

New Periodic Minimal Surfaces in H^3

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Abstract

We prove existence of new complete embedded minimal surfaces in H^3 having the symmetry of a regular tessellation by Coxeter orthoschemes. Each tetrahedron bounds a fundamental piece along four convex symmetry arcs. Its existence is proved by a conjugate surface construction.

1 Introduction.

It is a basic problem in minimal surface theory getting new examples of minimal surfaces in Riemannian manifolds. The existence question of the Plateau problem for the hyperbolic space H^3 was solved in 1943 by A. Lonseth [5], and M.A. Anderson [1] proved existence for arbitrary boundary curves at infinity using geometric measure theory. But only in very special cases more explicit complete minimal surfaces are known: the hyperbolic helicoid has an explicit formula, and its conjugate surface, the rotational symmetric catenoid, was determined by M.P. Do Carmo and M. Dajczer [2] solving the ODE for the meridian curve. As another special case we generalized in [7] the euclidean Enneper surfaces to H^3 . They are a two parameter family of minimal surfaces with a rotational symmetric metric, thus reducing the Gauß equation to an ODE for the conformal factor of the metric. In contrast to the euclidean Enneper surfaces, some of the hyperbolic analoga are embedded.

In this note we explain some ideas how we constructed new periodic minimal surfaces in H^3 . Fundamental pieces for the symmetry group of these surfaces are bounded by four planar symmetry lines, which lie on each of the four sides of a tetrahedron. Hyperbolic reflection in the totally geodesic hyperplane given by a tetrahedron's face will analytically continue the minimal surface piece across the symmetry line. If the tetrahedron is a Coxeter orthoscheme, then we obtain a complete embedded minimal surface via successive reflection in the sides of the tetrahedra. Inside each tetrahedron of the tessellation will lie a copy of the fundamental minimal surface piece.

The existence proof of the fundamental patch uses a conjugate surface construction which was first applied by B. Smyth [9] to construct patches in euclidean tetrahedra and later extended by H. Karcher, U. Pinkall and I. Sterling [3] to obtain new examples of compact embedded minimal surfaces in S^3 . Compared to the euclidean case the construction procedure in S^3 and in H^3 is much more involved, resulting from the fact that surface normals at corresponding points of a minimal surface and its conjugate surface are no longer parallel as it would be in \mathbb{R}^3 .

The surfaces we obtain in H^3 are analoga of the classical triply periodic minimal surfaces of H.A. Schwarz and E.R. Neovius in \mathbb{R}^3 , see figure 2. These two surfaces have fundamental cells in a regular cube and have all cubical symmetries. Therefore a fundamental piece of the surfaces lies in a Coxeter orthoscheme of the cube. We construct such pieces in H^3 in the Coxeter orthoschemes of all compact and noncompact regular polyhedra and additionally in many Coxeter orthoschemes which do not belong to tessellations with regular polyhedra, see Theorem. These additional tessellations have no analoga in \mathbb{R}^3 and S^3 .

All pictures of the hyperbolic space are conformal mappings of H^3 into the Poincare model, except of figure 4d where H^3 was mapped into the Klein model.

2 Symmetry Properties.

A curve on a surface in a space $M^3(c)$ of constant curvature c is called a *straight line*, if it is a geodesic in $M^3(c)$, and it is called a *planar line*, if it lies in a 2-dimensional totally geodesic submanifold of $M^3(c)$.

Minimal surfaces in spaces $M^3(c)$ have a very exquisite symmetry property. If they contain a straight line, then the surface is invariant by a 180° rotation around this line, and if they contain a planar geodesic, then the minimal surface is symmetric corresponding to this plane. These properties follow in from the Schwarz reflection principle of function theory and were generalized to minimal surfaces in $M^3(c)$ by H.B. Lawson [4].

According to Lawson there exists for every minimal immersion

$$F : M^2 \longrightarrow M^3(c)$$

of a simply connected piece M^2 of a Riemann surface into $M^3(c)$ a family of isometric minimal immersions

$$F^\theta : M^2 \longrightarrow M^3(c) \quad , \quad \theta \in (0, 2\pi)$$

with geometric data

$$g^\theta = g$$

$$S^\theta = D^\theta \cdot S,$$

where g (resp. g^θ) denotes the induced metric on M^2 by F (resp. F^θ) and S (resp. S^θ) denotes the Weingarten map of F (resp. F^θ). The existence of this family is proved by defining their geometric data with the equations above: for minimal surfaces S^θ is symmetric with trace $S^\theta = 0$, therefore the integrability conditions of Gauß and Codazzi are fulfilled for the pair (g^θ, S^θ) .

