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**ON UNIQUENESS IN THE
LARGE OF SOLUTIONS
OF EINSTEIN'S EQUATIONS
("Strong Cosmic Censorship")**

PIOTR T. CHRUSCIEL

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Chapter 1

Introduction

In the last ten years or so a considerable amount of work has been done to transform general relativity into a mathematically rigorous discipline. With the work of Christodoulou and Klainerman [29] on stability of Minkowski space-time, the work of Schoen and Yau [113] on the positive energy theorem, the work of Christodoulou [26] [27] on the gravitational collapse, the work of Newman [100] and others (*cf. e.g.* [55] and references therein) on Yau's Lorentzian splitting conjecture, the work of Bartnik [7] on maximal hypersurfaces in Lorentzian manifolds, general relativity has become a respectable field of mathematical research. For an analyst interested in differential geometry, general relativity turns out to be a rich source of various, sometimes extremely difficult, mathematical problems, encompassing all classical classes of partial differential equations — hyperbolic (*cf. e.g.* [49] [29] [53]), elliptic (*cf. e.g.* [19] [7] [9] [74] [127] [3] [2]), and even parabolic (*cf. e.g.* [8] [33]), as well as some difficult problems of the theory of dynamical systems (*cf. e.g.* [12] [121]). The aim of this paper is to present to a mathematically oriented reader one of the current research problems in general relativity — the problem of uniqueness in the large of solutions of Einstein's equations, also known under the baroque name of “strong cosmic censorship”.

In this chapter we discuss some old and new results on global structure of space-time.

In Section 1.1 some classical results on the Cauchy problem in general relativity are reviewed — emphasis is put on things which are not known (and which the author would like to know¹). In Section 1.2 the shortcomings of the classical singularity theory are pointed out. In Sections 1.3 and 1.4 the “strong cosmic censorship conjecture” and the “weak cosmic censorship conjecture” are discussed. In the Sections that follow, answers to some of the questions raised in the previous Sections are presented. In Section 1.5 results on cosmological space-times with spacelike Killing vectors are presented. In Section 1.6 some stability results are reviewed. In Section 1.7 the Robinson–Trautman space-times are discussed. We close this Chapter by discussing in Section 1.8 what restrictions one needs to impose on both Cauchy data and space-times so that generic predictability of Einstein’s theory can be eventually attained. In Chapter 2, proofs of the results discussed in Section 1.5, the symmetry group $U(1) \times U(1)$ non-including, are given. The “ $U(1) \times U(1)$ stability of strong cosmic censorship” in a neighbourhood of a $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner metric is proved in Chapter 3. Some miscellaneous results are collected in the Appendices. The motivation for this work and the results presented here are best illustrated by Tables 1.0.1 and 1.0.2 (“S.C.C.” stands for “Strong Cosmic Censorship”, “T.T.B.P.” for “Theorem-to-be-proved”, cf. Section 1.1). In table 1.0.1 all connected Lie groups which admit an effective action on a compact, connected, orientable three dimensional manifold and the appropriate manifolds are listed, following Fischer [45]².

Acknowledgements: The author wishes to express special thanks to V. Moncrief, R. Bartnik and J. Baez for many useful discussions. At various stages of work on the topics discussed here the author also benefited from comments from or discussions with D. Christodoulou, H. Friedrich, D. Garfinkle, J. Isenberg, M. MacCallum, J. Shatah and D. Singleton.

¹Questions similar to the ones raised here can also be found in Refs. [43] [94] [6] [123].

²It should be pointed out that the list of manifolds which admit a $U(1)$ action given in [45] is incomplete.

<i>Symmetry Group</i>	${}^3\Sigma$ <i>Topology</i>	<i>Supplementary Conditions</i>	<i>S.C.C. (T.T.B.P.)</i>
$SO(4)$ $SO(3) \times SO(3)$	S^3 \mathbb{P}^3		no vacuum metrics ³
$(U(1) \times SU(2))/D$	$L(p, 1)$ p — odd		no ⁴
$U(1) \times SO(3)$	$L(p, 1)$ p — even		no ⁴
	$S^1 \times S^2$		$\frac{1}{2}$ yes ⁵
$SU(2)$	$L(p, 1)$ p — odd		? (probably yes ⁶)
$SO(3)$	$L(p, 1)$ p — even		? (probably yes ⁶)
	$S^1 \times S^2$ $\mathbb{P}^3 \# \mathbb{P}^3$		$\frac{1}{2}$ yes ^{5, 7}
	\mathbb{P}^3 S^3/Γ		no vacuum metrics ⁸
$U(1) \times U(1) \times U(1)$	T^3		yes ⁹
$U(1) \times U(1)$	$L(p, q)$ $S^1 \times S^2$	$g_{\mu\nu} X_1^\mu X_2^\nu = 0$	yes ¹⁰
	T^3	$g_{\mu\nu} X_1^\mu X_2^\nu = 0$	yes ¹⁰
		$\epsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu \nabla^\rho X_a^\sigma = 0$	
		“small” data, $\epsilon_{\mu\nu\rho\sigma} X_1^\mu X_2^\nu \nabla^\rho X_a^\sigma = 0$	$\frac{3}{4}$ yes ¹¹
	$L(p, q)$ $S^1 \times S^2$ T^3		? ¹²
$U(1)$? ¹³

Table 1.0.1: Strong cosmic censorship, spatially compact case

<i>Symmetry Group</i>	${}^3\Sigma$ <i>Topology</i>	<i>Supplementary Conditions</i>	<i>S.C.C. (T.T.B.P.)</i>
$U(1) \times \mathbb{R}$	\mathbb{R}^3		yes ¹⁴
{1}	\mathbb{R}^3	“small”, smoothly conformally compact ¹⁵ “hyperboloidal” data	$\frac{1}{2}$ yes (yes to the future ¹⁶ , ? to the past)
		“small”, asymptotically flat data	yes ¹⁷
	$\mathbb{R} \times S^2$	Robinson–Trautman space-times	$m > 0$: no ¹⁸ $m < 0$: $\frac{1}{2}$ yes

Table 1.0.2: Strong cosmic censorship in vacuum ”minisuperspaces”, spatially open case.

³cf. page 25; \mathbb{P}^3 denotes the three dimensional real projective space.

⁴ D is the subgroup $\{(1, 1), (-1, -1)\}$, $L(p, q)$ denotes a lens space, recall that $L(1, 1) = S^3$. These are the Taub-NUT metrics, cf. Section 2.2.

⁵These are the Kantowski–Sachs cosmological models; locally, the metric is the Schwarzschild metric ”below the Schwarzschild horizon”, $r < 2m$, cf. Section 2.3; the space-times are extendible to, say, the past of ${}^3\Sigma$ and inextendible to its future.

⁶These are the Bianchi IX metrics, cf. page 26; no rigorous results on inextendability (strong cosmic censorship) have been established so far.

⁷ $\#$ denotes the connected topological sum.

⁸cf. Section 2.3; for a list of all possible groups Γ by which S^3 is divided, cf. [45].

⁹These are the Kasner metrics, cf. Section 2.4.

¹⁰These are the polarized Gowdy metrics; the X_a ’s, $a = 1, 2$, are Killing vectors, cf. Theorem 1.5.1.

¹¹Yes to the, say, past, probably yes to the future, cf. Theorem 1.5.2.

¹²Partial results on global existence are available, cf. Section 1.5, but no inextendibility results are known.

¹³cf. [45] for a list of possible manifolds and actions; no rigorous results on s.c.c. are known. To the list of manifolds with a $U(1)$ action given in [45] one should add the Eilenberg–Mac Lane spaces $K(\pi, 1)$ whose fundamental group has an infinite cyclic center (A.Fischer, private communication).

¹⁴This is an unpublished result of Christodoulou, cf. Theorem 1.5.3.

¹⁵By this we mean that $({}^3\Sigma, g)$ is conformally isometric to a compact manifold with boundary, with a

1.1 The spacelike Cauchy problem in General Relativity

Let us start with recalling some basic facts about the Cauchy problem in general relativity. The Cauchy data for vacuum Einstein equations consist of a triple (Σ, g, K) where (Σ, g) is a three dimensional Riemannian manifold¹⁹ with a metric of appropriate differentiability class (which we shall discuss later), K is a symmetric tensor on Σ (representing, roughly speaking, the time derivative of g), and g, K are assumed to satisfy a system of four coupled “constraint equations”:

$${}^3R = K^{ij}K_{ij} - (g^{ij}K_{ij})^2,$$

$$K^{ij}|_j - g^{kl}g^{ij}K_{kl}|_j = 0,$$

where 3R is the Ricci curvature scalar of g , $|$ denotes the Levi-Civita covariant derivative defined by the metric g , which arise as a consequence of the Gauss-Codazzi equations. A triple (M, γ, i) , where (M, γ) is Lorentzian manifold and $i : \Sigma \rightarrow M$ is an embedding of a three dimensional manifold Σ in M , is called a *development* of (Σ, g, K) if γ satisfies the vacuum Einstein equations:

$$R_{\mu\nu} = 0,$$

where $R_{\mu\nu}$ is the Ricci tensor of γ , and if

$$i^*\gamma = g, \quad i^*\tilde{K} = K,$$

metric which can be extended in a sufficiently differentiable way to a metric on a slightly larger manifold, cf. Section 1.6 and Appendix A for details.

¹⁶This result is due to Friedrich; no symmetry conditions are assumed, cf. Theorem 1.6.1.

¹⁷This result is due to Christodoulou and Klainerman; no symmetry conditions are assumed, cf. Theorem 1.6.2.

¹⁸ $m > 0$: no, cf. Theorem 1.7.2; $m < 0$: no to the future, yes to the past, cf. Theorem 1.7.1.

¹⁹Unless specified otherwise, “manifold” is used here to denote a connected, σ -compact (cf. e.g. [78]) manifold without boundary, the degree of differentiability of which can be in principle inferred from the context. We will not assume at the outset Hausdorffness of space-times, because of the existence of non-Hausdorff extensions of the globally hyperbolic region of the Taub-NUT space-time, cf. [64]. On the other hand we shall be conservative and always assume that the Riemannian manifold Σ on which Cauchy data (g, K) are defined is *always* Hausdorff.

where \tilde{K} is the extrinsic curvature tensor of $i(\Sigma)$ in M . We shall often say that (M, γ) is a development, whenever no need arises to make i explicit. Some authors add the requirement of global hyperbolicity in the definition of development, it should be stressed that we are *not* adding this restriction. From the invariance of Einstein equations under diffeomorphisms it follows that for every diffeomorphism $\Phi : N \rightarrow M$, if (M, γ) is a development of (Σ, g, K) , so is $(N, \Phi^* \gamma)$ (the appropriate embedding is $\Phi^{-1} \circ i$). A development will be called *maximal* if “there exists no larger development”; technically, if $\Phi : M \rightarrow M'$ is an isometric diffeomorphism²⁰ from M to $\Phi(M)$ and if the metric γ' on M' satisfies vacuum Einstein equations, then $M' = \Phi(M)$. (For completeness we prove the existence of maximal developments in Appendix C.1.) (M, γ) will be said *strongly maximal* if there exists no larger Lorentzian manifold (not necessarily satisfying the field equations) in which (M, γ) can be isometrically embedded). It should be stressed that *some maximal developments may fail to be strongly maximal*, because it could happen that there exist non-vacuum space-times $(\tilde{M}, \tilde{\gamma})$ extending a given space-time, while no vacuum extensions of (M, γ) exist. We shall say that (M, γ) is *strongly maximal to the future*, respectively *to the past*, if in any extension $(\tilde{M}, \tilde{\gamma})$ of (M, γ) the set of points $p \in \tilde{M}$, $p \notin M$ such that p lies to the future of M , respectively p lies to the past of M , is empty.

The theorem one would like to prove in general relativity is the following:

Theorem-to-be-proved 1 (TTBP) *For every three dimensional manifold Σ there exists a space of Cauchy data $X(\Sigma)$ and a class of four dimensional Lorentzian manifolds $\mathcal{M}(\Sigma)$ such that:*

1. *for all Cauchy data in a dense set of “generic Cauchy data” $Y(\Sigma) \subset X(\Sigma)$ the*

²⁰By an abuse of language a locally Lipschitz continuous bijection with locally Lipschitz continuous inverse will also be considered to be a diffeomorphism, if useful in the context. The usefulness of such a weak notion of diffeomorphism relies in the fact, that such transformations are “as regular +1” as the least regular of g and Φ^*g : more precisely, if both g and Φ^*g are in a Sobolev class H^{loc}_s , then Φ must be in H^{loc}_{s+1} (similar properties hold in C^k_{loc} or $C^{k,\alpha}_{loc}$ spaces). Therefore once the class of differentiability of the metrics one speaks about is fixed, one needs not to worry about the differentiability class of such “diffeomorphisms”.

Cauchy problem admits a unique maximal solution (M, γ) in $\mathcal{M}(\Sigma)$,

2. with (M, γ) — strongly maximal, moreover

3. all smooth pairs (g, K) satisfying the constraint equations and satisfying some completeness and/or fall-off conditions are in $X(\Sigma)$.

Condition 1 is the requirement of *generic predictability* of any theory. As we shall see later, there exist smooth initial data in general relativity for which it seems that neither uniqueness nor strong maximality of developments holds. The undefined condition of genericity is also meant to capture the fact, that Cauchy data for which uniqueness in the large holds are stable, and a precise statement should of course be implemented by a definition of “genericity” — a tentative possibility could be that $Y(\Sigma)$ is open and dense. Condition 2 — *strong maximality* — is a necessary condition to be able to predict the evolution of “things” as long as they can “potentially exist”: the existence of an extension (M', γ') of a maximal development (M, γ) would mean that there are observers which leave M to end up in M' in which, by definition of maximal development, the metric cannot satisfy the field equations. Condition 3 is a requirement for the applicability of the theory to all possible physical situations²¹; the proviso of some completeness and/or fall-off restrictions is necessary, as will be discussed in Section 1.8. At this stage the reader may wish to assume that if Σ is compact, then all smooth solutions of the constraint equations should be in $X(\Sigma)$; if Σ is not compact, than $X(\Sigma)$ could be required to contain *e.g.* all smooth solutions of the constraint equations with g_{ij} — asymptotically flat (*e.g.* in the sense of Theorem 1.6.2) in the “ends” of Σ . If the theorem-to-be-proved holds, then the Cauchy data for which uniqueness and/or strong maximality fail are unstable, and therefore their existence, though an unpleasant feature, does not seem to be a very serious threat to predictability of the theory. There are three possible attitudes to a theory in which theorem-to-be-proved does not hold:

²¹One could also adopt the point of view, that since there is only one universe accessible to our observations, namely the one we live in, it would actually suffice to have $Y(\Sigma) = X(\Sigma) = \text{one point}$ — the Cauchy data for the universe we live in. Such an approach does not seem to be very fruitful.

1. accept that the theory has poor predictability power, and cannot thus be considered as a fundamental one, or
2. isolate the set of Cauchy data, say $Z(\Sigma)$, for which uniqueness and strong maximality hold, and admit it as a prediction of the theory that no other Cauchy data than the ones in $Z(\Sigma)$ are admissible, or
3. renounce the idea, that unique predictability is a basic requirement for any satisfactory (nonquantum) physical theory.

Since none of these seems attractive to the author, he will hope for the best, namely that the theorem-to-be-proved can be proved indeed.

Let us recall some of the known results concerning the Cauchy problem for Einstein equations. The following theorem, due to Hughes, Kato and Marsden [72], is an improvement (as far as the differentiability hypotheses are concerned) of the pioneering work by Choquet-Bruhat [49] [17] (followed by some subsequent results by Hawking and Ellis [66], and Fischer and Marsden [46]; cf. [24] for an extensive review of this and other topics treated here):

Theorem 1.1.1 (Local existence) *Let (Σ, g, K) be an initial data set for vacuum Einstein equations. Suppose that Σ can be (locally finitely) covered by coordinate charts U_α , C^1 related to each other, such that $(g, K) \in H_s^{loc}(U_\alpha) \times H_{s-1}^{loc}(U_\alpha)$, $s > 5/2$. Then there exists a globally hyperbolic development (M, γ) of (Σ, g, K) , for which Σ is a Cauchy surface, with γ determined uniquely (up to isometry) by (g, K) ²². If Σ is Hausdorff, M can be chosen to be Hausdorff.*

This theorem guarantees the existence of some development in which the metric is uniquely determined by the data. It also tells us that the space-time constructed in the proof will be globally hyperbolic. Recall that (M, γ) is said to be globally hyperbolic if in

²²Recall that $f \in H_s^{loc}(U)$ if for every conditionally compact subset K of U $\|f\|_{H_s(K)} < \infty$, where H_s are the standard Sobolev spaces, $s \in \mathbb{R}$. s can be taken greater than or equal to 3 in the local existence theorem if one prefers integer values of s .

