PLATEAU'S PROBLEM FOR MINIMAL SURFACES WITH A CATENOIDAL END

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In a preceding paper R. Ye and the author have shown that every rectifiable Jordan curve in $\mathbb{R}^3$ bounds a minimal immersion of the punctured disc which stretches out to infinity and has a flat end, i.e. outside some ball the surface is the graph of a bounded function defined on some exterior domain in the asymptotic tangent plane [TY]. In the present paper we are concerned with the corresponding problem for minimal surfaces with a catenoidal end, which means that at infinity the surface is required to be the graph of a logarithmically growing function and therefore ressembles the shape of a half catenoid.

We introduce the following notation: C denotes the standard catenoid in $\mathbb{R}^3$, i.e.

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \sqrt{x_1^2 + x_2^2} = \cos h x_3\},$$

and T the closure of the non-simply connected component of $\mathbb{R}^3 \backslash C$; by $(\cos h)^{-1}$ we mean the positive branch of the inverse to $\cos h$, i.e. $(\cos h)^{-1}(\rho) = \ell n(\rho + \sqrt{\rho^2 - 1})$.

Setting $D_r := \{z \in \mathbb{C} || z| \leq r\}$ we state our result in the following.

Theorem. Let Γ be a rectifiable Jordan curve contained in T which generates $\pi_1(T)$. Then, for every $\lambda \in [-1, +1]$ there exists $\mu > 0$ and a conformal minimal immersion $u : D_1^0 \backslash \{0\} \to \mathbb{R}^3$ such that

(i) u extends continuously to $D_1 \backslash \{0\}$ and u maps ∂D_1 onto Γ topologically,

(ii) $|u(z)| \to +\infty (z \to 0)$,

(iii) for some $r > 0$, $u | D_r \backslash \{0\}$ is embedded and $u(D_r \backslash \{0\})$ is the graph of a function φ which is defined on an exterior domain in the (x_1, x_2)-plane and satisfies

$$(1) \qquad \begin{aligned} |\varphi(x_1, x_2)| &\leq \mu \text{ if } \lambda = 0, \\ |\varphi(x_1, x_2) - \lambda(\cos h)^{-1}(\rho / |\lambda|)| &\leq \mu \text{ if } \lambda \neq 0, \end{aligned}$$

where $\rho = \sqrt{x_1^2 + x_2^2}$.

It would be interesting to study the corresponding existence problem in the class of graphs, i.e. to solve the exterior Dirichlet problem for the minimal surface equation. Some new results in this direction have been proved by Krust [K].

The proof of the above theorem is based on the same principle as in the flat case: the solution is obtained as the limit of a sequence of expanding minimal annuli spanned by

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Γ and certain round circles Γ_R which this time, however, are not contained in a fixed plane but instead lie on suitable catenoids. The crucial area estimates (see Lemma 4 below) rely on the classical theory of field embeddings. Barrier arguments will play an important rôle in this paper.

The case $\lambda = 0$ in the theorem corrsponds to a flat end and therefore is a special case of the result in [TY]. By reflection across the plane $x_3 = 0$ the case of negative λ may be reduced to $\lambda > 0$ and it is therefore enough to consider this case. We set $A_r := D_r \backslash D_1^0$ and start with

Lemma 1. Let $\Gamma' \subset T$ be a further rectifiable Jordan curve disjoint from Γ and such that Γ and Γ' are homotopic in T. Then there exists $r > 1$ and a conformal minimal immersion $u : A_r^0 \to \mathbb{R}^3$ such that u extends continuously to A_r, $u(A_r) \subset T$, and $u/\partial D_1$ and $u/\partial D_r$ are one-to-one mappings onto Γ and Γ' respectively. Moreover, u has minimal area amongst all mappings $\widetilde{u} : A_r \to T$ satisfying the same boundary conditions.

