SOLITON GEOMETRY, KAC-MOODY ALGEBRAS AND THE YANG-BAXTER EQUATIONS

Ivan Sterling

1.INTRODUCTION AND EXAMPLES

This note is based on joint work with D.Ferus, F. Pedit and U.Pinkall [1].

Soliton geometry has been successfully applied to the study of linear Weingarten surfaces (e.g. $H=\frac{1}{2}$, $K=-1$, smoke rings (tubular surfaces), minimal, flat) in three dimensional Riemannian and Lorentzian space forms, minimal surfaces in S^4, minimal surfaces in the Lorentzian S^4 (which are almost in 1:1 correspondence with Willmore surfaces), harmonic maps in Lie groups (and symmetric spaces), isometric imbeddings $M^n(k) \to M^{2n-1}(k')$, etc.

Since we will be discussing a quite general method which applies to all the above examples it is important to also consider a specific case in order to get a feel for the difficulties involved in "setting the dials on the machine". The case we'll consider is that of minimal surfaces in S^4. For the sake of completeness we'll first review the classical cases to show how soliton PDE's arise natually in geometry.

2. NICE SURFACES COME FROM SOLITON EQUATIONS

Let H and K be the mean and Gauss curvatures.

If $f:\mathbb{R}^2 \to \mathbb{R}^3$ has $K=-1$ and is parametrized by asymptotic lines with $|f_x| = |f_y| = 1$ and $\omega(x,y)$ is the angle between the asymptotic lines (i.e. arccos $<f_x,f_y>$), then ω satisfies the sine-Gordon equation $\omega_{xy}= \sin \omega$.

If $f:\mathbb{R}^2 \to \mathbb{R}^3$ has $H=\frac{1}{2}$ and is parametrized by curvature lines with $<f_z,f_z> = 0$ (i.e. f is conformal) and $\omega(x,y)$ is $\log|f_x|$, the ω satisfies the sinh-Gordon equation $\omega_{z\bar{z}} + \frac{1}{2}\sinh(2\omega) = 0$.

Conversely every solution to the sine or sinh-Gordon equation completely determines an associated family of K or H-surfaces.

Similarly Willmore surfaces (critical points of the integral of H^2) give rise to a Toda system and smoke rings give rise to the non-linear Schrödinger equation.

3. PICTURES

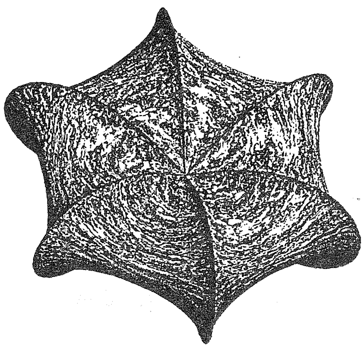

Figure 1

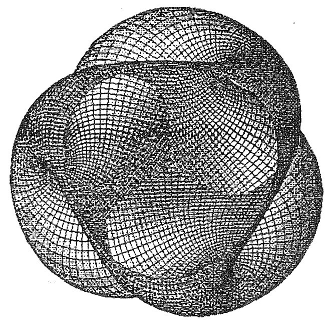

Figure 2

Figure 1 is a K-torus, computed by Melko and the author, due to Enneper. Hilbert proved there are no complete non-singular K-surfaces in $\mathbb{R}^3$. An important open problem is if there are any complete isometric immersions of $\mathbb{H}^3$ in $\mathbb{R}^5$. Figure 2 is one of Bryant's Willmore projetive planes. All Willmore S^2's and $\mathbb{R}\mathbb{P}^2$'s were classified by Bryant and are given by a Weierstraß formula. This $\mathbb{R}\mathbb{P}^2$ has the smallest possible Willmore integral among $\mathbb{R}\mathbb{P}^2$'s. Figure 3 is the famous Clifford torus. It is a long standing conjecture of Willmore that the Clifford torus minimizes the Willmore integral among tori. Figure 4 is one of Lawson's Willmore surfaces. This one, of genus 2, is the first in a sequence of peanut shaped Lawson surfaces of genus g. Kusner conjectures that the Lawson surfaces minimize the Willmore integral among surfaces of some fixed genus g. Figure 5 is a Platonic Willmore surface of genus three found by Karcher, Pinkall and the author. It has tetrahedral symmetry and is the first in a sequence of Platonic Willmore surfaces. So far these surfaces which are