M there exists a hypersurface Σ with the property that every inextendible causal curve meets Σ once and only once. The hypersurface Σ of this definition is called a Cauchy surface. The fact that (M, γ) so constructed is globally hyperbolic is welcome since it means that M is not “smaller than globally hyperbolic”, *i.e.* “no causal holes” are “dug” in M ; on the other hand the method of proof does not allow one to construct “larger than globally hyperbolic” space-times, *i.e.* space-times in which Cauchy horizons occur. (The notion of global hyperbolicity was first introduced by Leray in a slightly different form²³, resulting in his famous theorem that the Cauchy problem for linear hyperbolic systems with Cauchy data on a spacelike submanifold of a globally hyperbolic space-time is well posed for, say, smooth data, *i.e.* smooth data evolve to a unique globally defined solution on M . Let us also mention that it follows from the theorems of Geroch [57] and Seifert [114] that the notion of global hyperbolicity of a space-time coincides with the condition of “strict hyperbolicity” of the associated d’Alembert operator, *cf.* [71][Volume 3, Chapter 23].) As far as existence and uniqueness of *maximal globally hyperbolic* developments is concerned we have the following result²⁴ due to Choquet-Bruhat and Geroch [21], with some improvements on the differentiability conditions due to Hawking and Ellis [66]:

Theorem 1.1.2 (Uniqueness of maximal globally hyperbolic Hausdorff developments) *Let Σ be a three dimensional Hausdorff manifold. For every $(g, K) \in H_4^{loc}(\Sigma) \times H_3^{loc}(\Sigma)$ there exists a Hausdorff manifold M with a $C_{loc}^{1,1}$ metric²⁵ γ , which is maximal in the space of globally hyperbolic Hausdorff developments of (g, K) . M is diffeomorphic to $\Sigma \times \mathbb{R}$ and (M, γ) is unique (up to diffeomorphisms) in the class of globally hyperbolic Hausdorff manifolds (satisfying appropriate differentiability conditions).*

²³When discussing equivalence of various definitions of global hyperbolicity one should be careful about the differentiability conditions needed to establish them: the usual causality theory, as presented *e.g.* in [66], requires $C_{loc}^{1,1}$ metrics (first derivatives of the metric are Lipschitz continuous on conditionally compact subsets of coordinate charts); several (and maybe all) essential results can be recovered under the differentiability conditions considered in [41]. If the parameter s of the local existence theorem is larger than $5/2$ the metric will be $C_{loc}^{1,\alpha}$ with an appropriate $0 < \alpha < 1$, but not necessarily $C_{loc}^{1,1}$ so that the classical results require reexamination, and in fact several results seem not to go through due to technical problems. In the case of theorem 1.1.2 the manifold M can certainly be chosen small enough to be globally hyperbolic both in the original Leray sense and in the sense described above.

²⁴This theorem holds in considerably more general situations than vacuum Einstein equations, *cf.* [21].

²⁵The metric will actually have more regularity than $C_{loc}^{1,1}$, *cf. e.g.* [20] for details.

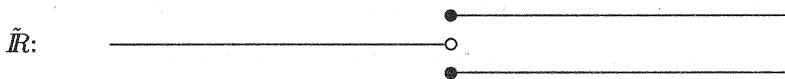


Figure 1.1.1: The bifurcating real axis.

Theorem 1.1.2 is unfortunately much weaker than what needs to be proved for “theorem” 1 to hold: uniqueness and maximality are guaranteed only within the class of Hausdorff globally hyperbolic developments, and nothing is said about *strong maximality*. It is easy to see that if one wishes to maintain uniqueness of M , some kind of requirement of Hausdorffness-type cannot be avoided: if one drops Hausdorffness altogether then if ${}^3\Sigma \times \mathbb{R}$ is a development, the manifold ${}^3\Sigma \times \tilde{\mathbb{R}}$ will also be a development, where $\tilde{\mathbb{R}}$ is the standard non-Hausdorff “bifurcating real axis”, Figure 1.1.1, and one can continue to produce manifolds with at least a countable infinity of different topologies by adding new bifurcation branches and removing some. On the other hand allowing some kind of weak violation of Hausdorffness — *e.g.*, as proposed by Hajiček, requiring non-existence of bifurcate causal curves in the space-time — may restore uniqueness of maximal developments of the Taub—Newman—Unti—Tambourino (Taub—NUT) space-time²⁶ (cf. *e.g.* [66]). Let us discuss this space-time in some more detail. The Taub—NUT space-time is a highly symmetric model for which space-sections are “initially” diffeomorphic to S^3 , “initially” meaning “in a neighborhood of some spacelike $(S^3)_o$ ” on which Cauchy data may be given, and these Cauchy data evolve to a space-time the global structure of which is well visualized by Figure 1.1.2 [85] [36]. The shaded region in Figure 1.1.2 is globally hyperbolic, and for any spacelike S^3 lying in this region it turns out to be the maximal globally hyperbolic Hausdorff development, as guaranteed by Choquet-Bruhat and Geroch. The unshaded region contains closed timelike geodesics. It can be shown [36] that there exist at least two different smooth ways (in fact even analytic) of at-

²⁶If it were possible to prove uniqueness of all solutions at the price of some form of violation of Hausdorffness of the topology of the resulting maximal developments, it could consider this as a prediction of the theory, namely that in some special, maybe highly unstable (*cf.* the discussion of Section 4) situations, violation of Hausdorffness of space-time occurs. It must be pointed out that it is not known, whether Hajiček’s non-Hausdorff extension of Taub—NUT space-time [64] is unique.

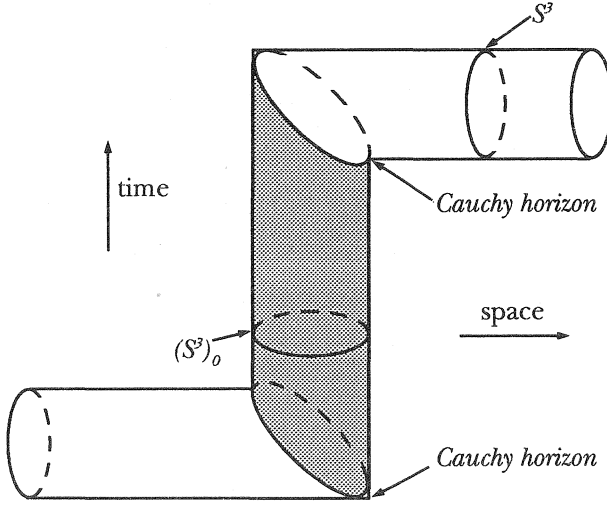


Figure 1.1.2: Taub-NUT space-time.

taching the acausal NUT region to the causal Taub region, one obtains in this way two *non-isometric developments* of the Taub region²⁷. The “corners” in Figure 1.1.2 at the junction of the causal and acausal regions are misleading because both the manifold and the metric are smooth at the “junction”. They are however correctly illustrating the fact that this junction acts as a singular surface for geodesics, the reader is referred to [85] [66] for more details. Let us note for further use that the hypersurface separating the globally hyperbolic region from the rest of the universe is called a *Cauchy horizon*. A similar behaviour is exhibited by the Misner space-time [84] (cf. also [66]), the globally hyperbolic region of which is diffeomorphic to $\mathbb{R} \times T^3$ ($T^3 = S^1 \times S^1 \times S^1$ = the three dimensional torus), and in a large class of space-times constructed by Moncrief [88] [89], in polarized Gowdy space-times which we discuss in more detail in Section 1.5; some flat space-times with Cauchy horizons are also presented in Appendix D. All these examples necessitate the proviso of genericity in TTBP.

The global hyperbolicity condition plays an essential role in the proof of the Choquet-

²⁷It would be of great interest to prove or disprove that the two known extensions are the only possibilities with, say, compact horizons.

Bruhat–Geroch theorem, because no systematic method of solving Einstein’s equations beyond Cauchy horizons is known (*cf.* however [89] [89] in the analytic case). Let us also note a discrepancy of differentiability conditions between Theorems 1.1.2 and 1.1.1 — it is not clear whether it can be removed. One may wonder whether one should really worry about this since one often encounters the point of view (*cf.* e.g. [66]) that we can assume without any loss of physical information that the metric is smooth. The argument is, that since we are not able to measure whether a quantity is either smooth or C^2 or, say, continuous, then, quantum considerations put apart, one can as well assume smoothness of physical fields. It seems that a logical conclusion of such a reasoning should be exactly the opposite, namely that in a physical theory one should assume only the minimal differentiability conditions under which the theory makes sense; let us illustrate this by the following considerations: by a well known result by Whitney [128], (*cf.* also [129]) every C^k manifold, $k \geq 1$, is C^k diffeomorphic to an analytic one, which implies that every field on a manifold can be approximated, locally to arbitrary accuracy, by analytic fields. Continuing the previously described line of thought, one could thus as well assume that all fields are analytic. Such a hypothesis leads immediately to at least one erroneous conclusion, namely that the solutions of the field equations are uniquely defined everywhere by their value on some open subset: the unique continuation property of analytic fields completely obscures the fundamental property of hyperbolic equations, that uniqueness of solutions holds within domains of dependence only. Another striking example in which completely wrong conclusions are drawn by assuming analyticity of fields is given by the Robinson–Trautman space–times (*cf.* Section 1.7): there exists a large family of analytic initial data for a Robinson–Trautman space–time which can be evolved both to the future and to the past of the initial surface. On the other hand for all data which are smooth but fail to be analytic, a Robinson–Trautman solution of the vacuum Einstein equations can exist only either to the future or to the past of the initial hypersurface, depending upon the sign of m (*cf.* Section 1.7). This clearly demonstrates that restricting oneself to analytic fields may be rather misleading. Analyticity put apart, we wish moreover to point out that *the assumption of global smoothness of solutions may*

be simply inconsistent with vacuum Einstein equations, or, more generally, with Einstein equations coupled with matter fields the energy-momentum tensor of which satisfies some positivity conditions. This is due to the possibility of breakdown of differentiability of the metric in the course of evolution²⁸. In order to discuss this in more detail, let us first turn to the basic question, what are the most general conditions under which Einstein equations make sense. The Ricci tensor may be written in the form

$$R_{\mu\nu} = \partial_\lambda A_{\mu\nu}^\lambda + B_{\mu\nu}$$

where A and B are expressions of the form

$$A_{\mu\nu}^\lambda = \sum g^{-1} \partial g, \quad B_{\mu\nu} = \sum g^{-1} \partial g g^{-1} \partial g,$$

where g^{-1} stands for the tensor $g^{\mu\nu}$ inverse to the metric tensor $g_{\mu\nu}$. If²⁹ $g \in H_1^{loc}(M)$ and $g^{-1} \in L_{loc}^\infty(M)$, the “distributional equation”

$$\forall \phi^{\mu\nu} \quad \int_M \{ \partial_\lambda \phi^{\mu\nu} A_{\mu\nu}^\lambda - \phi^{\mu\nu} B_{\mu\nu} \} = 0$$

is well defined, with $\phi^{\mu\nu}$ — symmetric C^1 tensor density of compact support (cf. [41] and [60] for a discussion of some properties of such metrics). This class of metrics is much larger than the one considered in the existence theorem. As long as the metric does not leave this class during evolution, one can interpret Einstein equations in the above sense³⁰. In the theory of partial differential equations it is often much easier to prove existence of some weak generalized solutions than to prove existence of solutions with high regularity and therefore existence of such weakly differentiable metrics seems highly plausible³¹. The problem with this kind of solutions is their potential non-uniqueness and it is likely that with such weak conditions the theorem-to-be-proved may not hold.

²⁸cf. [116] for an example of evolution equations in which smooth initial data cease to be differentiable during evolution, while globally defined weak solutions exist (the weak solutions fail, however, to be unique [117]). The example studied by Shatah may be of some relevance to our discussion, because harmonic maps are related to Einstein equations, cf. e.g [90] [87].

²⁹It is worthwhile noting that similar conditions arise naturally when studying the problem of weakest possible conditions for finite and well defined ADM mass, cf. e.g. [5].

³⁰One cannot exclude the possibility of being able in the future to formulate Einstein equations in an even larger class of metrics.

³¹A standard approximation argument should show the existence part of theorem 2.1 holds with (g, K) in $H_s^{loc} \times H_{s-1}^{loc}$, $s > 3/2$ (the resulting space-time metric will actually be still more regular than the requirements of well posedness of distributional Einstein equations pointed out above).

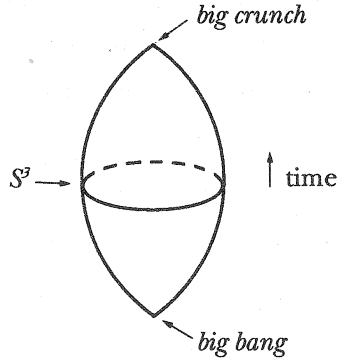


Figure 1.2.1: Friedman–Robertson–Walker space–time.

1.2 Singularities?

Although of fundamental importance, the Choquet-Bruhat–Geroch theorem gives only very poor information (global hyperbolicity, Hausdorffness) about the structure of the space–time whose existence and uniqueness it guarantees. An especially unpleasant feature of some maximal solutions of Einstein equations is the potential failure of strong maximality, i.e. *existence of extensions to space-times on which there is no way of ensuring satisfaction of field equations*. The possibility of existence of such extensions leads us to the question, what global properties generic solutions possess, and therefore to the problem of singularities. Soon after Einstein’s theory had been formulated it was realized that Einstein’s equations led to formation of singularities (cf. [123] for a review of the history of the problem). A standard example of such an occurrence is given by the Friedman–Robertson–Walker metrics on $\mathbb{R} \times S^3$ evolving from the “big bang” singularity to the “big crunch” singularity, cf. Figure 1.2. In this crude cosmological model Einstein theory describes very precisely what will happen to the inhabitants of the universe: in finite proper time they will be crushed to zero volume by infinite gravitational forces. One can rightly raise the objection, that the assumed dust model fails to describe physical reality near the singularities — to avoid such discussions we shall from now on restrict our

attention to vacuum Einstein equations. The question whether singularities generically occur in vacuum Einstein gravity has intrigued physicists for years, the general belief being that this problem has been satisfactorily settled by Penrose and Hawking in their celebrated singularity theorems. Let us present³² one of those [67] [66]:

Theorem 1.2.1 (Hawking–Penrose Singularity Theorem) *Suppose that Σ is a compact three dimensional manifold, let (g, K) be Cauchy data which admit $C_{loc}^{1,1}$ developments, let (M, γ) be any development in the $C_{loc}^{1,1}$ differentiability class and suppose that (M, γ) satisfies a genericity condition (for details, cf. [67] [66]). Then either*

1. (M, γ) contains a Cauchy horizon beyond which closed timelike curves occur, or
2. (M, γ) is geodesically incomplete.

We shall not discuss in detail what “generic data” means in this context, let us just mention that there is hope (and some evidence) that the hypothesis of genericity can be removed by adding a third possibility, namely that (M, γ) splits isometrically as $\mathbb{R} \times \Sigma^3$, with Σ^3 — flat (cf. [100] [55] and references therein). 1) seems to be a very interesting prediction of general relativity — we have seen an example of such a behaviour in the Taub-NUT space-time. We have chosen the somewhat artificial dynamical formulation of the above theorem to emphasize that there is no causality violation in a neighborhood of the hypersurface Σ , the closed timelike curves “develop” beyond Cauchy horizons, in particular the timelike loops which develop in Taub-NUT space-time beyond, say, a future Cauchy horizon, do not allow for any interference with the past of the observers on the other side of the horizon. It is widely believed that generically the second part of the alternative occurs, namely existence of incomplete geodesics. The problem with point 2) is, as already pointed out by Hawking and Penrose [67], that though it leaves one with the feeling that something goes wrong, it doesn’t say what actually is going wrong. In view of the weak differentiability conditions for existence of solutions of Einstein equations,

³²One would expect that the condition of genericity of (M, γ) required below might hold for generic Cauchy data, but this remains to be proved.

not to mention well posedness of the distributional equations, this theorem completely fails to indicate whether we are really going to face a situation in which either Einstein equations will fail or some other unwanted physical phenomena will occur. A breakdown of $C_{loc}^{1,1}$ character of the metric may lead to a blow up of the Riemann tensor³³, but even this need not be accompanied by any extremely unpleasant physical effects. A simple way of analyzing the effects of curvature on matter is via tidal forces induced by gravity, the effect of which is given by an integral of some components of the Riemann tensor along world lines. If the blow up preserves finiteness of such integrals no extraordinary physical effects will be observed. Rather than interpreting this theorem as a breakdown of general relativity, one can think of it as evidence for the need of considering weakly differentiable metrics. There is little doubt that some singularities that occur in general relativity cannot be removed by the introduction of coordinate systems in which the metric is only weakly differentiable, and it is important to isolate those from the other kind. Rather than having a theorem telling us that *something* goes wrong, one would like to have a result which tells us exactly *what* goes wrong, e.g.:

Theorem-to-be-proved 2 (TTBP-2) *Suppose that Σ is a three dimensional Riemannian manifold, let $(g, K) \in X(\Sigma)$, $X(\Sigma)$ as in TTBP and let (M, γ) be a maximal development of (Σ, g, K) . For all/most/some Γ 's, where Γ is either*

1. *a past/future incomplete inextendible timelike geodesic, or*
2. *a past/future incomplete inextendible null geodesic³⁴, or*
3. *a finite total acceleration future/past inextendible timelike curve of finite proper length, i.e. a curve for which (s — proper time along Γ)*

$$\int_{\Gamma} g_{\mu\nu} \frac{D^2 x^\mu}{ds^2} \frac{D^2 x^\nu}{ds^2} ds < \infty.$$

4. *a past/future inextendible timelike curve of finite proper length,*

³³Some results of this kind have been established by Clarke [41] [123] under supplementary hypotheses.