Proof. Since $\pi_1(T) = \pi_1(A_r)$, Γ generates $\pi_1(T)$ by hypothesis, and $\partial T = C$ is mean convex, the existence of a minimizing conformal branched annulus u spanning Γ and Γ' results as an application of the concept of incompressible surfaces, cf. [TT]. It follows from [O] and [GT] that u is immersed in the interior. □

We need some further notation:

$$\rho(x) := \sqrt{x_1^2 + x_2^2}, Z_R = \{x \in \mathbb{R}^3 | \rho(x) \leq R\},$$

$$C^+ := \{x \in C | x_3 > 0\},$$

$$C^+_{\lambda\mu} := \lambda C^+ + (0,0,\mu) \text{ where } 0 \leq \lambda \leq 1) \text{ and } \mu \in \mathbb{R},$$

$$T_{\lambda\mu} := \{x \in \mathbb{R}^3 | \rho(x) > 1, |x_3 - \lambda(\cosh)^{-1}(\rho(x)/\lambda)| \leq \mu\}.$$

We now choose $R_0 > 1$ such that $\Gamma \subset \mathring{Z}_{R_0}$ and define

$$\mu := 2(\cosh)^{-1}(R_0), \Gamma_R := C^+_{\lambda,-\mu} \cap \partial Z_R.$$

We may clearly find a number $R_1 \geq R_0$ such that $\Gamma_R \subset T$ for $R \geq R_1$. We then denote by $u_R : A_{r(R)} \to T$ some minimal annulus spanning Γ and Γ_R the existence of which is guaranteed by Lemma 1.

Lemma 2. $u_R(A_{r(R)}) \subset T_{\lambda\mu}$.

Proof. We consider the family of half catenoids $C^+_{\lambda\nu}$, $\nu \in \mathbb{R}$, which foliates $\mathbb{R}^3 \backslash Z_1$. Clearly $u_R(A_{r(R)}) \cap C^+_{\lambda\nu} = \emptyset$ if ν is suffienciently negative. Let us now increase the value of ν until $C^+_{\lambda\nu}$ contacts $u_R(A_{r(R)})$ for the first time, say at $\nu = \nu_0$. Since $\Gamma_R \subset C^+_{\lambda,-\mu}$ we must have $\nu_0 \leq -\mu$ and by the choice of μ we have $\Gamma \cap C^+_{\lambda\nu_0} = \emptyset$. Hence if $\nu_0 < -\mu$ then the surface u_R would lie on one side of $C^+_{\lambda\nu_0}$ and would contact $C^+_{\lambda\nu_0}$ at an interior point, contradicting the maximum principle. It follows that $\nu_0 = -\mu$ and hence the image of u_R lies above $C^+_{\lambda,-\mu}$. By a similar argument one shows that $u_R(A_{r(R)})$ lies below $C^+_{\lambda\mu}$. □

Let us now consider an open set $U \subset \mathbb{R}^3$ which is foliated by a smooth family S_μ of embedded minimal surfaces and let N be a smooth unit length vectorfield in U such that $N(p)$ is a normal vector to S_μ if $p \in S_\mu$. Let us pick some specific surface S_μ from the given family and let us assume that the trajectories of N starting from any point in some subregion V of U reach S_μ after finite time. Then there is a well defined, smooth projection map

$$\Pi_\mu : V \to S_\mu$$

which assigns to each point $p \in V$ the first intersection point of the trajectory starting from p with S_μ. It follows from the classical theory of field embeddings that Π_μ is area decreasing. In our special case this property of Π_μ results from Stokes's theorem and the observation that div $N = 0$. We shall now apply these considerations to the family $C^+_{\lambda\nu}, \nu \in \mathbb{R}$, in the open set $U = \mathbb{R}^3 \backslash Z_1$. In cylindrical coordinates

$$x_1 = \rho\cos\Theta, x_2 = \rho\sin\Theta, x_3 = z$$

the vector field N is given by

$$N(\rho, \Theta, z) = (\lambda/\rho, 0, -\frac{1}{\rho}\sqrt{\rho^2 - \lambda^2}).$$

We have

Lemma 3. The trajectory of N starting from a point in $T_{\lambda\mu}$ reaches $C^+_{\lambda,-\mu}$ before time $1 + 2\sqrt{2}\mu$. Hence the corresponding projection

$$\Pi_{-\mu} : T_{\lambda\mu} \to C^+_{\lambda,-\mu}$$

is well defined and smooth. Moreover, the estimate

$$\rho(\Pi_{-\mu}(p)) \le \rho(p) + \rho_0 \tag{2}$$

holds for every $p \in T_{\lambda\mu}$, where ρ_0 only depends on μ.