From this an important property follows for a geodesic γ on a minimal surface in $M^3(c)$:

$$F(\gamma) \text{ is a straight line} \iff F^{\pi/2}(\gamma) \text{ is a planar geodesic}$$

and the torsion of $F(\gamma)$ is equal to the curvature of $F^{\pi/2}$. This means that symmetry properties of a minimal surface F imply corresponding symmetry properties on the conjugate immersion $F^{\pi/2}$.

3 The Construction Procedure.

We will now describe the conjugate surface construction. At first we start with a tessellation of $M^3(c)$ with Coxeter orthoschemes. A Coxeter orthoscheme is a tetrahedron whose vertices P_1, P_2, P_3, P_4 have the property that $\text{span}(P_1, P_i) \perp \text{span}(P_i, P_4)$ for $i \in \{2, 3\}$ and whose other three dihedral angles are of the form $\pi/p, \pi/q, \pi/r$, with $p, q, r \in \mathbb{N}$.

Every combination (p, q, r) with

$$\sin \frac{\pi}{p} \sin \frac{\pi}{r} < \cos \frac{\pi}{q} \tag{3.1}$$

defines a Coxeter orthoscheme in H^3 . They all tessellate H^3 because they have natural dihedral angles of the form π/k .

Some tessellations with Coxeter orthoschemes include as subgroups the tessellation of H^3 by regular polyhedra, namely the platonic solids, since each regular polyhedron with the Schläfli symbol $\{p, q, r\}$ is divided by its symmetry group in congruent Coxeter orthoschemes (p, q, r) . The other tessellations contain Coxeter orthoschemes with vertices lying in or beyond infinity.

We now construct a minimal surface patch in a Coxeter orthoscheme meeting the faces by four planar symmetry lines as in figure 1. Then the symmetry group of the tetrahedron would build up a complete periodic minimal surface. The underlying method was first used by Smyth [9] to prove existence of three such patches in every euclidean tetrahedron.

To every such patch in a tetrahedron exists its conjugate surface which is bounded by a quadrilateral of four straight lines, and vice versa to every patch in a quadrilateral exists a

patch in a tetrahedron. So once we know the quadrilateral we solve the Plateau problem in this contour, conjugate, and have existence of the required patch in the tetrahedron.

To obtain the quadrilateral is straight forward in \mathbb{R}^3 : both patches are isometric and therefore the angles in corresponding vertices are the same. Additionally we have in that normals in corresponding points are parallel, especially the normals at all vertices of a quadrilateral are determined. In the case of four boundary arcs the angle condition and the normal parallelity, whose values can be read off the tetrahedron as dihedral angles and edge directions, uniquely determine the quadrilateral except of similiarity transformations. So the conjugate surface construction in \mathbb{R}^3 works as follows:

- given a tetrahedron
- angles of quadrilateral = dihedral angles of tetrahedron
- normals of quadrilateral = edge directions of tetrahedron
- quadrilateral is determined
- solve Plateau problem in the quadrilateral
- conjugate the solution.

This gives the solution to our free boundary value problem in the euclidean tetrahedron. The patch is usually unstable.

Karcher, Pinkall and Sterling [3] extended this construction to S^3 and proved existence of a minimal surface corresponding to each of the nine tesselations of S^3 into regular polyhedrons. We now prove the corresponding result in H^3 .

For the construction of the patch we proceed exactly as in the euclidean case with one major exception: in spaces of non zero curvature we have no longer global parallelity. That means we cannot speak any longer of parallel normals, and therefore we need other arguments to prove existence of a right quadrilateral.

Let us fix a hyperbolic Coxeter orthoscheme (p, q, r) with characteristic angles $\beta_1 := \pi/r$, $\beta_2 := \pi/p$, $\eta := \pi/q$. The patch we are looking for has angles $\pi/2, \pi/2, \pi/2$ and η and therefore the quadrilateral too. Such quadrilaterals are uniquely determined for example by the angles a_1, a_2 as in figure 1, such that we have a 2-parameter family of candidates.

