LOCAL SPLINE APPROXIMANTS

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Abstract

Local spline approximants offer a means for constructing finite difference formulae for numerical solution of PDEs. These formulae seem particularly well suited to situations in which the use of conventional formulae leads to non-linear computational instability of the time integration. This is explained in terms of frequency responses of the FDF.

1. Introduction

This is a brief summary of [12], the main theme of which is the study of a two parameter family of finite difference formulae (FDF) with the aim of determining which (if any) are useful for the numerical solution of PDEs.

The family of FDF in question arise from a scheme for local approximation by splines. There are several such schemes (see for example [1], [2], [4], [6], [7], [11] and [18]). The scheme discussed here is perhaps the simplest. It is certainly the oldest and has as its origins the theory of what used to be called osculatory interpolation [8], [10]. In terms of central B-splines it first appears in Schoenberg's well known papers [13] and [14].

Although this local spline approximation scheme has been with us for over 50 years it has received little attention from those interested in the numerical solution of PDEs (as yet I have found no references to its use). By way of explanation it should be pointed out that the work described in [12] made extensive use of REDUCE for computing a data base of FDF coefficients, as well as special purpose graphics programs for reading the data base and ploting transfer functions, frequency response functions, kernel functions, etc. Without these tools it would be difficult to study all but the simplest of schemes and those are not of high enough order of accuracy to be of value in the solution of PDEs.

2. Sampling kernels via B-splines

The local spline approximants used here are constructed from samples taken at the integers. A function $g(x)$ and its derivatives are approximated by the sampling sums,

$$\bar{g}^{(j)}(x) = \sum_n g(n)\phi^{(j)}(x-n), \qquad j = 0, \ldots, j_{max}. \tag{1}$$

The kernel $\phi(x)$ is a C^k spline (peicewise polynomial of degree $k+1$) with support $w+1$ and knots at the integers. We take $w+1$ even and let $supp(\phi(x)) = (-(w+1)/2, (w+1)/2)$. The kernel is then a linear combination of central B-splines of the form,

$$\phi(x) = \sum_{m=-r/2}^{r/2} a_m M_{k+2}(x-m), \qquad r = w - k - 1. \tag{2}$$

The constants a_m are fixed by the $r+1$ independent conditions $\bar{g}^{(0)}(0) = g(0)$ where $g(x) = x^q$, $q = 0, \ldots, r$.

For suitable band-limited functions and reasonable choices of the pair of integers (k, r) one finds that expressions (1) for $j = 0, 1, \ldots, j_{max}$ are useful approximations to $g(x)$ and its derivatives [4], [12].

On combining (1) and (2) one sees that local spline approximants of the above type provide us with central finite difference approximations $\bar{g}^{(j)}(m)$ for the derivatives $g^{(j)}(m)$ in terms of the w function values $g(m-(w-1)/2), g(m-(w-1)/2+1), \ldots, g(m+(w-1)/2)$. We call these spline-type FDF.

As an example we cite the case $(k, r) = (2, 2)$ which has appeared a number of times in the literature (e.g. [2] equation (6.10); [17] page 569). Here

$$\phi(x) = \frac{4}{3}M_4(x) - \frac{1}{6}(M_4(x-1) + M_4(x+1))$$

and corresponding central finite difference coeffecients are,

0th derivative: -1/36, 1/9, 5/6, 1/9, -1/36

1st derivative: 1/12, -2/3, 0, 2/3, -1/12

See [4] for plots of the above kernel and its Fourier transform.

3. Frequency responses of FDF

Numerical experiments have shown that spline-type finite difference formulae can be used in some situations where *compact* formulae are subject to non-linear *computational instability* [12]. Compact finite difference formulae are those conventionaly used (their coefficients can be found via differentiation of a Lagrange interpolation polynomial). Computational instability refers to the spurious growth (over a small number of time steps) of high frequency components of a solution, in particular, of so called 2-h waves [3],[15]. Computational instability is a non-linear phenomenon thought to be caused by aliasing. The following is an attempt to explain why spline-type FDF perform better in this respect.

There are at least two sources for aliasing error in finite difference computations. The first and obvious is the presence of non-linear terms in the equations to be solved. This is also the well known source of aliasing error for spectral collocation methods and is discussed at length in [3] and [5]. For both spectral and finite difference methods the way to avoid problems caused by this type of aliasing error is to band-limit the solution to some fraction of the Nyquist interval. For finite difference methods this can be done by periodicaly filtering the solution using a low-pass digital filter [9], [15]. The 0th derivative FDF given in the last section is such a filter. In fact the filters described in [16] for this purpose are all spline-type 0th derivative FDF. They may be obtained with $k = 1$ and $r = 1, 3, 5, \ldots$.