³⁴If the metric is not C^1 , a geodesic may eventually be defined as a curve possessing some extremal properties in appropriate classes of curves.

it holds that

???

The “finite total acceleration” curves are interesting, because they may be thought to represent observers which have at their disposition a finite amount of fuel for their rocket’s engine. It is not to be excluded that in generic situations “a lot of” geodesics will be complete either to the future, or to the past, or both.

Let us emphasize that the analysis of what goes wrong along incomplete geodesics is relevant to TTBP, at least in the $C_{loc}^{1,1}$ differentiability class of metrics, because of the following inextendability criterion:

A Maximality Criterion: *Let γ be a $C_{loc}^{1,1}$ Lorentzian metric on M . If on every incomplete timelike geodesic in M some curvature scalar blows up, then (M, γ) is strongly maximal in the class of Lorentzian space-times with $C_{loc}^{1,1}$ metrics.*

It follows that a proof of curvature blow-up is a useful step in the proof of TTBP. It should be noted that the above criterion also shows that a geodesically complete space-time is necessarily maximal in the class of space-times with $C_{loc}^{1,1}$ metrics. The lack of criteria for inextendibility for Lorentzian metrics in weak differentiability classes is at the origin of our inability to exclude extensions which are not $C_{loc}^{1,1}$. For instance, it is not known whether *e.g.* Minkowski space-time can or cannot be extended to a larger manifold with a continuous Lorentzian metric, though such a possibility seems rather unlikely to the author. This, and some other inextendability criteria, are proved in Appendix C.2.

1.3 Strong Cosmic Censorship?

One often encounters the opinion, that an important issue in general relativity is to establish the validity of the *strong cosmic censorship*, a hypothesis formulated by Penrose [103] (*cf.* also [70][Section 5.4]), which can tentatively be formulated as follows:

Strong Cosmic Censorship Conjecture (SCCC): *Every maximal Hausdorff development of a generic Cauchy data set (Σ, g, K) , with (Σ, g) — compact or asymptotically flat, is globally hyperbolic³⁵.*

This conjecture is often formulated in the $C_{loc}^{1,1}$ context, and a breakdown of $C_{loc}^{1,1}$ differentiability class of the metric on a globally hyperbolic manifold is considered as a breakdown of validity of the conjecture. Note that by the Choquet–Bruhat–Geroch theorem there is only one globally hyperbolic development, thus SCCC implies uniqueness in generic situations. The supposed importance of this conjecture is motivated by the belief that

1. a failure of $C_{loc}^{1,1}$ differentiability of the metric implies necessarily a breakdown of predictability of Einstein theory (non-uniqueness of solutions in the large), and
2. non-global hyperbolicity means existence of Cauchy horizons, and it is believed that there is no physically reasonable class of space-times in which one can uniquely solve Einstein equations beyond the Cauchy horizon.

As we have emphasized in the previous sections, we have no convincing evidence that 1) is justified. We have purposefully stated the SCCC without making the differentiability conditions explicit to simply disregard 1) and concentrate on 2). Let us recall some facts about *compact* Cauchy horizons: Isenberg and Moncrief have shown [75], under some technical conditions which can probably be removed, that an analytic space-time with a compact Cauchy horizon must admit a non-trivial isometry group. Moncrief [88] [89] has studied the analytic Cauchy problem posed on an analytic $U(1)$ symmetric horizon with S^3 and T^3 topology and has shown that this problem is well posed: more precisely, by proving a singular version of the Cauchy–Kowalevskaya theorem, he has shown that there exists a unique analytic development of appropriate analytic data on the horizon both into the causal and, what interests us more here, into the acausal region. It is not known whether this result is a pure accident of analyticity, or reflects the possibility

³⁵It should be stressed that we *do not* include the notion of global hyperbolicity in the definition of development, cf. the beginning of Section 1.1.

that the problem of evolving say, smooth data into the acausal region is well posed. Let us just point out at this stage that solvability of the analytic Cauchy problem is usually an excellent starting point for the proof of solvability of the smooth or the H^s Cauchy problem — what is needed is some kind of *a priori* estimates. *If one could establish generic unique solvability beyond those Cauchy horizons which can occur in space-times containing compact or asymptotically flat space-like hypersurfaces, it seems that this formulation of SCCC would cease to be a fundamental issue.* To the author's knowledge, no conclusive study of this problem has been undertaken yet. It must be noted that it follows from the proofs of the Penrose–Hawking theorems [67] [66] that causality violations will occur beyond Cauchy horizons. As emphasized previously, in no way does this lead to causality problems in the globally hyperbolic regions, and ending in an acausal region does not seem to be a worse fate for a universe than ending in a big crunch. It is often argued, quite convincingly (for references cf. e.g. [103]), that Cauchy horizons will be unstable (cf. also [91]). Whatever the status of generic solvability of Einstein equations beyond Cauchy horizons, it would be of interest to isolate the set of Cauchy data which lead to formation of Cauchy horizons:

Theorem-to-be-proved 3 (TTBP-3) *Let $\Sigma, X(\Sigma), Y(\Sigma), \mathcal{M}(\Sigma)$ be as in TTBP. Let $X_H(\Sigma)$ be the set of Cauchy data for which Cauchy horizons occur. Then*

1. $X(\Sigma) \setminus X_H(\Sigma)$ is dense and open ???,
2. The intersection $X_H(\Sigma) \cap Y(\Sigma)$ is empty ??? is dense in $X_H(\Sigma)$??? is equal to $X_H(\Sigma)$ for appropriately chosen $\mathcal{M}(\Sigma)$???

We would like to emphasize once again that

- if it were convincingly demonstrated that there is no sense in which one can have generic unique solvability of Einstein equations beyond the kind of Cauchy horizons which can occur by evolution from asymptotically flat or spatially compact initial data, then SCCC would be of fundamental importance to Einstein's theory of gravitation. Its failure would imply that TTBP does not hold.

- If generic unique solvability beyond Cauchy horizons as above can be established, SCCC becomes an interesting but not fundamental problem.

This last possibility seems to be highly unlikely to most authors (*cf.*, however, [50] and [95]).

1.4 (Weak) Cosmic Censorship?

Another famous conjecture due to Penrose [105] is the so-called *(weak) cosmic censorship hypothesis* (w.c.c.), which expresses the hope that for *generic collapsing isolated gravitational systems the singularities that might develop will be hidden beyond a smooth event horizon* (*cf.* also [70] [123] [66] for discussion, and for various technical formulations of the problem). Since 1976 the following examples of naked singularity formation have been found, we list them in chronological order of appearance:

1. Yodzis, Seifert and Müller zum Hagen have shown that “naked singularities” may occur during the collapse of spherically symmetric dust [131] ($T_{\mu\nu} = \rho u_\mu u_\nu$, with $\gamma_{\mu\nu} u^\mu u^\nu = -1$, where $T_{\mu\nu}$ is the energy-momentum tensor), or of spherically symmetric perfect fluids [131] [97] ($T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\gamma_{\mu\nu}$, $\gamma_{\mu\nu} u^\mu u^\nu = -1$), for a large class of equations of state $p = p(\rho)$. These are the so-called “shell-crossing” singularities, which arise because of crossing of shells of matter, and their occurrence is stable under spherically symmetric perturbations [97].
2. Steinmüller, King and Lasota [122] have noted that naked singularities can form during a radiative collapse of a star emitting “null³⁶ dust” ($T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\gamma_{\mu\nu} + \epsilon k_\mu k_\nu$, $\gamma_{\mu\nu} u^\mu u^\nu = -1$, $\gamma_{\mu\nu} k^\mu k^\nu = 0$), in a family of metrics considered by Demiański and Lasota [42].
3. Hiscock, Williams and Eardley have shown [68] that the implosion of null dust ($T_{\mu\nu} = \epsilon k_\mu k_\nu$, $\gamma_{\mu\nu} k^\mu k^\nu = 0$) can lead to the formation of so-called “shell-focusing”

³⁶A “null dust” can be obtained from *e.g.* a geometric optics approximation to a massless scalar field.

naked singularities³⁷ (the same result has been noticed independently by Papapetrou in [102], *cf.* also [125] and references therein for more information about these singularities).

4. Christodoulou has proved that the collapse of spherically symmetric dust ($p=0$) can also lead to the formation of a central naked singularity [25] of “shell-focusing” type; numerical evidence suggesting that this might occur has been previously obtained by Eardley and Smarr [43] (*cf.* also [98] for more information about these singularities). Numerical results of Ori and Piran [101] suggest that central naked singularities may also form for perfect fluids with equation of state $p = \kappa\rho$, $\kappa = \text{const.}$
5. Shapiro and Teukolsky [115] have presented numerical evidence suggesting that axially symmetric collapse of the Einstein–Vlasov system may lead to formation of naked singularities, and that this behaviour is stable under axially symmetric perturbations.
6. In his recent analysis of a self-gravitating massless scalar field, Christodoulou has shown [27] that *generic initial data* (of BMO class, prescribed on an outgoing light cone, *cf.* [27] for details), do not lead to the formation of naked singularities (thus both w.c.c. and also TTBP hold for the spherically symmetric Einstein–massless scalar field problem). He has also shown that there exists a set of codimension 1 of initial data for which naked singularities will occur.

It must be stressed that all the examples above suffer from the drawback of not being realistic, and from all of them Christodoulou’s scalar field seems to be the most acceptable one. The deficiencies of the numerical results are 1) the problem of numerical uncertainties arising when approaching the singularity, and 2) the difficulty of deciding, on the basis of a numerical solution of finite space extent and obtained only up to some finite

³⁷There are some reasons to believe that the central “shell-focusing” singularities are “worse” than the shell crossing ones, *cf.* [99] for details; the status of these assertions does not seem to be clear.

value of time, whether or not the solution in question has an event horizon³⁸ (recall that an event horizon is defined in terms of asymptotic structure [66], and thus in terms of the behaviour of the solution as both the space and time variables tend to infinity). The only result above which does not assume spherical symmetry is the one due to Shapiro and Teukolsky, and it has been pointed out by Rendall [108] that the presumed occurrence of a naked singularity might be an artefact of the singular character of the initial data³⁹ assumed for the Vlasov field in [115]. To give support to his suggestion Rendall draws attention to the recent results of Pfaffelmoser [106], who has shown that the Poisson–Vlasov system (which is the Newtonian equivalent of the Einstein–Vlasov system) has global solutions for smooth data, while the numerical results of Shapiro and Teukolsky show that some solutions of the Poisson–Vlasov system with Dirac- δ -type initial data blow up in finite time.

The original motivation for the formulation of the w.c.c conjecture seems to be the expectation, that formation of singularities may lead to unpredictability, *i.e.* non-uniqueness of solutions. If w.c.c. holds, then the potential regions of non-uniqueness will be hidden beyond event horizons, and predictability in the exterior world will be saved. This attitude is similar in spirit to that of TTBP, with the difference however that TTBP insists on predictability throughout the space–time. Clearly if one adopts the attitude that the main point of w.c.c. is predictability, then a proof of TTBP will make w.c.c. irrelevant. One might of course consider that a significant part of w.c.c. is the requirement of smoothness of the metric on the event horizon, or maybe even in a neighbourhood of the event horizon, in which case w.c.c. and TTBP may contain only partially overlapping information.

It should be noted that a rather different “curvature strength” approach to the w.c.c. problem has been initiated by Królak and Newman, *cf.* [80] [99] [110].

³⁸In spherically symmetric situations with matter with a sharp imploding boundary this problem does not arise because by Birkhoff’s theorem the metric is the Schwarzschild one in the vacuum region; no such information is available if spherical symmetry is not assumed.

³⁹Shapiro and Teukolsky model the Vlasov field by a swarm of particles, which corresponds to initial data for the Vlasov function f which are a sum of Dirac δ functions.

1.5 Some answers, in spatially compact space-times with two or more spacelike Killing vector fields.

TTBP, TTBP2 and TTBP3 seem to be a formidable challenge to analysts, and it must be said that the possibility of establishing them in the near future seems to be rather remote. When facing a difficult problem it is usually believed that insight can be gained by studying it under certain restrictive assumptions, and the obvious idea is to consider space-times with Cauchy data invariant under a smooth action of a group. The ideal would be to assume the smallest possible isometry group — $U(1)$ or \mathbb{R} . Although several remarkable simplifications occur in such space-times (cf. [90] and [16]) this problem also seems to be out of reach at the time of writing this review. Let us thus take the reverse approach, and start with the largest possible isometry group. In the remainder of this Section we shall consider the cosmological vacuum space-times only, unless explicitly specified otherwise, i.e. space-times which develop out of a compact, connected, orientable Cauchy surface. All possible groups acting on such manifolds have been listed by Fischer ([45], p. 334), cf. Table 1.0.1, p. 5, let us examine Fischer's list one by one:

1. $G = SO(4)$ or $SO(3) \times SO(3)$: it follows from the equations on p. 471 of [126] that no such vacuum metrics exist;
2. $G = (U(1) \times SU(2))/D, D = \{(1, 1), (-1, -1)\}$, or $G = U(1) \times SO(3)$, acting on lens spaces (recall that S^3 is a lense space: $S^3 = L(1, 1)$): as shown in Section 2.2, the Taub-NUT metrics exhaust the space of metrics with this symmetry group, thus TTBP certainly does not hold in this "minisuperspace" of metrics: all the maximal globally hyperbolic space-times in this class are non-uniquely extendible both to the future and to the past of the Cauchy surface (cf. [36] for a detailed discussion);
3. $G = U(1) \times SO(3)$ acting on $S^1 \times S^2$: by a generalization of the Birkhoff theorem (cf. Section 2.3) all such metrics are, locally, isometric to the " $r, 2m$ " Schwarzschild metric. It follows that the maximal globally hyperbolic development is inextendible

either to the future or to the past of the Cauchy surface, because of the $r = 0$ Schwarzschild singularity, and extendible either to the past or to the future of the Cauchy surface, because of the Schwarzschild event horizon $r = 2m$; thus TTBP “half-holds” in this space of metrics;

4. $G = SU(2)$ or $SO(3)$ with three dimensional principal orbits: these are the so-called Bianchi IX space-times. In spite of the fact that these space-times have been studied by several authors (cf. [12] and references therein; cf. also [81] for some recent results) no information of the kind looked for in TTBP — TTBP2 — TTBP3 seems to be available. The general belief seems to be that all such space-times except for the Taub-NUT family are curvature singular⁴⁰ both to the future and to the past of the Cauchy surface;
5. $G = SO(3)$ with two dimensional principal orbits: as discussed in Section 2.3, the generalized Birkhoff theorem shows that these metrics are, locally, the interior Schwarzschild metric⁴¹;
6. $G = U(1) \times U(1) \times U(1)$ acting on T^3 — these are the so-called Bianchi I space-times, which we discuss in detail in the Section 2.4. The picture that emerges is the following: generic maximal space-times in this class are globally hyperbolic and contain incomplete causal geodesics, however on those some scalars constructed out of curvature blow up to infinity, which shows that generic space-times of this class are inextendible in the $C_{loc}^{1,1}$ class of metrics, and which gives a version of TTBP2. Only a “zero measure set” of Bianchi I space-times contains horizons, therefore TTBP holds (in the class of Hausdorff Lorentzian manifolds with $C_{loc}^{1,1}$

⁴⁰There is an ongoing discussion whether the Bianchi IX dynamical system is chaotic or not; and it has even been claimed [12] that the orbits of the Bianchi IX system asymptotically tend to orbits of a dynamical system which contains a strange attractor. These assertions about chaos in Bianchi IX metrics do not seem to be sufficiently justified. Let us also mention that the positive Lyapunov exponents criterion of chaos, which is sometimes claimed to be fulfilled in Bianchi IX space-times [10] [15] (cf. however [69] for an opposite point of view) seems to be irrelevant for dynamical systems on non-compact manifolds (which is the case for Bianchi IX space-times). It would be of great interest to give a mathematically rigorous analysis of whether Bianchi IX cosmologies are chaotic or not.

⁴¹This excludes the S^3 and P^3 spatial topologies listed by Fischer, cf. Section 2.3.

metrics). Nothing is known about the possibility (or the impossibility) of extending the metrics as solutions of field equations beyond the “curvature singularities”, whenever they occur, in some weaker differentiability class of space-times. Since all the Bianchi I metrics are analytic they can be analytically extended as vacuum metrics beyond the Cauchy horizons (which can always be assumed to be compact), whenever they occur; the extensions fail to be unique, as in the Taub-NUT case.

Thus, as we see, vacuum spatially compact space-times for which the dimension of the isometry group is larger than or equal to three are fairly well understood, the only case in which our knowledge is unsatisfactory being the general Bianchi IX metrics. There is no doubt that this last case deserves a careful analysis, from a dynamical point of view it seems, however, more interesting to study space-times in which the dimension of the maximal isometry group is less than or equal to two: in all the cases listed above the dynamics reduces to ODE’s rather than PDE’s, and one is left with the feeling of missing something fundamental by turning wave phenomena off. One can therefore expect that more insight can be gained by studying those space-times for which the isometry group is the next group on Fischer’s list, namely $G_2 = U(1) \times U(1)$ ⁴². In the remainder of this section we shall consider space-times which evolve from a Cauchy data set (Σ, g, K) symmetric under a G_2 action, namely:

1. Σ is a manifold which admits a differentiable action of a G_2 , and
2. (g, K) are invariant under G_2 .