Given one of these candidate quadrilaterals then its Plateau solution leads via conjugation to a patch in a tetrahedron, which differs at most in the two angles β_1 and β_2 . So we have a map

$$Q(a_1, a_2, \eta) \longrightarrow (\tilde{\beta}_2, \eta, \tilde{\beta}_1)$$

from the set of quadrilaterals to the set of tetrahedrons, see figure 1. Since η was fixed this map is equivalent to a function

$$F_\eta(a_1, a_2) := (\tilde{\beta}_1, \tilde{\beta}_2).$$

The task now is to prove that the pair of angles (β_1, β_2) of the given Coxeter orthoscheme is in the image of F_η . Having shown continuity of F_η we proceed with a homotopy argument. We construct a closed contractible curve γ in the domain of F_η , whose image $(\tilde{\beta}_1(\gamma), \tilde{\beta}_2(\gamma))$ runs around the given pair (β_1, β_2) . Then contracting the curve will prove that (β_1, β_2) is in the image of F_η , and therefore prove the existence of a quadrilateral.

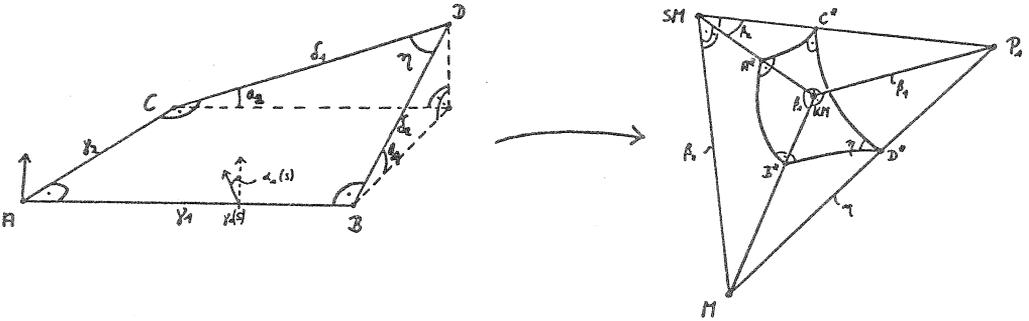


Figure 1: Isometry between the minimal patch in the quadrilateral and the minimal patch inside the tetrahedron.

The construction of the contractible curve uses further control over F_η . This is obtained by estimates of the following type:

- the torsion of a boundary line of the minimal surface in the quadrilateral is equal to the curvature of the corresponding planar conjugate arc. There is no direct control of the torsion or curvature, but we can estimate the turning angle of the normal compared to a parallel normal field along this straight boundary line of the quadrilateral using hyperbolic helicoids and bilinear interpolation surfaces as barriers. With this turning angles we control the corresponding planar curves in the tetrahedron and its dihedral angles, i.e. the image of F_η .

- the situation for infinitesimal small patches is comparable with the euclidean case, where F_η is the identity. In practice we have for some quadrilaterals that the corresponding tetrahedron has edges beyond infinity, such that the angles $\tilde{\beta}_1, \tilde{\beta}_2$ are no longer defined, even if (β_1, β_2) are well defined (this is the case when faces of the tetrahedron do not meet). To

cover these situation we do not consider the angles $\tilde{\beta}_1, \tilde{\beta}_2$ but consider instead their cosine. The image of the resulting new function F_η can be continuously extended to handle vertices beyond infinity (infinity means $\beta = 0$ or $\cos \beta = 1$) by defining F_η to be the hyperbolic cosine \cosh of the distance h of two sides of the tetrahedron. For $\beta = 0$ we have $h = 0$ and therefore $\cos \beta = 1 = \cosh h$. On the other side this enables us to prove existence of patches for tetrahedra with edges beyond infinity.

The appearance of this situation distinguishes the hyperbolic case from the spherical case.

Let (p, q, r) denote a hyperbolic Coxeter orthoscheme, i.e. p, q, r fulfill equation 3.1, then we have the following existence theorem:

Theorem: *There exist complete minimal surfaces in H^3 with the symmetry of tessellations given by*

- a) *all compact and non compact platonic polyhedra (see the list below)*
- b) *all Coxeter orthoschemes (p, q, r) with $q \in \{3, 4, \dots, 1000\}$ and small p and r .*
- c) *all "selfdual" Coxeter orthoschemes (p, q, r) with $p = r$, and additionally all rotational symmetric Coxeter orthoschemes with all four vertices in or beyond infinity (their Coxeter graph is $0 \dots 0^{20} \dots 0$, they have no (p, q, r) representation).*

The selfdual tessellations in c) lead to two different complete minimal surfaces of Schwarz and Neovius type which are both embedded.