The second source of aliasing error depends only on the particular set of finite difference formulae used. It shows up as *miss-matching* between the frequency responses of the FDF for different derivatives. The frequency response of a FDF for jth derivatives is defined by the following ratio of numerical and exact derivatives. Let $g(x) = e^{2\pi i f x}$ then

$$R^{(j)}(f) = \left(\frac{\overline{g}^{(j)}(x)}{g^{(j)}(x)} \right)_{x=0}$$

$$= \frac{1}{(2\pi i f)^j} \, \overline{g}^{(j)}(0), \qquad f \in (-1/2, 1/2).$$

For central FDF $R^{(j)}(f)$ is real and is symmetric about $f = 0$.

Fourier transforms of functions and their derivatives should be related by factors of $2\pi i f$. It is clear then that if there are significant differences between response functions within a set of FDF then numerical derivatives calculated using the formulae will be *inconsistent*.
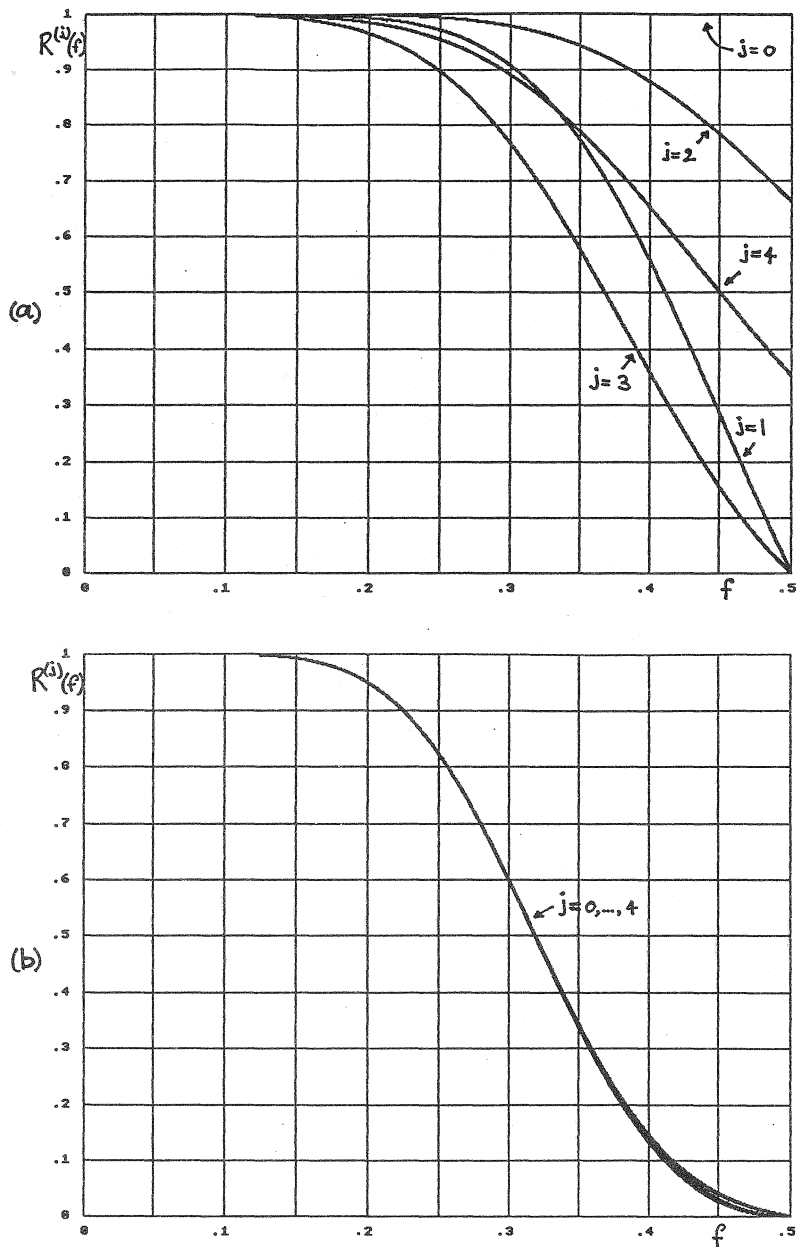
Figure 1 Typical frequency response functions $R^{(j)}(f)$ of central FDF for $j = 0, \ldots, 4$. (a) 9 point compact FDF. (b) 17 point spline-type FDF with $(k, r) = (9, 7)$

They will be inconsistent in the sense that there can exist no function band-limited to the Nyquist interval which has as its derivatives those calculated numerically.

Miss-matched frequency responses can be explained in terms of aliasing of the high frequency components of the kernel function [12]. For spline kernels these components decay as $f^{-(k+2)}$ since $\phi(x)$ is C^k. This causes matching between spline-type FDF to improve as one increases k.

Figures 1a and 1b show frequency response functions for 0th,...,4th derivatives. The general features of these curves are typical of compact and spline-type FDF respectively and show that spline-type FDF can be well matched even for high order derivatives. The finite difference coefficients for these and other spline-type FDF are given in [12] as well as plots of kernel functions and their derivativs.

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