Proof. In cylindrical coordinates the trajectories satisfy

$$\dot\rho = \lambda/\rho,\ \dot\Theta = 0,\ \dot z = -\sqrt{1 - (\lambda/\rho)^2}.$$

By direct integration we get $\rho(t) = \sqrt{\rho(0)^2 + 2\lambda t}$, in particular, ρ is increasing and $\rho(t) \ge 1$ for any initial point $p \in T_{\lambda\mu}$. If therefore $0 < \lambda \le \frac{1}{2}$ we obtain

$$z(t) - z(0) < -\int_0^t \sqrt{3/4}ds \le -2\mu \text{ for } t \ge 4\mu/\sqrt{3}. \tag{3}$$

If $\frac{1}{2} < \lambda \le 1$ then $\rho(1) > \sqrt{2}$ and hence

$$z(t) - z)0) < -\int_1^t \sqrt{1/2}ds \le -2\mu \text{ for } t \ge 1 + 2\sqrt{2}\mu. \tag{4}$$

Combining (3) and (4) it follows that the trajectory starting from p after time $1+2\sqrt{2}\mu$ has reached a position below $C^+_{\lambda,-\mu}$ and hence has hit $C^+_{\lambda,-\mu}$ before that time. It follows finally that

$$\rho(\Pi_{-\mu}(p)) \le \sqrt{\rho(p)^2 + 2\lambda(1+2\sqrt{2}\mu)} \le \rho(p) + 3\sqrt{\mu} + \sqrt{2},$$

proving (2) and the lemma. □

We are now ready to establish the local area estimates analogous to Lemma 2.2 in [TY], where however the linear orthogonal projection onto the x_1, x_2-plane has to be replaced by our $\Pi_{-\mu}$. Let us remark that the linear projection corresponds to the foliation by parallel planes. We define

$$a(\rho) := \text{ area } (C^+_{\lambda,0} \cap Z_\rho), \rho > 1.$$

By enlargening the constant R_1 chosen after Lemma 1 we may assume that

$$\Pi_{-\mu}(\Gamma) \subset Z_{R_1}. \tag{5}$$

With the notation

$$\Sigma(u,\rho) = (\Pi_{-\mu} \circ u)^{-1}(Z_\rho),$$

where u is any mapping into $T_{\lambda\mu}$, we then have

Lemma 4. (i) area $(u_R) \le a_1 + a(R) - a(R_1)$ for $R \ge R_1$, where $a_1 :=$ area (u_{R_1}).

(ii) area $\big(u_R|\Sigma(u_R,\rho_2)\backslash\Sigma(u_R,\rho_1)\big) \ge a(\rho_2) - a(\rho_1)$ for $R_1 \le \rho_1 < \rho_2 \le R$.

(iii) area $\big(u_R|\Sigma(u_R,\rho)\big) \le a(\rho) - a(R_1) + a_1$ for $R_1 \le \rho \le R$.

Proof. (i) We may join the surfaces u_{R_1} and $C^+_{\lambda,-\mu} \cap (Z_r\backslash Z_{R_1})$ along their common boundary component Γ_{R_1} to obtain an annular type surface v_R in T spanning Γ and Γ_R. By the minimality of u_R (Lemma 1) we have area $(u_R) \le$ area $(v_R) = a_1 + a(R) - a(R_1)$.