Proof: a) + b): The proof of a) and b) uses the conjugate surface construction in H^3 introduced above. The numerical values in b) were obtained by numerical estimates of an explicitly given function and computed only for these values of q .

c) These Coxeter orthoschemes have an axis of rotational symmetry. Neighbouring axis can be combined in two different ways to polygons such that their Plateau solution extend to complete minimal surfaces. Both surfaces are embedded since their Plateau patch is a hyperbolic graph. If the orthoschemes also generate a tessellation by regular polyhedra (selfdual examples of the list below) then the two minimal surfaces sit inside the polyhedron in the same way as the Schwarz and Neovius surface sit inside the cube.

List of compact and non-compact regular polyhedra in H^3

compact polyhedra

{3,5,3} selfdual	120° icosahedron	fig. 4c
{5,3,5} selfdual	72° dodecahedron	fig. 2c,2d
{5,3,4} dual	90° dodecahedron	

$\{4,3,5\}$ dual 72° cube fig. 3b-d

polyhedra with all vertices at infinity with their duals

$\{3,4,4\}$ $\{4,4,3\}$ 90° octahedron fig. 4a,4b

$\{3,3,6\}$ $\{6,3,3\}$ 60° tetrahedron

$\{4,3,6\}$ $\{6,3,4\}$ 60° cube fig. 3a

$\{5,3,6\}$ $\{6,3,5\}$ 60° dodecahedron

(the vertices of their duals lie on a horosphere, see figure 4b)

"polyhedra" with vertices and centers at infinity

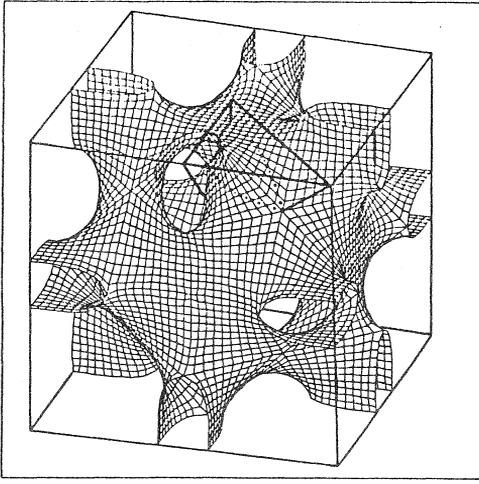
$\{6,3,6\}$ selfdual

$\{4,4,4\}$ selfdual

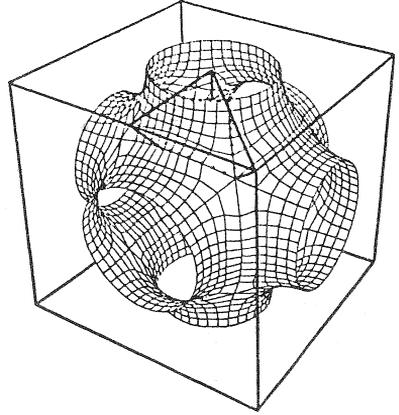
$\{3,6,3\}$ selfdual

4 Literature

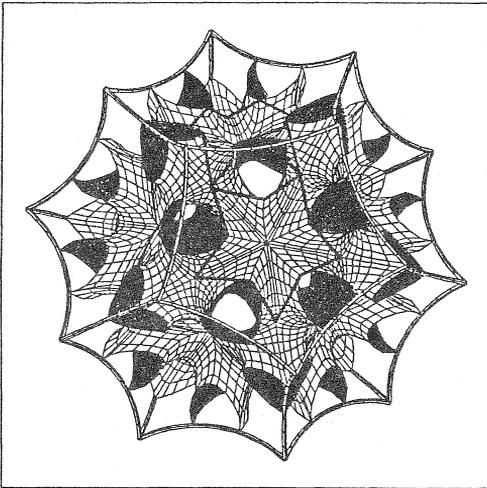
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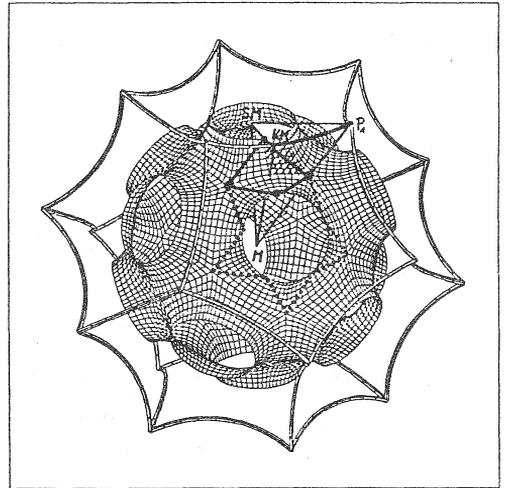
(a)