It follows that in any development of (Σ, g, K) there is a neighborhood of Σ on which G_2 acts by isometries, and in which the orbits of G_2 are spacelike. Such space-times have played an important role in the development of general relativity: the cylindrically symmetric Einstein–Rosen space-times [79] provided the first non-perturbative arena to discuss gravitational radiation, the “boost–rotation” symmetric space-times were the first example of asymptotically flat radiating space-times (cf. [11] and references therein),

⁴²The only remaining possibility on the list given in [45] is $G = U(1)$.

the $\mathbb{R} \times \mathbb{R}$ symmetric plane waves of Kahn and Penrose [77] gave the first example of singularity formation without spherical symmetry, finally Gowdy [61] used $U(1) \times U(1)$ symmetric metrics to exhibit radiation phenomena in spatially closed space-times. There has been recently a renewal of interest in these space-times, because of many interesting properties and because they have not been studied with sufficient detail and rigour in the original papers. One should mention a careful reexamination of the radiative properties of the boost-rotation symmetric space-times by Bičák and Schmidt [11], a study of plane waves by Ernst and Hauser [44], and an exhaustive study of polarized Gowdy metrics by Isenberg and Moncrief [76] which we shall discuss in more detail below. It is also worth mentioning some studies of polarized Gowdy metrics in the quantum gravity context, e.g. some recent work by Husain and Smolin [73].

Before presenting in detail the results on polarized Gowdy metrics derived in Refs. [76], [37] and [35], let us recall a few facts about G_2 isometric space-times. If Σ is a compact manifold on which a topological group acts effectively, then G must be a compact Lie group. It may be shown that if G_2 is the maximal isometry group of the metric, then $G_2 = U(1) \times U(1)$ (otherwise the real isometry group of the metric would be larger). It is also a standard result that if a compact connected orientable three dimensional manifold is acted upon by $G = U(1) \times U(1)$, then $\Sigma = T^3$ or S^3 or a lens space $L(p, q)$ or $S^2 \times S^1$, and the action of G is unique up to diffeomorphisms. Metrics on $L(p, q)$ can be identified with metrics on S^3 and need not be discussed separately in the applications we shall be concerned with here. The “polarized Gowdy space-times” form a small but non-trivial subset of the set of metrics on which $U(1) \times U(1)$ acts by isometries on spacelike Cauchy surfaces. By definition for these space-times a coordinate system exists such that the metric takes the form

$$ds^2 = f(t, \theta)(-dt^2 + d\theta^2) + g_a d\theta dx^a + g(t, \theta)(dx^1)^2 + h(t, \theta)(dx^2)^2, \quad (1.5.1)$$

$$\partial_\mu g_a = 0.$$

In the S^3 or $S^2 \times S^1$ case these metrics can be characterized as all $U(1) \times U(1)$ symmetric metrics for which

$$g_{\mu\nu} X_1^\mu X_2^\nu = 0, \quad (1.5.2)$$

where $X_a, a = 1, 2$ are appropriate Killing vectors; in the T^3 case they can be characterized by (1.5.2) and the requirement $c_a = 0$, where the constants c_a are defined in (1.5.3). We have the following result [37]:

Theorem 1.5.1 (Global structure of maximally developed polarized Gowdy spacetimes) *Let $X(\Sigma) = \{(g, K) \in C^\infty(\Sigma) \times C^\infty(\Sigma)\}^{43}$, with (Σ, g, K) — Cauchy data for a polarized Gowdy metric.,*

- *Let $\Sigma \approx T^3$. There exists an open dense subset $Y(\Sigma)$ of $X(\Sigma)$ such that, for every (g, K) in $Y(\Sigma)$, there exists a globally hyperbolic Hausdorff development (M, γ) , $M \approx \mathbb{R} \times T^3$, and a time orientation of (M, γ) with the following properties:*
 1. *on every past directed inextendible timelike curve the scalar $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ uniformly tends to infinity in finite proper time (in particular (M, γ) is past inextendible⁴⁴ in the set of Hausdorff manifolds with metrics of $C_{loc}^{1,1}$ differentiability class),*
 2. *(M, γ) is future geodesically complete (in particular (M, γ) is future inextendible in the set of Hausdorff manifolds with metrics of $C_{loc}^{1,1}$ differentiability class).*
- *Let $\Sigma \approx S^3$ or $\Sigma \approx S^2 \times S^1$. There exists an open dense subset $Y(\Sigma)$ of $X(\Sigma)$, with the property that for all Cauchy data in $Y(\Sigma)$ there exists a globally hyperbolic Hausdorff development (M, γ) for which on every past directed inextendible timelike*

⁴³Several claims of this theorem hold under much weaker differentiability conditions, which we shall not discuss here because the main issue — the proof of inextendability — requires $C_{loc}^{1,1}$ differentiability, and as far as Theorem 1.5.1 is concerned no new phenomena arise when assuming some Sobolev conditions on the initial data consistent with $C_{loc}^{1,1}$ differentiability of the metric instead of C^∞ data.

⁴⁴By this we mean that in any extension M' of M the implication $(p \in M' \text{ and } p \in J^-(\Sigma)) \Rightarrow (p \in M)$ holds.

curve the Riemann tensor blows to infinity both in the future and in the past in finite proper time (in particular (M, γ) is inextendible in the set of Hausdorff manifolds with metrics of $C_{loc}^{1,1}$ differentiability class).

It should be emphasized that this theorem ensures generic maximality and uniqueness within the $C_{loc}^{1,1}$ class, not only the globally hyperbolic $C_{loc}^{1,1}$ class — the global hyperbolicity of the maximal extensions follows from the theorem. Inextendibility in a space of metrics of weaker differentiability class is not known. This theorem is “TTBP” for this class of metrics and gives some information of the kind looked for in TTBP2. It seems to be the most precise *singularity theorem* known for a reasonably large class of metrics, especially in view of the sharp estimates on the blow up rates of the Riemann tensor one can obtain from the results proved in [76]. Theorem 1.5.1 shows, that in the S^3 or $S^2 \times S^1$ case the vacuum $U(1) \times U(1)$ symmetric picture is, generically, essentially the same as in the Friedman–Robertson–Walker dust-filled cosmological model, i.e. the “big bang to big crunch” scenario holds.

In the polarized Gowdy class one can obtain [37] [35] almost exhaustive information about the extendible metrics. Let us consider the T^3 case first. It should be noted that causal future geodesic completeness (and thus future inextendability) actually holds for all polarized Gowdy spacetimes with T^3 spatial sections, so that nothing remarkable happens for large t . On the other hand several interesting phenomena occur at the boundary $t = 0$. For all Cauchy data (g, K) in a certain nonempty closed subspace of $X(\Sigma) \setminus Y(\Sigma)$ the maximal globally hyperbolic development has the property that $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is bounded on M . For such (M, γ) there exist at least two nonisometric manifolds (M_a, γ_a) , $a = 1, 2$, with C^∞ metrics in which M can be isometrically embedded in such a way that $\overline{M}_a \setminus M$ is a Cauchy horizon⁴⁵. In this case a *necessary* condition for existence of vacuum extensions (M_a, γ_a) in the polarized Gowdy class is analyticity of (M, γ) : thus if smooth Cauchy data out of which a compact Cauchy horizon develops are not analytic, there will be no vacuum polarized Gowdy extension. This result is

⁴⁵ \overline{M}_a denotes the closure of M in M_a .

not as strong as one would wish since it does not exclude vacuum extensions which are not in the polarized Gowdy class. One can also show existence of Cauchy data out of which non-compact Cauchy horizons develop: a particularly interesting class of metrics can be constructed in which the Cauchy horizon has a countable infinity of connected components, across every one of which the metric can be extended as a solution of the vacuum Einstein equations in two inequivalent ways. One obtains in this way maximal globally hyperbolic spacetimes which admit a *countable infinity* of *inequivalent* vacuum extensions [36].

In the $S^2 \times S^1$ and S^3 very similar results hold, the only essential difference is, that on S^3 no compact Cauchy horizons are possible in this class of metrics. On $S^2 \times S^1$ polarized Gowdy space-times with a compact Cauchy horizon can be shown to exist, in such a case the Cauchy horizon must lie to one side of the Cauchy surface only, i.e. if the Cauchy horizon has a compact, non-empty, connected component to, say, the future of ${}^3\Sigma$, then $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ tends to infinity uniformly on all past incomplete causal curves. Again a necessary and sufficient condition for vacuum extendability is analyticity of the Cauchy data when a compact Cauchy horizon is present. Non-compact Cauchy horizons are possible both in the S^3 and $S^2 \times S^1$ case; for both topologies there exist space-times for which the Cauchy horizon has connected components both to the future and to the past of the Cauchy surface; for both topologies the Cauchy horizon may have an infinite number of connected components, and, finally, an infinite number of inequivalent vacuum extensions is possible for a large family of polarized Gowdy space-times with S^3 or $S^2 \times S^1$ space sections.

Since polarized Gowdy metrics are of “zero measure” in the space of all Gowdy metrics, a natural question to ask is what happens if one relaxes the polarization condition. Let us recall some general features of vacuum metrics with two commuting Killing vectors $X_a, a = 1, 2$. In such a space-time one can form two scalar functions

$$c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta, \quad (1.5.3)$$

and one shows that in vacuum [58] the c_a 's must be constant. It follows immediately from this and the definition of the c_a 's that if there is a symmetry axis (i.e. a subset on which one of the Killing vectors vanishes) then the c_a 's must both vanish (since they are constant and since the right hand side of (1.5.3) vanishes on the axis, they must vanish when axes occur) if the space-time is connected (they may, of course, also vanish if no symmetry axes exist). A space-time in which the c_a 's vanish will be called a *Gowdy space-time*. In this case Einstein equations simplify considerably and it can be shown that they reduce to a system of equations which can either be interpreted as a harmonic map from a three dimensional space-time to the two-dimensional hyperboloid H_2 of constant curvature [32] [87], or a harmonic map from a two-dimensional space-time to the same target space with a "source term". (A map $\phi : M \rightarrow N$ between two Riemannian (or pseudo-Riemannian) manifolds $(M, g), (N, h)$ is called harmonic if it is a formal stationary point of the "energy" integral:

$$E(\phi) = \int_M g^{\mu\nu}(x) h_{AB}(\phi(x)) \partial_\mu \phi^A \partial_\nu \phi^B d\mu_g(x).$$

Once the harmonic map problem is solved, the components of the metric are obtained either by algebraic manipulations or by line integrals. In the T^3 problem one is left to study a simple generalization of the harmonic map equations from $\mathbb{R} \times S^1$ to H_2 when both c_a 's vanish, and we have the following results (the first part of this theorem is due to Moncrief [87]; the second is proved in Chapter 3):

Theorem 1.5.2 ($U(1) \times U(1)$ stability of the singularity of the $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner metric) *Suppose that $\Sigma \approx T^3$, set $X(\Sigma) = \{(g, K) \in H_2(\Sigma) \times H_1(\Sigma),$ with $(\Sigma, g, K) — Cauchy data for a Gowdy space-time\}$. One can choose a time orientation on Σ in such a way that:*

1. *there exists a globally hyperbolic Hausdorff future development (M, γ) of (Σ, g, K) which can be covered by a family of Cauchy hypersurfaces $\Sigma_\tau, \tau \in \mathbb{R}^+$, such that $\lim_{\tau \rightarrow \infty} d(\Sigma_\tau, \Sigma) = \infty$, where d is the Lorentzian distance between sets,*

2. there exists $\epsilon > 0$ such that for all initial data satisfying $\|(g - g_o, K - K_o)\|_X < \epsilon$ where (g_o, K_o) are Cauchy data for a Kasner metric with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, there exists a globally hyperbolic Hausdorff development of (g, K) for which on every past directed past inextendible timelike curve the scalar $|R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}|$ tends to infinity in finite proper time (thus (M, γ) is future inextendible in the class of Hausdorff manifolds with $C_{loc}^{1,1}$ metrics).

This theorem is almost the equivalent of theorem 1.5.1 for general Gowdy metrics on T^3 with small initial data⁴⁶ — future inextendability is missing, though strongly suggested by 1 above. When $\Sigma = S^3$ or $S^2 \times S^1$ additional difficulties occur because the space of orbits of the isometry group is not a manifold. These have been recently overcome in a related model by Christodoulou [28], namely for non-linear Einstein–Rosen waves⁴⁷:

Theorem 1.5.3 (Geodesic completeness of (non-linear) Einstein–Rosen space-times) *Let $\Sigma \approx \mathbb{R}^3$. For all cylindrically symmetric $(g, K) \in X(\Sigma) = \{(g, K) \in H_3^{loc}(\Sigma) \times H_2^{loc}(\Sigma), (g, K) \text{ satisfying certain decay conditions}^{48} \text{ for large } r\}$ there exists a unique globally hyperbolic, timelike and null geodesically complete (and therefore strongly maximal), Hausdorff development.*

One of the steps of the proof of Theorem 1.5.3 is to show that the constraint equations imply non-existence of trapped surfaces in Σ (cf. [32][Corollary 5.2]). This is a necessary condition for null geodesic completeness, as follows from a well-known singularity theorem due to Penrose [66]. Using the methods of proof of Theorem 1.5.2 and the results proved by Christodoulou one can via a “patching method” establish some properties of the maximally developed Gowdy space-times with topology $\mathbb{R} \times S^3$ and $\mathbb{R} \times S^1 \times S^2$, cf. [32]. The up to date known results are not as sharp as theorems 1.5.1–1.5.2.

⁴⁶One can also establish “probably generic” existence of singularities in the past without restriction on the size of the data [38], these results do not however have such a simple formulation as the ones presented here for small data.

⁴⁷It should be stressed that theorem 1.5.3 is concerned with the *nonlinear* Einstein–Rosen metrics, and *not* the “polarized” ones in which the evolution equations reduce to a linear equation.

1.6 Spatially open space-times: some small data results.

In cosmological considerations one often assumes the existence of compact spacelike hypersurfaces. However, when studying isolated gravitational systems, Σ cannot be assumed compact. On non-compact manifolds a natural class of metrics which describe isolated gravitating systems are the asymptotically flat metrics — in a sense to be made precise in Theorems 1.6.1 and 1.6.2 — and under the hypothesis of asymptotic flatness the only known results concerning global properties of “big classes” of solutions of Einstein equations are

1. the results on Robinson–Trautman space-times discussed in Section 1.7,
2. some results on analytic “boost-rotation” symmetric metrics due to Bičák and Schmidt [11], and
3. some “small data” results, which will be discussed in detail below

(recall that the Einstein–Rosen metrics discussed in Theorem 1.5.3, although “asymptotically flat” in a 2+1 dimensional sense, are not asymptotically flat in a 3+1 dimensional sense, because of translation invariance along the z -axis). The idea of “small data” results is to fix some space-time (M, γ_o) , maximal globally hyperbolic developement of some data (Σ, g_o, K_o) , and try to prove that for (g, K) sufficiently close to (g_o, K_o) in some norm the global properties of maximal developments (M, γ) of (Σ, g, K) will mimic those of (M, γ_o) . The first class of results of this type has been proved some five years ago by Friedrich [52], with or without a cosmological constant, and also for the Einstein–Yang–Mills system [53]; here we shall consider the vacuum case with zero cosmological constant only. Friedrich’s approach takes advantage of the fact that conformal transformations can map infinite domains into finite ones, reducing in this way the global in time stability problem to a much simpler short time stability problem for conformally rescaled fields. Suppose thus that a (spatially non-compact) space-time (M, γ) can be

conformally mapped into a spatially compact space-time $(\tilde{M}, \tilde{\gamma})$ (this can be done *e.g.* for the Minkowski space-time, with $(\tilde{M}, \tilde{\gamma})$ — the "Einstein cylinder", $\tilde{M} \approx \mathbb{R} \times S^3$, *cf. e.g.* [66] or [52]). After such an infinite compression the conformal factor which relates the physical metric γ to the "unphysical" metric $\tilde{\gamma}$, $\gamma_{\mu\nu} = \Omega^{-2} \tilde{\gamma}_{\mu\nu}$, will vanish on the boundary $\partial M \equiv \mathcal{I}$ of M in \tilde{M} , which introduces singular terms when one naively rewrites Einstein equations for γ in terms of $\tilde{\gamma}$ and Ω . Friedrich has observed that one can write a well posed system of equations for the conformally rescaled fields which is regular even at points at which Ω vanishes, and which is equivalent to vacuum Einstein equations on the set $\Omega > 0$ (*cf.* also [22] [23] for a different "conformally regular" system of equations), which leads to the following [52] [54]:

Theorem 1.6.1 (Future stability of the "hiperboloidal initial value problem")

Let (g_o, K_o) be the data induced on the unit hyperboloid $\Sigma = \{(t, x, y, z) \in \mathbb{R}^4 : t = \sqrt{1 + x^2 + y^2 + z^2}\} \approx \mathbb{R}^3$ from the flat metric of Minkowski space-time. Consider the space X of Cauchy data (g, K) such that

1. *(g, K) are smoothly conformally compactifiable⁴⁹, i.e. there exist a smooth compact Riemannian manifold $(\tilde{\Sigma}, \tilde{g})$ with boundary, with $\text{Int}(\tilde{\Sigma}) \approx \Sigma$, where $\text{Int}(\cdot)$ is the interior of \cdot , and a smooth (up to boundary) non-negative function Ω on $\tilde{\Sigma}$, vanishing only on $\partial\tilde{\Sigma}$, with $d\Omega(p) \neq 0$ for $p \in \partial\tilde{\Sigma}$, such that we have*

$$g_{ij} = \Omega^{-2} \tilde{g}_{ij} ,$$

and the fields

$$\tilde{L}^{ij} \equiv \Omega^{-3} (g^{ik} g^{jl} K_{kl} - \frac{1}{3} g^{\ell m} K_{\ell m} g^{ij}) , \quad \tilde{K} \equiv \Omega g^{ij} K_{ij}$$

are smooth (up to boundary) on $\tilde{\Sigma}$;

⁴⁹In [52] one assumes, roughly speaking, that $(\Omega, \tilde{g}, \tilde{L}, \tilde{K}) \in H_k(\tilde{\Sigma}) \oplus H_k(\tilde{\Sigma}) \oplus H_{k-1}(\tilde{\Sigma}) \oplus H_{k-1}(\tilde{\Sigma})$, $(\Omega_n, d_{\beta\gamma\delta}^2, f_{\alpha\beta}) \in H_{k-1}(\tilde{\Sigma}) \oplus H_{k-2}(\tilde{\Sigma}) \oplus H_{k-2}(\tilde{\Sigma})$, $k \geq 6$; it is rather clear that by not too difficult technical improvements of the existence theorems used in [52] this threshold can be relaxed to $k \geq 5$ and probably even to $k \geq 4$.