(ii) The map $\Pi_{-\mu} \circ u_R$ is a homeomorphism from $\partial D_{r(R)}$ onto Γ_R and we conclude, also using (5), that the topological degree of $\Pi_{-\mu} \circ u_R$ with respect to any point $p \in C^+_{\lambda,-\mu} \cap (Z_R\backslash Z_{R_1})$ must be ± 1. Hence $\Pi_{-\mu} \circ u_R$ covers $Z_R\backslash Z_{R_1}$ and by the area decreasing property of $\Pi_{-\mu}$ the assertion (ii) follows. Setting $\rho_1 = \rho$ and $\rho_2 = R$ in (ii) we obtain (iii) from (i) and (ii) by subtraction. □

Once the uniform local area estimates for the surfaces u_R are established we find it technically convenient to replace $\Pi_{-\mu}$ again by the linear orthogonal projection P from $\mathbb{R}^3$ onto the (x_1, x_2)-plane, in particular, since this allows a direct application of the results of [TY]. In accordance with the notations of the latter paper we introduce the sets

$$\Omega(u_R,\rho) = (P \circ u_R)^{-1}(Z_\rho),$$

$$A(u_R,\rho) = \text{ the connected component of } \Omega(u_R,\rho) \text{ containing } \partial D_1.$$

It follows from Lemma 3, (2), that

$$\Omega(u_R,\rho) \subset \Sigma(u_R, \rho + \rho_0)$$

and hence we obtain from Lemma 4, (iii) the estimate

(6) $$\text{area }\left(u_R|\Omega(u_R,\rho)\right) \leq a(\rho+\rho_0)+a_1-a(R_1)$$

for $R_1 \leq \rho \leq R$. By direct calculation one finds $a(\rho) = \pi\left(R\sqrt{R^2-\lambda^2}+\lambda^2\ell n\frac{1}{\lambda}(R+\sqrt{R^2-\lambda^2})\right)$. In particular, there is a constant c_0 such that

(7) $$\text{area }\left(u_R|\Omega(u_R,\rho)\right) \leq c_0\rho^2 \text{ for } R_1 \leq \rho \leq R.$$

Let us now introduce Dirichtlet's energy

$$E(u) = \frac{1}{2}\int\int |\nabla u|^2$$

and remark that for conformal mappings like u_R Dirichlet energy and area coincide. We also remark that by the maximum principle $A(u_R,\rho)$ is an annular region with two boundary components, one of them of course ∂D_1. For abbreviation we define

$$\partial^* A(u_R,\rho) = \partial A(u_R,\rho)\backslash\partial D_1.$$

On the basis of the estimate (7) the proof of the next lemma is identical with that for Lemma 2.2 and Lemma 2.3 in [TY] and may therefore be omitted.

Lemma 5. There is a positive constant c such that

(i) dist $\left(\partial^* A(u_R,\rho),\partial^* A(u_R,2\rho)\right) \geq c$ for $R_1 \leq \rho \leq \frac{1}{2}R$,

(ii) dist $\left(\partial D_1,\partial^* A(u_R,\rho)\right) \geq c\ell n\rho$ for $2R_1 \leq \rho \leq R$,

(iii) $E(u_R|A_r) \leq c_0 \exp{(r/c)}$.

On the basis of the last lemma we can now perform the limit $R \to +\infty$. The arguments are the same as those in Proposition 2.1 in [TY] except that the "condition of cohesion" used in the proof of the equicontinuity of $u_R|\partial D_1$ is in the present situation a direct consequence of the incompressibility of the surfaces u_R in T. We formulate the result as

Proposition. For any sequence $R_k \to +\infty$ there is a subsequence of u_{R_k}, denoted by v_k, and a conformal, harmonic map $u \in C^0(A,\mathrm{IR}^3)\cap C^\infty(\mathring{A},\mathrm{IR}^3)$, $A := \mathbb{C}\backslash\mathring{D}_1$, with the following properties:

(i) v_k converges to u in $C^0(A_r)$ for every $r > 1$ and in $C^\infty(D_r\backslash D_s)$ for all $r > s > 1$,

(ii) $u : \partial A \to \Gamma$ is one-to-one,

(iii) $u(A) \subset T_{\lambda\mu}$,

(iv) u has least area in the following sense: for any $r > 1$ and any map $v \in C^0 \cap H_2^1(A_r,T)$ such that $v|\partial D_1$ parametrizes Γ monotonically and $v|\partial D_r = u|\partial D_r$ it follows that area $(u|A_r) \leq$ area (v),

(v) dist $(\partial D_1, \partial^* A(u,\rho)) \geq c\ell n\rho$,

(vi) $E\big(u|\Omega(u,\rho)\big) \leq a(\rho+\rho_0) - a(R_1) + a_1$.