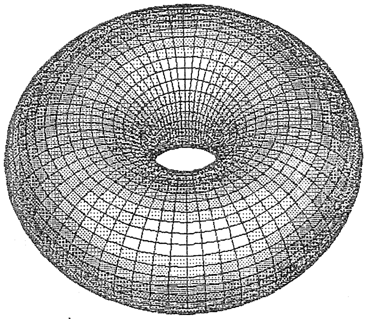

Figure 3

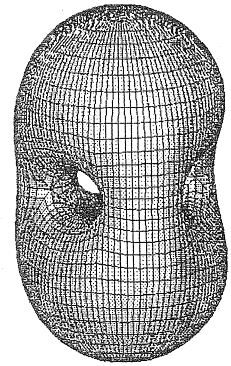

Figure 4

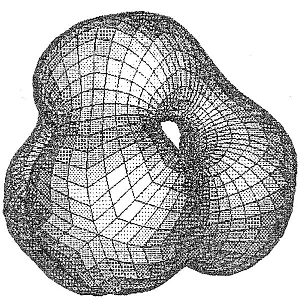

Figure 5

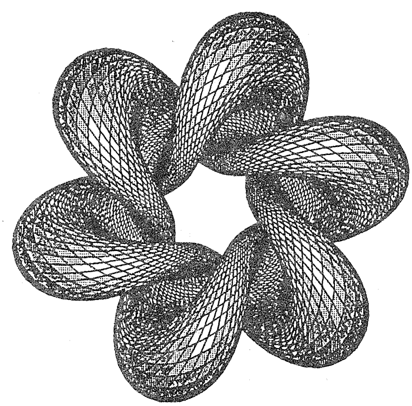

Figure 6

computed using a trick of Smyth´s by solving a Plateau problem cannot yet be found using the soliton methods discussed here. We will be discussing surfaces like that shown in Figure 6. This particular torus is due to Ferus and Pedit. It is especially nice because it is embedded. The ODE´s discussed here are in fact very explicit, can be plugged into a computer, solved using say Runga-Kutta and give nice pictures.

4. MAIN THEOREM

MAIN THEOREM. (Ferus,Pedit,Pinkall & Sterling) *Classification of minimal tori in* $S^4(1)$ *and Willmore tori (of finite type) in* R^3.

The classication is divided roughly into three parts:

(i) a recipe which constructs minimal surfaces,

(ii) showing that all minimal tori come from the recipe,

(iii) telling when the recipe yields tori.

Parts (ii) and (iii) will not be discussed here. In general the recipe produces quasi-periodic minimal surfaces and one needs to study the period matrix and Jacobian of the spectral curve associated to the surface in order to control periodicity properties. Part (ii) amonts to finding an explicit sequence of solutions to the Jacobi equation, an elliptic system of PDE´s on the surface. In the case of a compact torus, we can have only finitely many linearly independent solutions (in general if this is true we say the surface is of finite type) and this turns out to imply that the surface comes from the recipe.

5. MINIMAL SURFACES IN $S^4(1)$

Let $f: \mathbb{R}^2 \to S^4(1) \subset \mathbb{R}^5$. Then f is conformal if and only if $<f_z,f_z> = 0$ and a conformal f is minimal if and only if $f_{z\bar{z}} = 0$.

Definition We say $F(x,y) \in SO(5)$ is an adapted moving frame of f if $f=Fe_0$ and Fe_1, Fe_2 span the tangent space to f where $e_0,\dots,e_4$ is the canonical basis of $\mathbb{R}^5$.

This gives the Frenet equations: $F_z = FA$, $F_{\bar{z}} = F\bar{A}$, $A: \mathbb{R}^2 \to so(5,\mathbb{C})$.

And the integrability conditions (the Maurer-Cartan equations): $A_{\bar{z}} - \bar{A}_z = [A,\bar{A}]$.

In order to translate our geometric problem into an algebraic one we must sort out what is special about the A for an adapted moving frame of a minimal f. In order to do this we need to decompose $so(5,\mathbb{C})$ as follows: Let

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in SO(5) \text{ , } Q^4 = I,$$

$$M_k = \{Y \in SO(5,\mathbb{C}) \mid QYQ^{-1} = i^k Y\},\ k = 0,\dots,3.$$

The M_k´s are the eigenspaces of AdQ with eigenvalues i^k. For example:

$$M_0 = \left\{ \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & y & 0 \end{array} \right) \middle| \; x,y \in \mathbb{C} \right\}.$$

PROPOSITION 1. *F is an adapted moving frame of a minimal f if and only if*

$$A: \mathbb{R}^2 \longrightarrow M_0 \oplus M_1.$$

Definition $A: \mathbb{R}^2 \longrightarrow so(5,\mathbb{C})$ is called admissible if

$$A = A_0 + A_1 \,,\; A_k: \mathbb{R}^2 \longrightarrow M_k,$$

$$A_{\bar{z}} - \bar{A}_z = [A, \bar{A}]\,.$$

Given an admissible A we solve the Frenet equations for F and the first component $f = Fe_0$ will be a minimal surface. Furthermore we have

PROPOSITION 2. (Enneper´s associated family of minimal surfaces) *If A is admissable, then so is*

$$A(\lambda) = A_0 + \lambda A_1 \,,\; \lambda \in \mathbb{C},\; |\lambda| = 1.$$

We consider this entire associated family as a single object. A loop in $so(5,\mathbb{C})$.