2. the Weyl tensor $C^\alpha_{\beta\gamma\delta}$ of the four-dimensional metric, formally calculated from (g, K) using vacuum Einstein equations, vanishes at the conformal boundary $\partial\tilde{\Sigma}$;
3. there exist fields Ω_n and Ω_{nn} , smooth (up to boundary) on $\tilde{\Sigma}$, which we identify with tetrad components in the directions normal to $\tilde{\Sigma}$ of the gradient, respectively the Hessian, of Ω , such that

$$(\Omega_n^2 - \tilde{g}^{ij}\Omega_i\Omega_j)|_{\partial\tilde{\Sigma}} = 0 ,$$

and the tensor field

$$e_{\alpha\beta} = \nabla_\alpha \nabla_\beta \Omega - \frac{1}{4} \tilde{\gamma}^{\mu\nu} \nabla_\mu \nabla_\nu \Omega \tilde{\gamma}_{\alpha\beta}$$

vanishes at $\partial\tilde{\Sigma}$.

Set $d^\alpha_{\beta\gamma\delta} \equiv \Omega^{-1} C^\alpha_{\beta\gamma\delta}$, $f_{\alpha\beta} = \Omega^{-1} e_{\alpha\beta}$. There exists $\epsilon > 0$ such that for all $(g, K) \in X$ satisfying

$$\begin{aligned} & \| \Omega - \Omega_o \|_{H_6(\tilde{\Sigma})} + \| \tilde{g}_{ij} - \tilde{g}_{oij} \|_{H_6(\tilde{\Sigma})} + \| \tilde{L}_{ij} - \tilde{L}_{oij} \|_{H_5(\tilde{\Sigma})} + \| \tilde{K} - \tilde{K}_o \|_{H_5(\tilde{\Sigma})} \\ & + \| d^\alpha_{\beta\gamma\delta} \|_{H_4(\tilde{\Sigma})} + \| \Omega_n - \Omega_{on} \|_{H_5(\tilde{\Sigma})} + \| f_{\alpha\beta} - f_{o\alpha\beta} \|_{H_4(\tilde{\Sigma})} < \epsilon \end{aligned}$$

(where $H_\ell(\tilde{\Sigma})$ is the Sobolev space of tensors on $\tilde{\Sigma}$ which are square integrable together with all the derivatives up to order ℓ on $\tilde{\Sigma}$ with respect to the Riemannian measure $d\mu_{\tilde{g}}$ of the metric \tilde{g} , and Ω_o , \tilde{g}_{oij} , etc., denote the corresponding quantities for Minkowski space-time), the maximal globally hyperbolic development (M, γ) is future null and timelike geodesically complete, hence (M, γ) is strongly maximal to the future.

By the very nature of Friedrich's construction (cf. the discussion of Example 4 in Section 1.8) the above theorem gives only "50 %" of TTBP – it guarantees global uniqueness to the future of Σ only, nothing is known about the possibility of supplementing the Cauchy data on Σ by Cauchy data on the part of \mathcal{I} which lies to the past of Σ to obtain global uniqueness to the past. Moreover, the following features of this Theorem deserve further investigation: 1) the rather high differentiability conditions needed for stability, 2) the hypothesis of the vanishing of the Weyl curvature on the conformal boundary $\partial\tilde{\Sigma}$

— the so called Weyl tensor condition, 3) the hypothesis of the vanishing of $e_{\alpha\beta}$ at the conformal boundary, 4) the independence of the various hypotheses above. Let us recall that the Weyl tensor condition has been shown by Penrose [104] to be necessary for C^k , $k \geq 3$, differentiability of the conformally rescaled fields at the conformal boundary of a space–time, but some new results of Christodoulou and Klainerman [29] suggest that the Weyl tensor condition needs not to hold in generic space–times obtained by evolution from asymptotically flat (at spacelike infinity) initial data (thus generic \mathcal{I} ’s obtained in this way will probably *not* be C^3). As discussed in detail in Appendix A, the Cauchy data sets satisfying the Weyl tensor condition also turn out to be *non generic* in the space of solutions of the constraint equations which can be constructed by the conformal method. These drawbacks of Friedrich’s theorem are more than compensated by the (relative) simplicity of the method. It has recently been shown in [3] that the vanishing of the space components e_{ij} of $e_{\alpha\beta}$ at $\partial\tilde{\Sigma}$ and smoothness of \tilde{g}_{ij} imply the vanishing of the Weyl tensor at $\partial\tilde{\Sigma}$, under the supplementary hypotheses that the extrinsic curvature of $\tilde{\Sigma}$ is pure trace on $\partial\tilde{\Sigma}$ ($\tilde{L}_{ij}|_{\partial\tilde{\Sigma}} = 0$), and the Cauchy surface Σ has constant extrinsic curvature ($g^{ij}K_{ij} = \text{const}$).

Friedrich’s construction uses Cauchy data which are “asymptotically flat at null infinity”. A different stability result, with Cauchy data “asymptotically flat at spatial infinity” has been proved recently [29] [30] by Christodoulou and Klainerman:

Theorem 1.6.2 (Nonlinear stability of Minkowski space–time) *Let $p \in \Sigma \approx \mathbb{R}^3$, $a > 0$, consider the quantity*

$$Q(a, p) = a^{-1} \int_{\Sigma} \left\{ \sum_{\ell=0}^1 (d_p^2 + a^2)^{\ell+1} |\nabla^{\ell} Ric|^2 + \sum_{\ell=0}^2 (d_p^2 + a^2)^{\ell} |\nabla^{\ell} K|^2 \right\} d\mu_g, \quad (1.6.1)$$

where d_p is the geodesic distance function from p , Ric is the Ricci tensor of the metric g , $d\mu_g$ is the Riemannian measure of the metric g and ∇ is the Riemannian connection of g . Let

$$Q_* = \inf_{a>0, p \in \Sigma} Q(a, p).$$

There is an $\epsilon > 0$ such that if $Q_ < \epsilon$, then the maximal globally hyperbolic development of (Σ, g, K) is geodesically complete, thus strongly maximal.*

Christodoulou and Klainerman supplement this important and extremely difficult theorem by detailed information on the asymptotic behaviour of the gravitational field in various regimes, under the hypothesis, however, of stronger than required above fall-off of the initial data. Due to lack of space, and because these results are only loosely related to the problem of uniqueness in the large, we shall not discuss them here, the reader is referred to [29] for more details. It is worthwhile noting that the condition of finiteness of $Q(a, p)$ defined in (1.6.1) will be satisfied if *e.g.* there exists a coordinate system covering the complement of a compact set such that

$$g_{ij} - \delta_{ij} = O(r^{-1/2-\epsilon}), \quad \partial_{i_1} g_{ij} = O(r^{-3/2-\epsilon}), \quad \dots, \quad \partial_{i_1} \dots \partial_{i_3} g_{ij} = O(r^{-7/2-\epsilon}),$$

$$K_{ij} = O(r^{-3/2-\epsilon}), \quad \dots, \quad \partial_{i_1} \partial_{i_2} K_{ij} = O(r^{-7/2-\epsilon}), \quad \epsilon > 0.$$

In terms of rates of decay of the metric to the flat one, these are the well known conditions for a finite and well defined ADM mass [5] [31] of the initial Cauchy data set. Under the condition that $Q(a, p)$ is finite, Christodoulou and Klainerman have also been able to establish the important fact, that the maximal globally hyperbolic development of (Σ, g, K) contains “a neighbourhood of i^∞ ”, *cf.* [29] for more details.

1.7 Beyond the spacelike Cauchy problem: Robinson–Trautman space-times.

In the standard dynamical formulation of general relativity one considers space-times which develop from Cauchy data prescribed on a spacelike hypersurface. This situation seems natural for addressing questions such as existence and uniqueness of solutions, but there may be other settings in which such questions make sense, since space-times can be constructed by various other methods, *e.g.* some solution generating techniques. A natural extension of the spacelike Cauchy problem is the characteristic initial value problem,

in which the initial data are prescribed on a null rather than spacelike hypersurface. In this context the only general result available (in the vacuum) is a local existence theorem for data prescribed on two null transversally intersecting hypersurfaces [96] [107]. The analytic initial value problem with Cauchy data given on a Cauchy horizon has been shown to be well posed in the vacuum by V. Moncrief [88] [89]. In the non-vacuum case, the characteristic initial value problem for a spherically symmetric self-gravitating scalar field has been studied by D. Christodoulou [26], the initial hypersurface being the light cone of a point. In this section we shall discuss an interesting class of metrics which evolve from singular data prescribed on a null hypersurface — the Robinson–Trautman (RT) space-times. There are several interesting features exhibited by the RT metrics: the evolution of the metric is unique in spite of a “naked singularity”; suprisingly, Einstein equations reduce to a single parabolic fourth order equation in this class of metrics. From a physical point of view the RT metrics can be thought of as representing an isolated gravitationally radiating system — in fact these metrics were the first ones to be found, describing such a situation [109]. By definition the Robinson–Trautman space-times can be foliated by a null hypersurface orthogonal shear free geodesic congruence. It has been shown by I. Robinson and A. Trautman that in such a space-time there always exists a coordinate system in which the metric takes the form

$$ds^2 = -\Phi du^2 - 2du dr + r^2 e^{2\lambda} \overset{\circ}{g}_{ab} dx^a dx^b, \quad x^a \in {}^2M, \quad \lambda = \lambda(u, x^a), \quad (1.7.1)$$

$$\overset{\circ}{g}_{ab} = \overset{\circ}{g}_{ab}(x^a), \quad \Phi = \frac{R}{2} + \frac{r}{12m} \Delta_g R - \frac{2m}{r}, \quad R = R(g_{ab}) \equiv R(e^{2\lambda} \overset{\circ}{g}_{ab}),$$

m is a constant which is related to the total Bondi mass of the metric, R is the Ricci scalar of the metric $g_{ab} \equiv e^{2\lambda} \overset{\circ}{g}_{ab}$, and $({}^2M, \overset{\circ}{g}_{ab})$ is a smooth Riemannian manifold which we shall assume to be a two dimensional sphere (other topologies are considered in [34]). The Cauchy data for an RT metric consist of $\lambda_0(x^a) \equiv \lambda(u = u_0, x^a)$, which is equivalent to prescribing the metric $g_{\mu\nu}$ of the form (1.7.1) on the null hypersurface $\{u = u_0, x^a \in {}^2M, r \in (0, \infty)\}$, which extends up to a curvature singularity at $r = 0$ (the scalar $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ diverges at $r = 0$ as r^{-6}).

The global structure of RT space-times turns out to be different, depending upon the

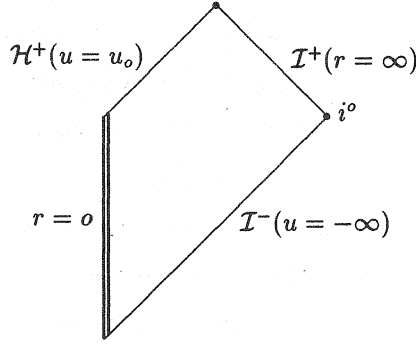


Figure 1.7.1: $m < 0$

sign of m : let us discuss the negative m case first. Recall that for $m < 0$ it is natural to consider a backwards (in u) initial value problem rather than a forwards initial value problem; alternatively one could think of u as being a retarded null coordinate ($u \sim "r + t"$) rather than an advanced one ($u \sim "r - t"$). The results of B. Schmidt [112] (cf. also [48]) and of reference [33] imply:

Theorem 1.7.1 (The global structure of negative-mass RT space-times) *For any $\lambda_0 \in C^\infty(^2M)$ there exists a unique RT space-time with a complete i^0 (in the sense of [4]), a complete \mathcal{I}^- and "a piece of \mathcal{I}^+ ", as shown in Figure 1.7.1, moreover*

1. ${}^4\mathcal{M}$ is smoothly extendible through \mathcal{H}^+ ,
2. If λ_0 is not analytic there exist no vacuum RT extension through \mathcal{H}^+ .

There are several interesting features of this result. Because of the singularity $r = 0$ in the initial data one could wonder whether any solutions of the Einstein equations would exist at all: it turns out that solutions exist either in the backwards or in the forward direction in u , depending upon the sign of m , moreover they are unique in the Robinson–Trautman class⁵⁰. Let us recall that the "weak cosmic censorship" conjecture, discussed in Section

⁵⁰It may be possible that there exist vacuum solutions with the same data at $u = u_0$ which are not in the Robinson–Trautman class.

1.4, can be formulated as the statement that the past of \mathcal{I}^+ is determined uniquely by the initial data. In the negative m case we are solving a backwards⁵¹ initial value problem, *i.e.* a “*final*” value problem, so in this context the cosmic censorship hypothesis should be reformulated as the requirement that the future of \mathcal{I}^- be determined uniquely by the “*final*” data. Theorem 1.7.1 establishes such a fact (in the RT class of vacuum metrics) for RT space-times with negative mass⁵². The information about the structure at i° is also rather interesting, though it must be said that its interest is somewhat diminished by the negativity of the ADM mass there. (It may be of some relevance to note that the limit as $u \rightarrow -\infty$ of the Bondi mass coincides with the ADM mass, as expected).

The generic non-extendability of the metric through \mathcal{H}^+ in the vacuum RT class is rather suprising, and seems to be related to a similar non-extendability result for compact non-analytic Cauchy horizons in the polarized Gowdy class, *cf.* [37]. Since it may well be possible that there exist vacuum extensions which are not in the RT class, this result does not convincingly demonstrate a failure of Einstein equations to propagate generic data forwards in u in such a situation; however, it certainly shows that the forward evolution of the metric via Einstein equations breaks down in the class of RT metrics. One can think of this as a “50%” failure of TTBP in the space of negative mass RT metrics: generic (in the sense: non-analytic initial data) RT metrics are extendible but *not RT vacuum extendible* beyond \mathcal{H}^+ , and thus in the terminology of Section 1.1 the space-time of Figure 1.7.1 is *maximal* but not *strongly maximal* in the RT class.

In the positive mass case the results of [112] [124] [33] [34] [39] show the following:

Theorem 1.7.2 (The global structure of positive-mass RT space-times) *For any $\lambda_0 \in C^\infty(S^2)$ there exists a Robinson–Trautman space-time $(^4M, g)$ with a “half-complete”*

⁵¹One can of course put everything upside down, changing u to $-u$, which then becomes an “advanced null coordinate” rather than a retarded one, and what was a backwards initial value problem becomes a standard one.

⁵²In this formulation of weak cosmic censorship (w.c.c.) no mention is made of horizons. One should recall that in the usual form of w.c.c. one “hides” singularities under a horizon to “hide” the regions of potential non-uniqueness of solutions. In the RT case uniqueness holds regardless of “nakedness” of the singularity $r = 0$.

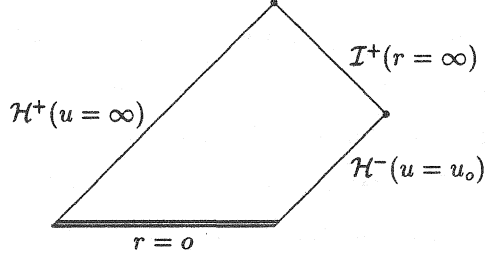


Figure 1.7.2: $m > 0$.