In the examination of the asymptotical behavior of the solution to the exterior Plateau problem obtained in the above proposition we use some arguments which are different from those in [TY] and, in fact, make the proof considerably simpler. The proof of statement (ii) in Lemma 6 below is essentially due to a student of the author, Th. Nehring [N], who, in his Diploma thesis studied the exterior problem under free boundary conditions.

Lemma 6. (i) $A(u,\rho)$ is bounded for each ρ,

(ii) $|P \circ u(z)| \to +\infty (z \to \infty)$.

Proof. Since u maps ∂D_1 onto Γ topologically, u is also incompressible in T and therefore (i) follows by the same argument as in Lemma 3.2 of [TY]. The statement in (ii) is equivalent to the boundedness of the sets $\Omega(u,\rho)$ for any fixed ρ. We claim that any component of $\Omega(u,\rho)$ is bounded. Otherwise, by (i) and Propsition, (v), it would have to intersect $\partial^* A(u,\sigma)$ for all sufficiently large σ, in particular for $\sigma > \rho$, an absurdity. Let us now assume that $\Omega(u,\rho)$ were unbounded. Since $\Omega(u,\rho) \subset \Omega(u,2\rho)$ and the components of $\Omega(u,2\rho)$ are also bounded, it follows that there is an infinite sequence of different components $\Omega_{2\rho,k}$ of $\Omega(u,2\rho)$, $k \in \mathbb{N}$, $\Omega_{2\rho,k} \neq A(u,2\rho)$, such that each $\Omega_{2\rho,k}$ contains some component of $\Omega(u,\rho)$, in particular $\Omega_{2\rho,k}$ contains some point w with $|P \circ u(w)| \leq \rho$. From this and the monotonicity formula [S] we conclude that area $(u, \Omega_{2\rho,k}) \geq \pi\rho^2$ for all k. This contradicts the finiteness of area $\big(u|\Omega(u,2\rho)\big)$, see Proposition, (vi). □

The next lemma is elementary.

Lemma 7. Let M be a surface without boundary, $v : M \to \mathbb{R}^3$ an immersion, $p_0 \in M$, $r > 0$, and $0 < \epsilon < \frac{1}{2}$ such that the unit normal n of v satisfies $|n(p) - n(p_0)| < \epsilon$ for all $p \in M$ and such that the geodesic disc (w.r.t. the induced metric) $B_r(p) \subset M$ is compact. Then $v(M)$ contains the Cartesian graph of a function f defined on the Euclidean disc $D_{r/\sqrt{2}}(v(p_0))$ in the tangent plane at $v(p_0)$. Furthermore, $|f(x)| \leq 2\epsilon|x - v(p_0)|$ for $x \in D_{r/\sqrt{2}}(v(p_0))$.

Lemma 8. With a suitable choice of the orientation the unit normal n of u converges to $e = (0,0,1)$ if $z \to \infty$.

Proof. Lemma 3.3 of [TY] is directly applicable and gives that the total Gauss curvature of u is finite on $A \backslash D_{r*}$ for some $r^* > 1$. It is well known that n is a holomorphic map to S^2 and the total Gauss curvature is its mapping area. The latter being finite it follows from Picard's theorem that n can only have a removable singularity at $z = \infty$. Hence $N := \lim_{z \to \infty} n(z)$ exists. Let us now assume that $N \neq e$. Let ϵ, and ρ be positive numbers, $0 < \epsilon < \frac{1}{2}$. We may then choose $r > 1$ such that u is immersed on $A \backslash D_r$ and