(b)

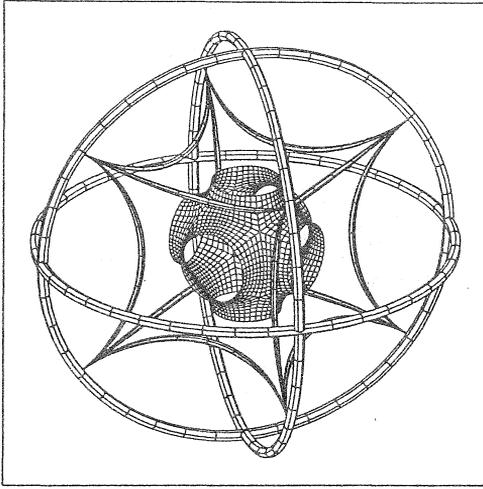


(c)

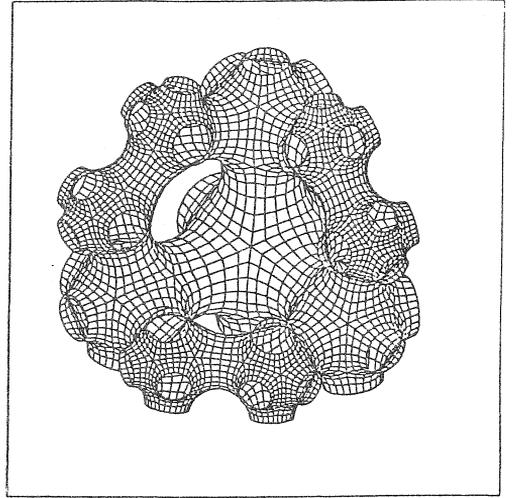


(d)

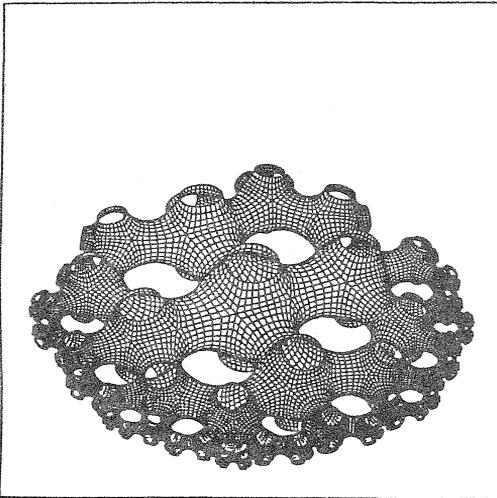
Figure 2: The classical triply periodic minimal surfaces in of E.R Neovius (a) and H.A. Schwarz (b) with a building block in a cube, and corresponding minimal surfaces in H^3 : the selfdual hyperbolic 72° dodecahedron contains a cell of a Neovius type surface with holes to each edge, and a cell of a Schwarz type surface with holes to each face. Both cells extend via successive reflections in the faces of the dodecahedron to complete embedded minimal surfaces in H^3 .



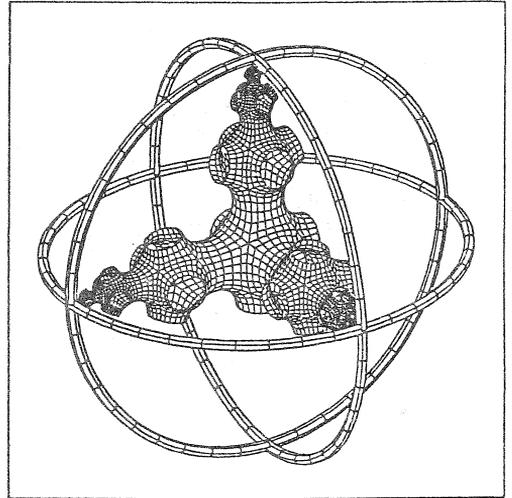
(a)



(b)



(c)



(d)

Figure 3: Figures (a), (b) and (c) show different parts of complete minimal surface with the symmetry of a tessellation of H^3 by 72° cubes. The minimal surfaces in (d) sits in a 60° cube with vertices in infinity.