\mathcal{I}^+ , the global structure of which is shown in Figure 1.7.2, moreover

1. $(^4\mathcal{M}, g)$ is smoothly extendible to the past through \mathcal{H}^- , if however λ_0 is not analytic no vacuum Robinson–Trautman extensions through \mathcal{H}^- exist.
2. There exist an infinite number of non-isometric vacuum Robinson–Trautman C^5 extensions⁵³ of $(^4\mathcal{M}, g)$ through \mathcal{H}^+ , which are obtained by gluing to $(^4\mathcal{M}, g)$ any other positive mass Robinson–Trautman spacetime, as shown in Figure 1.7.3.
3. There exist an infinite number of C^{117} vacuum RT extensions of $(^4\mathcal{M}, g)$ through \mathcal{H}^+ — one such extension can be obtained by gluing a copy of $(^4\mathcal{M}, g)$ to itself, as shown in Figure 1.7.3.
4. For any $6 \leq k \leq \infty$ there exists an open set \mathcal{O}_k of Robinson–Trautman space-times (in a C^k topology on the set of RT Cauchy data on a hypersurface $u_0 = 0$) for which no C^{123} extensions beyond \mathcal{H}^+ exist, vacuum or otherwise. For any u_0 there exists an open ball \mathcal{B}_k around the Cauchy data for the Schwarzschild metric such that $\mathcal{O}_k \cap \mathcal{B}_k$ is dense in \mathcal{B}_k .

The picture that emerges from Theorem 1.7.2 is the following: generic initial data lead to a space-time which has no RT vacuum extension to the past of the initial surface, even

⁵³By this we mean that the metric can be C^5 extended beyond \mathcal{H}^+ ; the extension can actually be chosen to be of $C^{5,\alpha}$ differentiability class, for any $\alpha < 1$.

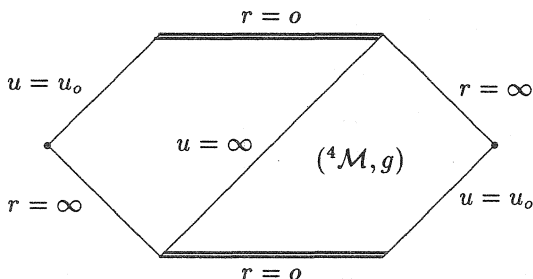


Figure 1.7.3: Vacuum RT extensions beyond \mathcal{H}^+

though the metric can be smoothly extended (in the non-vacuum class); and generic “small⁵⁴ initial data” lead to a space-time for which no smooth vacuum RT extensions exist beyond \mathcal{H}^+ . This shows that considering smooth extensions across \mathcal{H}^+ leads to non-existence, and giving up the requirement of smoothness of extensions beyond \mathcal{H}^+ leads to non-uniqueness. It follows that TTBP completely fails in the class of positive mass Robinson–Trautman metrics. It should be recognized that this might be thought of as demonstrating only some pathological aspects of the Robinson–Trautman conditions, rather than some real features of the theory.

1.8 Necessary conditions for uniqueness in the large.

In this section we shall discuss, by means of examples, some natural restrictions which one may wish to impose on the spaces $X(\Sigma)$ and $\mathcal{M}(\Sigma)$ introduced in TTBP.

Example 1: Let (Σ, g, K) be Cauchy data for a cylindrically symmetric polarized metric:

$$ds^2 = e^{2(U-A)}(-dt^2 + dr^2) + e^{-2U}r^2 d\phi^2 + e^{2U}dz^2 \quad (1.8.1)$$

$$U = U(t, r), \quad A = A(t, r),$$

$$\Sigma = \{t = 0\} \approx \mathbb{R}^3.$$

⁵⁴It is rather clear from the results of [39] that generic RT space-times will not be smoothly extendible across \mathcal{H}^+ , without any restrictions on the “size” of the initial data; but no rigorous proof is available.

For metrics of the form (1.8.1) vacuum Einstein equations essentially reduce to a single linear wave equation (in the flat Minkowski metric) for U ,

$$\square U = \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right] U = 0, \quad (1.8.2)$$

$$x = r \cos \phi, \quad y = r \sin \phi,$$

and the Cauchy data reduce to Cauchy data for $U : \varphi = \varphi(r) = U(0, r)$, $\psi = \psi(r) = \frac{\partial U}{\partial t}(0, r)$. (Given any solution U of the wave equation (1.8.2), the function A appearing in the metric can be found by elementary integrations, cf. e.g. Section 3.5.).

Let $\rho = \rho(t, r) \in \mathcal{D}'(\mathbb{R}^3)$ be a distribution on \mathbb{R}^3 , such that $\text{supp } \rho \cap \tilde{\Sigma} = \emptyset$, where $\tilde{\Sigma} = \{(t, x, y) \in \mathbb{R}^3 : t = 0\}$, define U_ρ as the unique solution of the problem

$$\square U_\rho = \rho, \quad (1.8.3)$$

$$U_\rho(t = 0) = \varphi (= U(t = 0)), \quad \frac{\partial U_\rho}{\partial t}(t = 0) = \psi = \left(\frac{\partial U}{\partial t}(t = 0) \right).$$

(U_ρ will exist if e.g. $\rho \in H_m(\mathbb{R}^3)$, $(\varphi, \psi) \in H_k(\tilde{\Sigma}) \oplus H_{k-1}(\tilde{\Sigma})$, for any $m, k \in \mathbb{R}$ (in particular if $f \in C^0(\mathbb{R})$), then the distribution ρ given by

$$\rho = f(t) \delta_0 \in H_m(\mathbb{R}^3), \quad m < -3/2 \quad (1.8.4)$$

is an allowed distribution.) Let M_ρ be the interior of $\mathbb{R}^4 \setminus \{(t, x, y, z) : \rho(t, x, y) \neq 0\}$, let γ_ρ be the metric (1.8.1) with $U = U_\rho$, and an appropriate A — the family (M_ρ, γ_ρ) is thus a family of vacuum space-times parametrized by the set of functions $\rho \in \mathcal{D}'(\mathbb{R}^3)$ subject to the restrictions above, each member (M_ρ, γ_ρ) being a vacuum development of (Σ, g, K) , and it is easy to see that for different ρ 's one will in general obtain non-isometric space-times.

Obviously the non-uniqueness described here arises from the fact that we have put some “matter” ρ in our space-time, and pretended “it is not there” by removing from our manifold the regions where matter was present. This example shows that in order to achieve any kind of uniqueness it is natural to consider these developments (M, γ) only, which contain as a subset the maximal globally hyperbolic development (M_0, γ_0) of the

data, in other words, that there exists an isometric embedding of (M_0, γ_0) into (M, γ) . Such a restriction, when imposed in the example above, would exclude all the M_ρ 's except for the space-time obtained by solving (1.8.3) with $\rho = 0$.

The space-times (M_ρ, γ_ρ) obtained from ρ of the form (1.8.4) provide a family of examples which suggest that non-uniqueness of solutions might arise in space-times with naked singularities. As is shown in Appendix E, given a smooth $f(t)$ such that $0 \notin \text{supp } f$, and smooth φ and ψ , there exists a unique solution U_ρ of (1.8.3) with ρ given by (1.8.4) which is smooth on $\mathbb{R}^3 \setminus \text{supp } \rho$. If we set $\text{supp } f$ to be, say, the interval $[1, 2]$, and we vary f , we will obtain an infinite dimensional family of non-isometric space-times with a “naked singularity sitting on the set” $\{t \in [1, 2], x = y = 0\}$. The arbitrariness of f represents an arbitrariness introduced by the singularity, thus f can be thought of as a “boundary condition at the singularity”. All these space-times are of course excluded by the criterion that we consider only these developments which are “at least as large as the maximal globally hyperbolic development”, they seem however to indicate that the occurrence of real singularities might lead to behaviour which is difficult to control.

Example 2: Let (Σ, g, K) be the initial data for Minkowski space-time on an open unit ball: $\Sigma = B(1) \subset \mathbb{R}^3$, $g_{ij} = \delta_{ij}$, $K_{ij} = 0$. As has been shown by Bartnik [8] (Σ, g, K) may be extended in an infinite number of ways to a Cauchy data set⁵⁵ $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$, $\tilde{\Sigma} = \mathbb{R}^3$. The maximal globally hyperbolic development (M, γ) of (Σ, g, K) is the set $\{-1 < t < 1, 0 \leq r < 1 - |t|\} \subset \mathbb{R}^4$ with the Minkowski metric, and any globally hyperbolic development $(\tilde{M}, \tilde{\gamma})$ of $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will provide an extension of (M, γ) . This example shows that in TTBP it is natural to restrict our attention to *inextendible* Cauchy data sets (Σ, g, K) — such a condition would exclude the behaviour described there.

There are at least two ways for a Cauchy data set (Σ, g, K) to be inextendible: one is to assume that (Σ, g) is complete, another possibility is the occurrence of a singularity at “what would have been $\partial\Sigma$ ”, let us consider the latter first:

⁵⁵ $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ can even be chosen to be asymptotically flat.

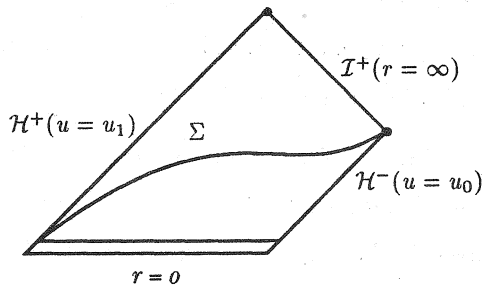


Figure 1.8.1: ${}^2M \approx S^2$, $m > 0$.

Example 3: Fix $u_0 < u_1 \leq \infty$, consider some smooth space-like hypersurface $\Sigma \subset [u_0, u_1] \times \mathbb{R} \times S^2$, as shown in Figure 1.8.1, let (g, K) be the data induced on Σ for a Robinson–Trautman space-time obtained by prescribing some smooth function $\lambda \in C^\infty(S^2)$ at $u = u_0$ (cf. Section 1.7). The hypersurface Σ is inextendible through its “left corner”, Figure 1.8.1 because of the singularity at $r = 0$ of Robinson–Trautman metrics. Theorem 1.7.2 shows that there exist an infinite number of $C^{1,7}$ extensions to the future of Σ of the maximal globally hyperbolic development of (Σ, g, K) , which is, in Robinson–Trautman coordinates (u, r, θ, φ) , the set $(u_0, u_1) \times \mathbb{R} \times S^2$ (note that if $u_1 < \infty$ then there exists a neighbourhood of the hypersurface $\{u = u_1\}$ in which the metric is uniquely defined in the vacuum Robinson–Trautman class by (Σ, g, K) , this fails, however, at the horizon $\mathcal{H}^+ = \{“u = \infty”\}$). This example suggests that in TTBP it may not be possible to allow for Cauchy data (Σ, g, K) which are inextendible through “ $\partial\Sigma$ ” because of “singularities sitting on $\partial\Sigma$ ”. It should be, however, pointed out that although this behaviour is generic in the class of Robinson–Trautman space-times, the Robinson–Trautman space-times themselves are not generic, and it cannot be excluded that this kind of non-uniqueness might disappear in generic situations.

Example 4: Let (Σ, g, K) be “hyperboloidal initial data”, as described in Appendix A, in particular (Σ, g) is a complete Riemannian manifold; suppose moreover that (Σ, g, K) is “smoothly conformally compactifiable” and that the hypotheses of Theorem 1.6.1 hold

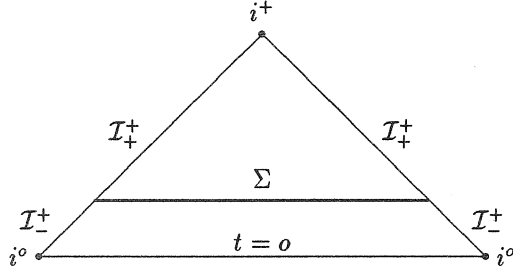


Figure 1.8.2: A “hyperboloidal” initial data surface.

(*cf. e.g.* [3], or Section 1.6 and Appendix A for more details). We can choose the time orientation in such a way that the maximal vacuum globally hyperbolic development (M, γ) of (Σ, g, K) contains at least “a piece” \mathcal{I}_+^+ of \mathcal{I}^+ , where \mathcal{I}_+^+ is the part of \mathcal{I}^+ to the future of Σ , *cf.* Figure 1.8.2 (*cf.* Theorem 1.6.1 and [52]). Using *e.g.* the techniques of Ref. [107] one can show⁵⁶ that supplementing (Σ, g, K) by appropriate smooth data on \mathcal{I}_-^+ — the part of \mathcal{I}^+ in the past of Σ — one can find a vacuum metric on a neighbourhood \mathcal{O} of $\Sigma \cup \mathcal{I}_-^+$. There is arbitrariness in the choice of the “missing data on \mathcal{I}_-^+ ”, and different data⁵⁷ will lead to non-isometric extension of (M, γ) to the past of Σ . This example shows that even the requirement of completeness of (Σ, g) is not sufficient in TTBP.

Let us close this section by emphasizing that it seems natural to require that

- The space $\mathcal{M}(\Sigma)$ of Lorentzian manifolds introduced in TTBP should contain only those developments (M, γ) of (Σ, g, K) which contain the unique maximal globally hyperbolic Hausdorff development of (Σ, g, K) ;
- if Σ is compact, then all the metrics in $X(\Sigma)$ should be complete;
- for non-compact Σ , all metrics in $X(\Sigma)$ should be complete and *e.g.* asymptotically

⁵⁶H. Friedrich, private communication.

⁵⁷By choosing (Σ, g, K) to be the data induced on the standard hyperboloid in Minkowski space-time one can by this method construct a curious space-time which is the Minkowski space-time to the future of a hyperboloid, and not-Minkowski to its past.

flat in the sense of Theorem 1.6.2.

Chapter 2

“Highly symmetric” space-times

In this chapter we shall describe the spaces of maximal globally hyperbolic (Hausdorff) spaces-times for which the groups $SO(3) \times U(1)$, $SO(3)$ (two-dimensional principal orbits) and $U(1) \times U(1) \times U(1)$ act by isometries on some compact, connected and orientable Cauchy surface. We start by proving the well known result, that symmetries of Cauchy data lead to symmetries of the space-time:

2.1 From symmetric Cauchy data to symmetric space-times.

In this Section we shall show that the existence of symmetries of Cauchy data implies the existence of symmetries of the space-time. Let us start with a “Killing vector approach” to this problem. The result that follows is essentially due to Moncrief [86] [Sections IV, V], we present the proof here for completeness (the proof below also seems to be somewhat simpler than the one in [86]; a similar proof can be found in [47]):

Theorem 2.1.1 *Let (M, γ) be a vacuum globally hyperbolic space-time, with a time function t , let $\Sigma_\tau = \{p \in M : t(p) = \tau\}$. Let \mathring{X}^μ be a vector field on M defined in a neighbourhood of Σ_0 such that for $p \in \Sigma_0$ we have*

$$\left(\nabla_\alpha \mathring{X}_\beta + \nabla_\beta \mathring{X}_\alpha \right) (p) = 0, \quad (2.1.1)$$

$$\nabla_\sigma \left(\nabla_\alpha \dot{X}_\beta + \nabla_\beta \dot{X}_\alpha \right) (p) = 0, \quad (2.1.2)$$

where ∇ is the covariant derivative of the metric γ . There exists a vector field X on M satisfying

$$\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0, \quad (2.1.3)$$

$$p \in \Sigma_0 \quad X_\alpha(p) = \dot{X}_\alpha(p), \quad \nabla_\alpha X_\beta(p) = \nabla_\alpha \dot{X}_\beta(p). \quad (2.1.4)$$

Proof. Let X^α be the unique solution of

$$\square X^\alpha = 0 \quad (2.1.5)$$

satisfying (2.1.4). (Because (M, γ) is globally hyperbolic, a solution of (2.1.4)–(2.1.5) will exist on M , if *e.g.*, in local coordinates, we have $\partial_{\alpha_1} \dots \partial_{\alpha_i} \gamma_{\mu\nu} \in L_{\text{loc}}^\infty(\Sigma_t)$, $0 \leq i \leq k+2$, $k \in \mathbb{N} \cup \{0\}$, where $L_{\text{loc}}^\infty(\Sigma_t)$ denotes the space of functions defined almost everywhere on Σ_t which are measurable and essentially bounded on every compact subset of Σ_t , and if $\partial_{i_1} \dots \partial_{i_\ell} \dot{X}_\alpha|_{\Sigma_0} \in L_{\text{loc}}^2(\Sigma_0)$, $0 \leq \ell \leq k+1$, $\partial_{i_1} \dots \partial_{i_j} (\partial_t \dot{X}_\alpha)|_{\Sigma_0} \in L_{\text{loc}}^2(\Sigma_0)$, $0 \leq j \leq k$; under these conditions we will have $(\partial_{\alpha_1} \dots \partial_{\alpha_i} X_\alpha)|_{\Sigma_t} \in L_{\text{loc}}^2(\Sigma_t)$ for all t , $0 \leq i \leq k+1$). Let us note that (as is well known) it follows from (2.1.3) that

$$\nabla_\alpha \nabla_\beta X_\gamma = R_{\lambda\alpha\beta\gamma} X^\lambda, \quad (2.1.6)$$

so that (2.1.5) is a necessary condition for (2.1.3) to hold in a vacuum space-time. We shall show that (2.1.1), (2.1.2), (2.1.4) and (2.1.5) and the fact that (M, γ) in vacuum imply (2.1.3). Set

$$A_{\alpha\beta} = \nabla_\alpha X_\beta + \nabla_\beta X_\alpha.$$

From (2.1.5) and from

$$\nabla^\lambda R_{\alpha\beta\gamma\lambda} = 0$$

one obtains

$$\square A_{\alpha\beta} + 2 R_{\lambda\beta\gamma\alpha} A^{\gamma\lambda} = 0. \quad (2.1.7)$$

Under the conditions on the metric and on \dot{X}_α outlined above with $k \geq 1$ (which will hold if *e.g.* $\gamma_{\mu\nu}$ and \dot{X}^μ are smooth) it follows that every solution of (2.1.7) with zero

initial data vanishes identically; and the vanishing of the initial data for $A_{\alpha\beta}$ follows from (2.1.1)–(2.1.2). \square

Corollary 2.1.1 *Under the hypotheses of Theorem 2.1.1, suppose that on Σ_0 there exists a smooth vector field Y such that*

$$\mathcal{L}_Y g = \mathcal{L}_Y K = 0,$$

where \mathcal{L}_Y denotes a Lie derivative. There exists a smooth vector field X on M such that

$$\mathcal{L}_X \gamma = 0,$$

$$p \in \Sigma_0 \quad X(p) = i_{\Sigma_0*} Y,$$

where i_{Σ_0} is the embedding of Σ_0 in M .

