \mathcal{H}_R \to \mathcal{H}'_R \quad \text{and} \quad S : \mathcal{H}_S \to \mathcal{H}'_S; \tag{4.1}$$

indeed, these are the canonical unitary isomorphisms. The operator R itself has a unique extension to a bounded linear operator

$$R : \mathcal{H}_R \to \mathcal{H}'_R, \tag{4.2}$$

because $|\langle \phi, 1 \rangle| \leq \|\phi\|_R$.

The closed subspaces

$$\dot{\mathcal{H}}_R \equiv \{\, \phi \in \mathcal{H}_R : \langle \phi, 1 \rangle = 0 \,\} \quad \text{and} \quad \dot{\mathcal{H}}'_R \equiv \{\, f \in \mathcal{H}'_R : \langle f, 1 \rangle = 0 \,\}$$

are the kernels of the projection $\phi \mapsto |\Gamma|^{-1}\langle \phi, 1 \rangle$ viewed as an operator on $\mathcal{H}_R$ and $\mathcal{H}'_R$, respectively, so

$$\mathcal{H}_R = \dot{\mathcal{H}}_R \oplus \mathrm{span}\{1\} \quad \text{and} \quad \mathcal{H}'_R = \dot{\mathcal{H}}'_R \oplus \mathrm{span}\{1\}.$$

In fact, these are *orthogonal* direct sums, because by (3.4),

$$(\phi|1)_R = \langle \phi, 1 \rangle \quad \text{for all } \phi \in \mathcal{H}_R. \tag{4.3}$$

THEOREM 4.1. *The kernel and image of the operator (4.2) are given by*

$$\ker R = \mathrm{span}\{1\} \quad \text{and} \quad \mathrm{im}\, R = \dot{\mathcal{H}}'_R.$$

Proof Since $C^2(\Gamma)$ is dense in $\mathcal{H}_R$, it follows from (3.4) that $\ker R$ consists of the constant functions. If $f \in \mathrm{im}\, R$, say if $f = R\phi$, then $\langle f, 1 \rangle = \langle R\phi, 1 \rangle = \langle R1, \phi \rangle = 0$ so $f \in \dot{\mathcal{H}}_R$. Conversely, suppose $f \in \dot{\mathcal{H}}_R$, then because R_1 in (4.1) is invertible, there exists a unique $\phi \in \mathcal{H}_R$ such that $f = R_1\phi$. Using (4.3),

$$0 = \langle f, 1 \rangle = \langle R_1\phi, 1 \rangle = (\phi|1)_R = \langle \phi, 1 \rangle,$$

so $f = R_1\phi = R\phi \in \mathrm{im}\, R$. $\qquad\qquad\square$

The results above show that the linear operator

$$R_0 : \dot{\mathcal{H}}_R \to \dot{\mathcal{H}}'_R,$$

defined by $R_0 \phi = R\phi$ for $\phi \in \dot{\mathcal{H}}_R$, is an isometric isomorphism. In fact, because

$$\langle R_0 \phi, \psi \rangle = (\phi | \psi)_R \quad \text{for all } \phi, \psi \in \dot{\mathcal{H}}_R,$$

one can view R_0 as the canonical unitary isomorphism from the Hilbert space $\dot{\mathcal{H}}_R$ onto its dual.

It is possible to obtain further information about $\mathcal{H}_R$ and $\mathcal{H}_S$ by showing that R and S are classical elliptic pseudodifferential operators of order $+1$ and -1, respectively. In particular, it can be shown that

$$\mathcal{H}_R = H^{1/2}(\Gamma) = \mathcal{H}'_S \quad \text{and} \quad \mathcal{H}_S = H^{-1/2}(\Gamma) = \mathcal{H}'_R,$$

where $H^s(\Gamma)$ denotes the usual Sobolev space of order $s \in \mathbf{R}$. When $n = 2$, these results can be proved quite easily using Fourier series — e.g., see McLean [13], [14].

5. BOUNDARY INTEGRAL EQUATIONS

There are two standard methods of reformulating the *Dirichlet problem*,

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{on } \Omega^+, \\ u^+ &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.1}$$

as a boundary integral equation of the form

$$S\phi = f \quad \text{on } \Gamma. \tag{5.2}$$

The *indirect method* consists of choosing $f \equiv g$, then $u = V\phi$ on Ω^+ by (2.2) and (2.10). The *direct method* is based on Theorem 2.2: take $f \equiv -\frac{1}{2}(g - Tg)$, then the solution of (5.2) is the unknown Cauchy data, i.e., $\phi = u_\nu^+$, and so $u = Wg - V\phi$ on Ω^+.

Next, consider the *Neumann problem*,

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{on } \Omega^+, \\ u_\nu^+ &= g \quad \text{on } \Gamma, \end{aligned} \tag{5.3}$$

noting that if u is a solution, then so is $u + c$ for any constant c. Also, applying the divergence theorem to the vector field ∇u yields

$$\int_{\Omega^+} \nabla^2 u \, dx = - \int_\Gamma u_\nu^+ \, ds,$$

so a necessary condition for the existence of a solution to (5.3) is that

$$\langle g, 1 \rangle = 0. \tag{5.4}$$

There are two standard reformulations of (5.3) as a boundary integral equation with a side condition:

$$R\phi = f \quad \text{on } \Gamma, \quad \text{and} \quad \langle \phi, 1 \rangle = 0. \tag{5.5}$$

By Theorem 4.1, there is a unique solution ϕ provided f satisfies

$$\langle f, 1 \rangle = 0. \tag{5.6}$$

For the indirect method, one chooses $f \equiv -g$, then (5.4) implies (5.6), and $u = W\phi$ on Ω^+ by (2.2) and (2.10). As with the Dirichlet problem, the direct method is based on Theorem 2.2: let $f \equiv -\frac{1}{2}(g + T^t g)$, then (5.4) implies (5.6) because, using (2.11) and (3.3),

$$\langle f, 1 \rangle = \langle -\tfrac{1}{2}(g + T^t g), 1 \rangle = -\langle g, \tfrac{1}{2}(1 + T1) \rangle = -\langle g, (W1)^+ \rangle = -\langle g, 1 \rangle.$$

The solution of (5.5) is $\phi = u^+ - c$, where $c = |\Gamma|^{-1} \langle u^+, 1 \rangle$, so $u = W\phi - Vg + c$ on Ω^+.

The preceding remarks are rather informal, and to make any precise statments about existence and uniqueness it is necessary to specify appropriate function spaces. Very briefly, the following is the case; cf. Costabel [3] or Costabel and Wendland [6]. With the Dirichlet problem (5.1), one assumes $g \in H^{1/2}(\Gamma)$, then there is a unique solution $u \in H^1(\Omega^+)$. For both the direct and indirect methods, $f \in H^{1/2}(\Gamma) = \mathcal{H}_S'$ so the boundary integral equation (5.2) has a unique solution $\phi \in H^{-1/2}(\Gamma) = \mathcal{H}_S$. With the Neumann problem, one assumes g belongs to $H^{-1/2}(\Gamma)$ and satisfies (5.4), then a solution $u \in H^1(\Omega^+)$ exists, and is unique up to an arbitrary constant term. For both the direct and indirect methods, $f \in \dot{\mathcal{H}}_R'$ so (5.5) has a unique solution $\phi \in \dot{\mathcal{H}}_R$. Note that $H^1(\Omega^+)$ is the energy space for the Dirichlet bilinear form D_{Ω^+}, and that in general the boundary values u^+ and u_ν^+ must be interpreted as traces on Γ.

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