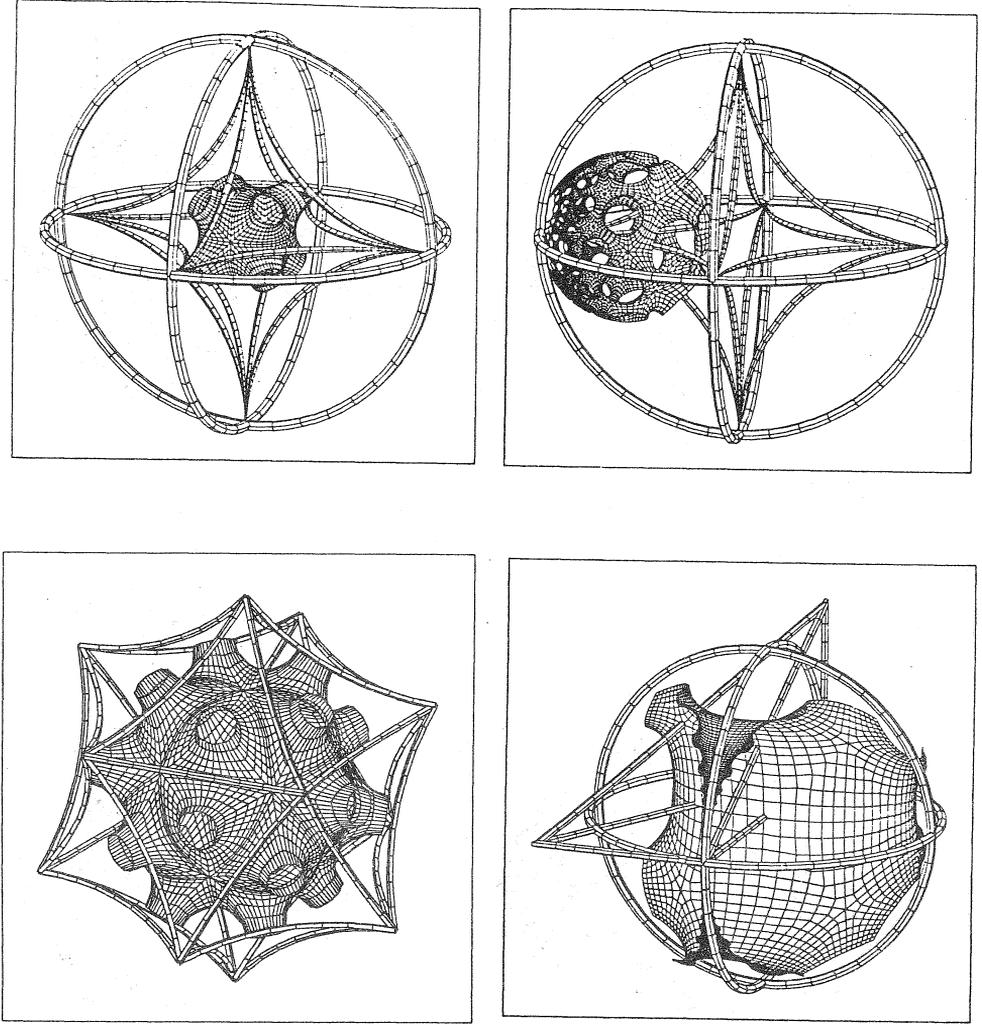


Figure 4: Minimal surface in a 90° octahedron (a) and in its dual cell (b). The dual cell has its midpoint in infinity, and its vertices lie on a horosphere. (c) shows an icosahedron with a minimal surface, and (d) a Coxeter orthoscheme with two vertices beyond and two in infinity. It has an axis of rotational symmetry, dihedral angles $\pi/2, \pi/2, \pi/2, \pi/q$, and a Coxeter graph $0 \dots 0^2 0 \dots 0$