Proof. Let us rewrite eqs. (2.1.1)–(2.1.2) in 3+1 notation; let n^α be the unit normal to the slicing Σ_τ ($n^\alpha n_\alpha = -1$), define

$$\beta_\sigma = n^\alpha \nabla_\alpha n_\sigma, \quad g_{\mu\nu} = \gamma_{\mu\nu} + n_\mu n_\nu, \quad K_{\mu\nu} = g_\mu^\alpha g_\nu^\beta \nabla_\alpha n_\beta,$$

$$x = -X^\alpha n_\alpha, \quad Y^\alpha = g^\alpha_\beta X^\beta \quad (\implies X^\alpha = x n^\alpha + Y^\alpha),$$

$$D_n x = n^\mu \nabla_\mu x, \quad D_n Y^\alpha = g^\alpha_\beta n^\mu \nabla_\mu Y^\beta.$$

Using this notation, (2.1.1) can be rewritten in the form

$$\mathcal{L}_Y g_{\alpha\beta} = -2x K_{\alpha\beta}, \tag{2.1.8}$$

$$D_n x = -\beta_\sigma Y^\sigma, \tag{2.1.9}$$

$$D_n Y^\alpha = K^\alpha_\beta Y^\beta + D^\alpha x - x \beta^\alpha, \tag{2.1.10}$$

where D is the covariant derivative operator of the Riemannian metric $g_{\alpha\beta}$ induced by $\gamma_{\alpha\beta}$ on Σ_τ . When (2.1.8)–(2.1.10) hold, one finds that 1) the equations obtained by projecting all indices in (2.1.2) on Σ_τ are identically satisfied, 2) the equations obtained from (2.1.2) by projecting one of the indices along n and the remaining on Σ_τ are equivalent to

$$\mathcal{L}_Y K_{\alpha\beta} + D_\alpha D_\beta x = \left({}^3R_{\alpha\beta} + K K_{\alpha\beta} - K_{\alpha\mu} K_\beta{}^\mu \right) x, \tag{2.1.11}$$

where $K = g^{\alpha\beta} K_{\alpha\beta}$, and ${}^3R_{\alpha\beta}$ is the Ricci curvature of the metric $g_{\alpha\beta}$, 3) and finally the equations obtained from (2.1.2) by projecting more than one index along n^μ involve second time derivatives of x, Y^α in a way consistent with eq. (2.1.5) if (2.1.8)–(2.1.11) hold. If (g, K) are invariant under the flow of a vector field Y , then we can set $X^\mu(0, \cdot) = Y^\mu(\cdot)$, $(x(0, \cdot) = -X^\mu n_\mu(0, \cdot) = 0)$ and use (2.1.9)–(2.1.10) to determine $n^\mu \nabla_\mu X^\alpha(0, \cdot)$, obtaining thus Cauchy data for equation (2.1.5), which satisfy those of the equations (2.1.1)–(2.1.2) which do not contain second time derivatives of X^α . Solving (2.1.5) we will obtain a Killing vector field on any globally hyperbolic development of $({}^3\Sigma, g, K)$. \square

When considering symmetric data sets, it is natural to ask the following:

1. are discrete symmetries of $({}^3\Sigma, g, K)$ preserved under evolution?
2. suppose that we have a group G acting on ${}^3\Sigma$, which leaves both g and K invariant; can we define an action of G on some development of $({}^3\Sigma, g, K)$?

The argument that follows answers both of these questions, at least when ${}^3\Sigma$ is compact:

Theorem 2.1.2 *Let $({}^3\Sigma, g, K)$ be a smooth Cauchy data set, with ${}^3\Sigma$ — compact, suppose that a group G acts smoothly on ${}^3\Sigma$*

$$G \times {}^3\Sigma \ni (g, p) \longrightarrow \phi_g(p) \in {}^3\Sigma,$$

and we have

$$(\phi_g^* g_{ij}, \phi_g^* K_{ij}) = (g_{ij}, K_{ij}).$$

For any development (M, γ) of $({}^3\Sigma, g, K)$ there exists a (globally hyperbolic) neighbourhood $\mathcal{O} \subset \mathcal{M}$ of ${}^3\Sigma$, and an action Ψ of the group G on \mathcal{O} ,

$$G \times \mathcal{O} \ni (g, p) \longrightarrow \Psi_g(p) \in \mathcal{O},$$

such that

$$\Psi_g^* \gamma = \gamma.$$

Moreover, there exists a diffeomorphism $\psi : \mathcal{O} \leftrightarrow (-\tau, \tau) \times^3 \Sigma$ such that we have

$$\Psi_g = \Psi \circ \Phi_g \circ \Psi^{-1},$$

where Φ_g is the following action of G on $(-\tau, \tau) \times^3 \Sigma$:

$$(-\tau, \tau) \times^3 \Sigma \ni (t, p) \mapsto \Phi_g(t, p) = (t, \phi_g(p)), \quad (2.1.12)$$

and the hypersurfaces $\{t\} \times^3 \Sigma$, $t \in (-\tau, \tau)$, are space-like.

Remark. Theorem 2.1.2 will still hold if $(g_{ij} K_{ij})(0, \cdot) \in H_k(\Sigma_0) \oplus H_{k-1}(\Sigma_0)$, $k \geq 4$. One would expect the result to be true under the condition $k > 5/2$, or, say $k \geq 3$, the proof presented here, however, fails if $k = 3$. This is due to the fact that the differentiability of the map Ψ constructed below is not better than this of γ , which in turn leads to a differentiability class of $\tilde{\gamma} \equiv (\Psi^{-1})^* \gamma$ worse by one as compared to γ . If $k = 3$ the differentiability of $\tilde{\gamma}$ is not high enough to guarantee uniqueness of solutions, and the argument breaks down.

Proof. Let $\mathcal{D}({}^3\Sigma)$ be the domain of dependence of ${}^3\Sigma$ in (\mathcal{M}, γ) ; replacing \mathcal{M} by $\mathcal{D}({}^3\Sigma)$ if necessary we may assume that $\mathcal{M} = \mathcal{D}({}^3\Sigma)$, and thus \mathcal{M} is globally hyperbolic. Let t be the unique solution of the problem

$$\square_\gamma t = 0, \quad (2.1.13)$$

$$t|_{{}^3\Sigma} = 0, \quad n^\mu \partial_\mu t|_{{}^3\Sigma} = 1, \quad (2.1.14)$$

where \square_γ is the scalar wave operator of the metric γ , $\square_\gamma = \nabla^\alpha \nabla_\alpha$. Let \mathcal{O} be any neighbourhood of ${}^3\Sigma$ on which $\gamma_{\mu\nu} \nabla^\mu t \nabla^\nu t < 0$, by compactness of ${}^3\Sigma$ there exists $\sigma > 0$ such that $M_\sigma = \{p \in M : t(p) < \sigma\} \subset \mathcal{O}$. Let Σ_t denote the level sets of t , we can use the integral curves of ∇t to identify M_σ with $(-\sigma, \sigma) \times \Sigma_0$. Let $\overset{\circ}{g}$ be any smooth Riemannian G -invariant metric on ${}^3\Sigma$, on M_σ we can define the Lorentzian metric

$$\overset{\circ}{\gamma} = \overset{\circ}{\gamma}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \overset{\circ}{g}.$$

Note that the action (2.1.12) preserves $\overset{\circ}{\gamma}$:

$$\Phi_g^* \overset{\circ}{\gamma} = \overset{\circ}{\gamma}. \quad (2.1.15)$$

Consider the following initial value problem for a map $\Psi : ((-\tau, \tau) \times^3 \Sigma, \gamma) \rightarrow ((-\sigma, \sigma) \times^3 \Sigma, \dot{\gamma})$,

$$\square(\gamma, \dot{\gamma}) \Psi = 0, \quad (2.1.16)$$

$$\Psi(t = 0, \cdot) = id_{\Sigma}(\cdot), \quad n^\mu \frac{\partial \Psi^\alpha}{\partial x^\mu}(t = 0, \cdot) = \delta_0^\alpha, \quad (2.1.17)$$

where $\square(\gamma, \dot{\gamma})$ is the Lorentzian harmonic map operator; in local coordinates

$$\square(\gamma, \dot{\gamma}) \Psi^\gamma = \gamma^{\alpha\beta} \left(\partial_\alpha \partial_\beta \Psi^\gamma - \Gamma_{\alpha\beta}^\lambda(\gamma) \frac{\partial \Psi^\gamma}{\partial x^\lambda} + \Gamma_{\mu\nu}^\lambda(\dot{\gamma}) \frac{\partial \Psi^\mu}{\partial x^\alpha} \frac{\partial \Psi^\nu}{\partial x^\beta} \right), \quad (2.1.18)$$

where $\Gamma_{\beta\gamma}^\alpha(h)$ denotes the Christoffel symbols of a metric h . There exists $\tau > 0$ such that there exists a unique smooth¹ solution of the problem (2.1.16)–(2.1.17) defined on $(-\tau, \tau) \times^3 \Sigma$. Note that from (2.1.13) and (2.1.14) we have

$$\gamma^{\alpha\beta} \Gamma_{\alpha\beta}^t(\gamma) = 0, \quad \Gamma_{\alpha\beta}^t(\dot{\gamma}) = \Gamma_{t\beta}^\alpha(\dot{\gamma}) = 0,$$

and uniqueness of solutions of (2.1.16) implies $\psi^0 = t$. This shows that, decreasing τ if necessary, there exist diffeomorphisms $\psi_t :^3 \Sigma \leftrightarrow^3 \Sigma$ such that

$$(-\tau, \tau) \times^3 \Sigma \ni (t, p) \leftrightarrow \Psi(t, p) = (t, \psi_t(p)),$$

and from (2.1.17) we have

$$\psi_0 = id_{\Sigma}, \quad n^\mu \frac{\partial \psi_t}{\partial x^\mu} \Big|_{t=0} = 0. \quad (2.1.19)$$

On M_τ we can define an action of the group G as follows:

$$g \in G, \quad \Psi_g = \Psi^{-1} \circ \Phi_g \circ \Psi.$$

The claim that the maps Ψ_g are isometries of γ is equivalent to the statement that the maps Φ_g are isometries of the metric $\tilde{\gamma} \equiv (\Psi^{-1})^* \gamma$. The covariance of the equation (2.1.18) under changes of coordinates in the source space implies that the identity map

$$id_{M_\tau} : ((-\tau, \tau) \times^3 \Sigma, \tilde{\gamma}) \leftrightarrow ((-\tau, \tau) \times^3 \Sigma, \dot{\gamma})$$

¹It is not too difficult to show, using *e.g.* the methods of [20], that if $\partial_{\alpha_1} \dots \partial_{\alpha_j} \gamma_{\mu\nu}(t, \cdot) \in H_{k-j}^{\text{loc}}(^3\Sigma_t)$, $0 \leq j \leq k$, $k \geq 3$, then there will exist a solution of (2.1.16)–(2.1.17) satisfying $\partial_{\alpha_1} \dots \partial_{\alpha_j} \Psi^\mu(t, \cdot) \in H_{k-j}^{\text{loc}}(^3\Sigma_t)$.

also satisfies (2.1.18) which in local coordinates reads

$$id_{\mathcal{M}_\tau}^\alpha(x^\mu) = x^\alpha \implies$$

$$A^\sigma \equiv \square(\tilde{\gamma}, \overset{\circ}{\gamma}) id_{\mathcal{M}_\tau}^\sigma = \tilde{\gamma}^{\alpha\beta} \left(\Gamma_{\alpha\beta}^\sigma(\overset{\circ}{\gamma}) - \Gamma_{\alpha\beta}^\sigma(\tilde{\gamma}) \right) = 0. \quad (2.1.20)$$

As is well known (*cf. e.g.* [20]) Einstein equations with conditions (2.1.18) are a well posed hyperbolic system for the metric $\tilde{\gamma}$, the solutions being determined uniquely by the initial data (\tilde{g}, \tilde{K}) , with (\tilde{g}, \tilde{K}) — obtained by appropriately transforming (g, K) . In our case, (2.1.19) implies $(g, K) = (\tilde{g}, \tilde{K})$. Consider now the metric $\tilde{\gamma}_g = \Phi_g^* \tilde{\gamma}$. $\tilde{\gamma}_g$ satisfies vacuum Einstein equations, and by coordinate-invariance of (2.1.20) under a simultaneous change of coordinates for both the metric $\overset{\circ}{\gamma}$ and $\tilde{\gamma}$ it follows that A^γ defined by (2.1.20) satisfies $A^\gamma = 0 = \tilde{\gamma}_g^{\alpha\beta} \left(\Gamma_{\alpha\beta}^\gamma(\overset{\circ}{\gamma}) - \Gamma_{\alpha\beta}^\gamma(\tilde{\gamma}_g) \right)$ (and we have used $\Phi_g^* \overset{\circ}{\gamma} = \overset{\circ}{\gamma}$). The initial data for $\tilde{\gamma}_g$ are given by

$$(\tilde{g}_g, \tilde{K}_g) = (\phi_g^* \tilde{g}, \phi_g^* \tilde{K}) = (\phi_g^* g, \phi_g^* K) = (g, K) = (\tilde{g}, \tilde{K}),$$

and uniqueness implies

$$\Phi_g^* \tilde{\gamma} = \tilde{\gamma} \implies \Psi_g^* \gamma = \gamma.$$

□

It must be stressed that while the Killing vectors argument, Theorem 2.1.1, proves the existence of Killing vector fields defined on the whole of (M, γ) , provided (M, γ) is globally hyperbolic, Theorem 2.1.2 shows only existence of a neighbourhood of ${}^3\Sigma$ on which the group G acts. It is easily seen that the hypothesis of compactness of Σ is necessary: consider the space-time $M = \{(t, x) \in \mathbb{R}^2 : |t| < f(x)\}$ with the metric $\gamma_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2$, where $0 < f$ is a differentiable function such that $|df/dx| < 1$, and $\lim_{|x| \rightarrow \infty} f(x) = 0$. (M, γ) is globally hyperbolic, the translations in x are isometries of the Cauchy data at $t = 0$, but the action does not extend to M . Similarly it can be seen that for spatially compact space-times the action of the symmetry group given by Theorem 2.1.2 needs not to extend beyond a neighbourhood of the Cauchy surface: consider the space-time $M = \{(t, x) \in \mathbb{R} \times S^1 : |t| < 1 + (\sin x)/2\}$, with the metric

$-dt^2 + dx^2$, and we have identified S^1 with $[0, 2\pi]_{\text{mod } 2\pi}$. The translations in x do not extend beyond the strip $\{|t| < 1/2\}$.

It is a remarkable fact, that isometries of Cauchy data always extend to the maximal globally hyperbolic developments, regardless of whether the Cauchy surface is compact or not. Before proving this, let us restate the Choquet–Bruhat — Geroch theorem 1.1.2 in a form more suitable for our further applications:

Theorem 2.1.3 *Let (Σ, g, K) be a Cauchy data set, where Σ is a Hausdorff manifold and $g, K \in C^\infty(\Sigma)$. There exists a C^∞ , vacuum, Hausdorff, globally hyperbolic development (M, γ, i) of (Σ, g, K) such that for every smooth, Hausdorff, globally hyperbolic development $(\tilde{M}, \tilde{\gamma}, \tilde{i})$ of (Σ, g, K) there exists an isometric embedding $\Psi : \tilde{M} \rightarrow M$ satisfying*

$$\Psi \circ \tilde{i} = i .$$

Any developement (M, γ) satisfying the above will be called a *maximal globally hyperbolic developement*. It is clear from the above that maximal developements are unique, up to isometry, and inextendible in the class of smooth, Hausdorff, globally hyperbolic spacetimes. We shall also need the following:

Lemma 2.1.1 *Let (M, γ) be a smooth, Hausdorff, connected Lorentzian manifold, let $\Psi : M \rightarrow M$ be a smooth map such that*

$$\Psi^* \gamma = \gamma , \quad \Psi|_S = id \quad (S \neq \emptyset) ,$$

where S is either an open set, or a non-everywhere-null submanifold of codimension 1; in this last case we moreover assume that Ψ is orientation-preserving. Then

$$\Psi = id .$$

Remark: Note that in the case when S is a submanifold one does not need to assume any kind of completeness of S .