$$|n(z) - N| < \epsilon \quad \forall z \in A \backslash D_r.$$

By Lemma 6, (ii) we can choose a point $w \in A \backslash D_r$ such that

$$|P(p)| \geq 2\rho, p := u(w)$$

and such that $A\backslash D_r$ contains a compact geodesic disc $B_\rho(w)$. It follows then from Lemma 7 that $u(A\backslash D_r)$ contains the Cartesian graph of a function f defined on the disc $D_{\rho/\sqrt{2}}(p)$ in the tangent plane of u at p and that $|f(x)| \leq 2\epsilon|x-p|$ for $x \in D_{\rho/\sqrt{2}}(p)$. We may clearly choose a point $q \in D_{\rho/\sqrt{2}}(p)$ such that

$$q_3 - p_3 \geq (\rho/\sqrt{2})\sqrt{1-N_3^2}$$

and hence, denoting by y the point in the graph of f lying over q we obtain from Lemma 7 that

$$(8) \qquad y_3 - p_3 \geq q_3 - p_3 - |y-q| \geq (\rho/\sqrt{2})(\sqrt{1-N_3^2} - 2\epsilon).$$

On the other hand y and p belong to the set $T_{\lambda\mu}$ and setting $s := |P(p)|$, $t := |P(y)|$ we conclude from the definition of $T_{\lambda\mu}$ that

$$|y_3 - p_3| \leq \lambda|(\cos h)^{-1}(t/\lambda) - (\cos h)^{-1}(s/\lambda)| + 2\mu$$

$$= \lambda\left|\ell n \frac{t/\lambda + \sqrt{(t/\lambda)^2 - 1}}{s/\lambda + \sqrt{(s/\lambda)^2-1}}\right| + 2\mu \leq \lambda \ell n \frac{2\max(s,t)}{\min(s,t)} + 2\mu.$$

Since $|s-t| \leq |p-y| \leq (1+2\epsilon)\rho/\sqrt{2} \leq \sqrt{2}\rho$ we obtain, taking into account that $s \geq 2\rho$,

$$\begin{aligned} |y_3 - p_3| &\leq \lambda \ell n \tfrac{s+\sqrt{2}\rho}{s-\sqrt{2}\rho} + \lambda \ell n 2 + 2\mu \\ &\leq \lambda \ell n \tfrac{1+\sqrt{2}/2}{1-\sqrt{2}/2} + \lambda \ell n 2 + 2\mu, \end{aligned}$$

what clearly is incompatible with (8) if ϵ is sufficiently small and ρ sufficiently big. It follows that $N_3 = \pm 1$. □

Proof of Theorem. Statement (i) follows from Proposition, (ii), and (ii) from Lemma 6 (ii). It remains to show (iii) and the immersed character of u. We conclude from Lemma 6, (ii) and from Lemma 8 that there is a $\rho > 1$ such that $P \circ u|A\backslash\Omega(u,\rho)$ is a local diffeomorphism with Jacobian of fixed sign. Since the map $P \circ u$ is proper its topological degree is well defined and hence $d := |\deg\left(P \circ u|A\backslash\Omega(u,\rho), x\right)|$ is a positive constant for $x \in \mathbb{R}^2\backslash D_\rho$. It follows that

$$(9) \qquad \text{area}\left(u|\Omega(u,R)\backslash\Omega(u,\rho)\right) \geq d\pi(R^2 - \rho^2)$$

for all $R > \rho$. From (9) and Proposition, (vi) we infer that $d = 1$ since $a(R) \leq (\pi + \epsilon)R^2$ for arbitrary small $\epsilon > 0$ and sufficiently large R. This proves that $u|A\backslash\Omega(u,\rho)$ is the graph of a function φ defined on $R^2\backslash D_\rho$. The estimate (1) follows from the fact that $u(A) \subset T_{\lambda\nu}$. By a classical result of Ossermann [O] u has no true branch points in $\mathring{A}$ since u is locally area minimizing (Proposition, (iv)).

Concerning false branch points we remark that $u|\Omega(u, \rho)$ with ρ as above is an annulus type incompressible minimal surface in T with $u|\partial\Omega(u, \rho)$ injective. It follows from the result in [GT] that $u|\Omega(u, \rho)$ has no false branch points either.

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