6. LOOP ALGEBRAS

Let $\mathfrak{g} = so(5,\mathbb{C})$.

Let $\tilde{\mathfrak{g}} = \{\lambda \longrightarrow X(\lambda) = \sum_{k=-\delta}^{\delta} \lambda^k X_k \mid X_k \in \mathfrak{g}\}$.

Let $\tilde{\mathfrak{g}}_Q = \{X \in \tilde{\mathfrak{g}} \mid X_k \in M_k\}$.

And let $\tilde{\mathfrak{g}}_Q^{\mathbb{R}} = \{X \in \tilde{\mathfrak{g}} \mid X_{-k} = \bar{X}_k\}$.

We can now give our recipe to construct minimal f in $S^4(1)$ by ODE´s.

THEOREM 1. *Choose any* $X_0 = \sum_{k=-\delta}^{\delta} \lambda^k X_k \in \tilde{\mathfrak{g}}_Q^{\mathbb{R}}$ *with* $\delta \equiv 1 \bmod 4$, *then*

(a) There is a unique $X: \mathbb{R}^2 \longrightarrow \tilde{\mathfrak{g}}_Q^{\mathbb{R}}$ *such that*

$$X_z = [X, i(\tfrac{1}{2}X_{\delta-1} + \lambda X_\delta)]$$

$$X(0,0) = X_0.$$

(b) $A = i(\tfrac{1}{2}X_{\delta-1} + \lambda X_\delta)$ *satisfies* $A_{\bar{z}} - \bar{A}_z = [A, \bar{A}]$.

7. R-MATRICES

Theorem 1 is a special case of a general proposition (given below). We need to introduce two new ideas, R-matrices (i.e. solutions to the Yang-Baxter's equation) and ad-invariant vector fields.

Let $\mathfrak{g}$ be any Lie algebra.

Definition A linear map $R:\mathfrak{g} \to \mathfrak{g}$ is called an R-matix if for some $\alpha \in \mathbb{C}$

$$R([RX,Y] + [X,RY]) - [RX,RY] = \alpha[X,Y] \quad \text{for all } X,Y \in \mathfrak{g}.$$

Example $\mathfrak{g} = \tilde{\mathfrak{g}}_Q$, $X = \Sigma\, \lambda^k X_k$, $RX = \frac{i}{2} \Sigma\, (\text{sign } k)\, \lambda^k X_k$.

Note $i(\frac{1}{2}X_{\delta-1} + \lambda X_\delta) = (R + \frac{i}{2})(\lambda^{1-\delta}X)$.

Definition A vector field J on $\mathfrak{g}$ is called ad-invariant if (D is the directional derivative)

$$(D_{[X,Y]}J)_X = [J_X,Y] \text{ for all } X,Y \in \mathfrak{g}.$$

Example $J_X = \lambda^{1-\delta}X$.

MAIN PROPOSITION. *If $R:\mathfrak{g} \to \mathfrak{g}$ is an R-matrix, $J,\tilde{J}$ ad-invariant and $\mu,\tilde{\mu} \in \mathbb{C}$, then*

(a) The system

$$X_s = [X, (R + \mu)J_X]$$
$$X_t = [X, (R + \tilde{\mu})\tilde{J}_X]$$

is integrable ($X_{st} = X_{ts}$).

(b) $A = (R + \mu)J_X$, $\tilde{A} = (R + \tilde{\mu})\tilde{J}_X$ *satisfy* $A_{\bar{z}} - \tilde{A}_z = [A,\tilde{A}]$.

By plugging our examples for R and J_X into the Proposition one obtains Theorem 1. Thus Theorem 1 is proved as soon as the Proposition is. The proof of the Proposition is a direct calculation which falls out from the definitions.

REFERENCE

[1] D. Ferus, F. Pedit, U. Pinkall and I. Sterling, Minimal tori in S^4. Preprint No. 261, TU Berlin (1990).

Technische Universität Berlin
Fachbereich Mathematik
Straße des 17.Juni 135
D-1000 Berlin 12
Germany