Proof: Suppose first that S is an open set, let \tilde{S} be the largest open set such that $\Psi|_{\tilde{S}} = \text{id}$. Suppose that \tilde{S} is not closed, thus there exists $p \in \partial\tilde{S}$, let \mathcal{O} be any neighbourhood of p with a local coordinate system such that $x^\mu(p) = 0$, smoothness of Ψ implies, in local coordinates,

$$\Psi^\mu(0) = 0, \quad \frac{\partial\Psi^\mu}{\partial x^\nu}(0) = \delta_\nu^\mu, \quad \partial_{\sigma_1} \cdots \partial_{\sigma_1} \Psi^\mu(0) = 0. \quad (2.1.21)$$

From $\Psi^*\gamma = \gamma$ one has

$$\gamma_{\alpha\beta}(x) = \gamma_{\mu\nu}(\Psi(x)) \frac{\partial\Psi^\mu}{\partial x^\alpha} \frac{\partial\Psi^\nu}{\partial x^\beta}, \quad (2.1.22)$$

$$\frac{\partial^2\Psi^\mu}{\partial x^\alpha \partial x^\beta} = \Gamma_{\alpha\beta}^\sigma(x) \frac{\partial\Psi^\mu}{\partial x^\sigma} - \Gamma_{\nu\rho}^\mu(\Psi(x)) \frac{\partial\Psi^\nu}{\partial x^\alpha} \frac{\partial\Psi^\rho}{\partial x^\beta}, \quad (2.1.23)$$

where Γ denotes the Christoffel symbols of the metric γ . Setting $A_\beta^\alpha \equiv \frac{\partial\Psi^\alpha}{\partial x^\beta}$, from the equation (2.1.23) one obtains the following system of ODE's along rays emanating from the origin:

$$\begin{aligned} \frac{d\Psi^\mu}{dr} &= A_\beta^\mu x^\beta / r, \quad r = \left(\sum (x^\alpha)^2\right)^{1/2}, \\ \frac{dA_\beta^\mu}{dr} &= \left(\Gamma_{\alpha\beta}^\sigma(x) A_\sigma^\mu - \Gamma_{\nu\rho}^\mu(\Psi(x)) A_\alpha^\nu A_\beta^\rho\right) x^\alpha / r, \end{aligned}$$

and the initial conditions (2.1.21) together with uniqueness of solutions of systems of ODE's imply $\Psi^\mu = x^\mu$ in \mathcal{O} , which leads to a contradiction, and shows that $\partial S = \emptyset$, thus $S = M$.

Suppose now that S is a hypersurface, let $p \in S$ be such that S is not null in a neighbourhood of p , let \mathcal{O} be a neighbourhood of p with a coordinate system $(x, y) = (x, y^1, \dots, y^n)$ such that $S \cap \mathcal{O} = \{x = 0\}$ and $\gamma_{xi}(0, y) = 0$; note that $\gamma_{xx}(0, y) \neq 0$. From (2.1.23) it follows that along the curves $y = \text{const}$ we have

$$\begin{aligned} \frac{d\Psi^\mu}{dx} &= A_x^\mu, \\ \frac{dA_\beta^\mu}{dx} &= \Gamma_{x\beta}^\sigma(x, y) A_\sigma^\mu - \Gamma_{\nu\rho}^\mu(\Psi(x, y)) A_x^\nu A_\beta^\rho, \end{aligned}$$

and $\Psi|_S = \text{id}$ implies

$$\Psi^i(0, y) = y^i, \quad \Psi^x(0, y) = 0, \quad \frac{\partial\Psi^i}{\partial y^j}(0, y) = \delta_j^i, \quad \frac{\partial\Psi^x}{\partial y^j}(0, y) = 0,$$

which together with (2.1.22) gives

$$\frac{\partial \Psi^\alpha}{\partial x^\beta}(0, y) = \delta_\beta^\alpha ,$$

and $\Psi|_{\mathcal{U}} = \text{id}$ for some neighbourhood \mathcal{U} of p follows. $\mathcal{U} = M$ follows by part 1 of this Lemma. \square

Corollary 2.1.2 *Let (Σ, g, K) be smooth Cauchy data on a Hausdorff manifold Σ , let $\phi : \Sigma \rightarrow \Sigma$ be a smooth diffeomorphism, set $\tilde{g} = \phi^*g$, $\tilde{K} = \phi^*K$. Let (M, γ, i) , respectively $(\tilde{M}, \tilde{\gamma}, \tilde{i})$ be the maximal globally hyperbolic (vacuum, Hausdorff, smooth) development of (Σ, g, K) , respectively of $(\Sigma, \tilde{g}, \tilde{K})$. There exists a diffeomorphism $\Phi : \tilde{M} \rightarrow M$ such that $\Phi^*\gamma = \tilde{\gamma}$, and*

$$\Phi \circ \tilde{i} = i \circ \phi .$$

Proof: By definition we have $(\phi^{-1})^*\tilde{g} = g$, $(\phi^{-1})^*\tilde{K} = K$, thus $(\tilde{M}, \tilde{\gamma}, \tilde{i} \circ \phi^{-1})$ is a development of (Σ, g, K) . By maximality of (M, γ, i) it follows from Theorem 2.1.3 that there exists an isometric embedding $\Phi : \tilde{M} \rightarrow M$ such that $\Phi \circ \tilde{i} \circ \phi^{-1} = i$. A similar argument using maximality of $(\tilde{M}, \tilde{\gamma}, \tilde{i})$ shows that there exists an isometric embedding $\Psi : M \rightarrow \tilde{M}$ such that $\Psi \circ i \circ \phi = \tilde{i}$. One thus has $\Psi \circ \Phi \circ \tilde{i} = \tilde{i}$, $(\Psi \circ \Phi)^*\tilde{\gamma} = \tilde{\gamma}$, so that $\Psi \circ \Phi$ is an isometry which is the identity on $\tilde{i}(\Sigma)$, therefore $\Psi \circ \Phi = \text{id}$ by Lemma 2.1.1. This shows that Φ is invertible, and the result follows. \square

The main result of this Section is the following:

Theorem 2.1.4 *Let (Σ, g, K) be smooth Cauchy data on a Hausdorff manifold Σ , let (M, γ, i) be the maximal globally hyperbolic (vacuum, Hausdorff, smooth) development of (Σ, g, K) , suppose that a group G acts on (Σ, g, K) by smooth isometries:*

$$G \times \Sigma \ni (g, p) \rightarrow \phi_g(p) \in \Sigma$$

*($\phi_g^*g = g$, $\phi_g^*K = K$). There exists an action of G on M ,*

$$G \times M \ni (g, p) \rightarrow \Phi_g(p) \in M ,$$

such that

$$\forall g \in G \quad \Phi_g^* \gamma = \gamma, \quad \Phi_g \circ i = i \circ \phi_g.$$

Proof: Let (M_g, γ_g, i_g) be a maximal globally hyperbolic development of $(\Sigma, \phi_g^* g, \phi_g^* K)$. By Corollary 2.1.2 there exists a diffeomorphism $\Psi_g : M_g \rightarrow M$ such that $\Psi_g \circ i_g = i \circ \phi_g$, $\Psi_g^* \gamma = \gamma_g$. Since $\phi_g^* g = g$, $\phi_g^* K = K$, Theorem 2.1.3 moreover implies the existence of a diffeomorphism $\Xi_g : M_g \rightarrow M$ such that $\Xi_g \circ i_g = i$, $\Xi_g^* \gamma = \gamma_g$. Consider the diffeomorphism $\Phi_g = \Psi_g \circ \Xi_g^{-1} : M \rightarrow M$. We have

$$\Phi_g^* \gamma = (\Xi_g^{-1})^* \Psi_g^* \gamma = (\Xi_g^{-1})^* \gamma_g = \gamma,$$

thus the Φ_g 's are isometries, moreover

$$\Phi_g \circ i = \Psi_g \circ \Xi_g^{-1} \circ i = \Psi_g \circ i_g = i \circ \phi_g.$$

Lemma 2.1.1 easily implies $\Phi_{gh} \circ \Phi_h^{-1} \circ \Phi_g^{-1} = \text{id}$, thus $\Phi_{gh} = \Phi_g \circ \Phi_h$. Finally continuity of Φ_g in g follows from the continuous dependence upon Cauchy data (on compact subsets) of the solutions of the initial value problem for Einstein equations. \square

2.2 $SO(3) \times U(1)$ symmetric space-times, ${}^3\Sigma = L(p, 1)$.

Consider the set of space-times with Cauchy-data on compact, connected, orientable manifolds ${}^3\Sigma$, invariant under an effective action of the group $G = SO(3) \times U(1)$ or $G = (SU(2) \times U(1))/D$, $D = \{(-1, -1), (1, 1)\}$, with three-dimensional principal orbits. In this Section we shall outline the proof of the claim, that this set consists of the Taub-NUT metrics. As discussed by Fischer [45], ${}^3\Sigma$ is necessarily S^3 or a lens space $L(p, 1)$, and the action of G is unique up to equivalence. Identifying $U(1)$ with the subgroup of $SU(2)$ consisting of matrices of the form $\text{diag}(e^{i\alpha}, e^{-i\alpha})$, and identifying $SU(2)$ with S^3 in the standard way, it follows that up to homomorphism of G and diffeomorphism of ${}^3\Sigma$ the action of G on ${}^3\Sigma$ is given by

$$G \times {}^3\Sigma \ni (g = (g_1, g_2), p) \rightarrow gp = g_1 p g_2^{-1} \in {}^3\Sigma, \quad (2.2.1)$$

$$g_1 \in SO(3) \quad \text{or} \quad SU(2), \quad g_2 \in U(1).$$

When $G = [SU(2) \times U(1)]/D$, and/or when ${}^3\Sigma = L(p, 1)$, $p \neq 1$, appropriate equivalence relations in (2.2.1) should be taken into account². For any $p \in {}^3\Sigma$ there exists a neighbourhood \mathcal{O}_p of p diffeomorphic with a neighbourhood U_e of the identity of $SU(2) \approx S^3$, using this diffeomorphism any G -invariant metric on ${}^3\Sigma$ can be pulled-back to define a G -invariant metric on U_e . Let $X_i(e)$ be any basis of $T_e SU(2)$, let

$$X_i(g) = (R_g)_* X_i(e),$$

where R_g is the right action of $SU(2)$ on itself. Since the right and left actions commute, the vector fields X_i are left-invariant. Also, by definition of the adjoint representation ad ,

$$(R_{h^{-1}})_* X_i = ad_h X_i.$$

As is well-known, the ad representation of $SU(2)$ acts on $T_e SU(2) \approx \mathbb{R}^3$ by rotations, and we can choose $X_3(e)$ so that $U(1) = \text{diag}(e^{i\alpha}, e^{-i\alpha})$ acts as rotations around the $X_3(e)$ -axis. Let $\langle \cdot, \cdot \rangle$ be a metric on $U_e \subset SU(2)$, invariant under the action (2.2.1). The $SU(2)$ invariance implies that the functions

$$g_{ij}(p) = \langle X_i, X_j \rangle|_p$$

are p -independent. The $U(1)$ invariance implies that g_{ij} is a 2-covariant tensor on \mathbb{R}^3 invariant under rotations around the “ z -axis”, which by straightforward considerations leads to

$$g_{ij} \equiv \text{diag}(e^a, e^a, e^b),$$

for some $a, b \in \mathbb{R}$. It follows that any G -invariant metric on S^3 is of the form

$$g = e^a[(\omega^1)^2 + (\omega^2)^2] + e^b(\omega^3)^2, \quad (2.2.2)$$

where the ω^i 's are left invariant forms on S^3 , dual to the vectors X_i as defined above. The argument presented in Ref. [12], Chapter II, Section 3, shows that the vacuum

²Equation (2.2.1) defines an action of $SU(2) \times U(1)$ on ${}^3\Sigma$, this action is however not effective.

dynamics does not lead to a rotation of the eigen-axes of the $U(1)$ action, thus in a vacuum $SO(3) \times U(1)$ or $[SU(2) \times U(1)]/D$ symmetric space-time an appropriate choice of time τ and of the ω^i 's leads to

$$ds^2 = -d\tau^2 + e^{a(\tau)}[(\omega^1)^2 + (\omega^2)^2] + e^{b(\tau)}(\omega^3)^2, \quad (2.2.3)$$

for some functions $a(\tau)$, $b(\tau)$. In Section 8.2 of Ref. [111] all solutions of Einstein equations with a metric of the form (2.2.3) have been found—these are the Taub-NUT metrics.

2.3 $SO(3)$ symmetry, 2-dimensional principal orbits.

In this Appendix we shall describe the family of vacuum, maximal globally hyperbolic space-times 4M which admit compact, connected, orientable Cauchy surfaces ${}^3\Sigma$ with Cauchy data invariant under an $SO(3)$ action with two dimensional principal orbits³. The results presented below are a global version of the generalized Birkhoff theorem (*cf.* Appendix B to [66]).

As has been shown in Ref. [45], one has ${}^3\Sigma \approx S^1 \times S^2$ or S^3 or \mathbb{P}^3 or $\mathbb{P}^3 \# \mathbb{P}^3$ (connected sum of two projective spaces), and the action of $SO(3)$ is in each case unique up to homomorphism of $SO(3)$ and diffeomorphism of ${}^3\Sigma$. $S^1 \times S^2$ as well as S^3 or \mathbb{P}^3 or $\mathbb{P}^3 \# \mathbb{P}^3$ are of the form

$${}^3\Sigma = ([0, 1] \times S^2) / \sim, \quad (2.3.1)$$

where

- in the $S^1 \times S^2$ case the relation “ \sim ” identifies $\{0\} \times S^2$ with $\{1\} \times S^2$ via the identity map from S^2 to S^2 ,
- in the $\mathbb{P}^3 \# \mathbb{P}^3$ case “ \sim ” identifies $\{0\} \times S^2$ with itself via the antipodal map from S^2 to S^2 , similarly for $\{1\} \times S^2$,

³The author is grateful to dr G. Cieciura for useful discussions about the results presented here.

- in the S^3 case “ \sim ” shrinks $\{0\} \times S^2$ to a point p_0 and $\{1\} \times S^2$ to a point p_1 ,
- in the \mathbb{P}^3 case “ \sim ” shrinks $\{0\} \times S^2$ to a point p_0 and identifies $\{1\} \times S^2$ with itself via the antipodal map.

From [45] it follows that in all cases there exist coordinates $(\psi, (\theta, \phi))$ on $[0, 1] \times S^2$ such that the action of $SO(3)$ consists of ψ -independent rotations of S^2 , which implies that any $SO(3)$ invariant metric on ${}^3\Sigma$ in these coordinates takes the form

$$ds^2 = f^2(\psi) d\psi^2 + A^2(\psi)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.3.2)$$

Let $({}^3\Sigma, g_{ij}, K_{ij})$ be any $SO(3)$ invariant Cauchy data, let $({}^4M, \gamma)$ be the maximal globally hyperbolic development thereof, let t be a time function in a neighborhood \mathcal{O} of ${}^3\Sigma$ such that for $\tau \in (t_1, t_2)$ the group $SO(3)$ acts on the level sets $\mathcal{I}_\tau \equiv \{p : t(p) = \tau\}$ of t by isometries (*cf.* Section 2.1). By Lie dragging along the normals to \mathcal{I}_τ the above coordinates $(\psi, (\theta, \phi))$ can be extended to \mathcal{O} , and one finds that in this coordinate system the metric takes the form

$$ds^2 = -F^{-2}(t, \psi) dt^2 + X^2(t, \psi) d\psi^2 + Y^2(t, \psi)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.3.3)$$

for some (strictly) positive functions F, X , and a non-negative function Y ; Y strictly positive for $\psi \in (0, 1)$, and

- Y strictly positive if ${}^3\Sigma \approx S^1 \times S^2$ or ${}^3\Sigma \approx \mathbb{P}^3 \# \mathbb{P}^3$,
- if ${}^3\Sigma \approx S^3$ the area function Y vanishes at $\psi = 0$ and $\psi = 1$ only,
- if ${}^3\Sigma \approx \mathbb{P}^3$ the function Y vanishes at $\psi = 0$ only.

In order to analyze the constraint equations it is useful to introduce the following “null” derivatives:

$$Y_\pm \equiv \frac{1}{X} \frac{\partial Y}{\partial \psi} \pm F \frac{\partial Y}{\partial t}.$$

We have the following:

Lemma 2.3.1 *Let Cauchy data for a metric of the form (2.3.3) satisfy the vacuum constraint equations, suppose that there exists ψ_+ such that*

$$Y_+(0, \psi_+) = 0 .$$

Then

$$\begin{aligned} \psi < \psi_+ &\Rightarrow Y_+(0, \psi) < 0 , \\ \psi > \psi_+ &\Rightarrow Y_+(0, \psi) > 0 . \end{aligned} \quad (2.3.4)$$

The same statement holds with the subscript “+” replaced by the subscript “-”.

Proof: From equations (A.2)-(A.3) on p. 370 in [66] we have

$$\frac{\partial Y_{\pm}}{\partial \psi} = h_{\pm} Y_{\pm} + \frac{X}{2Y} , \quad (2.3.5)$$

$$h_{\pm} \equiv \pm F \frac{\partial X}{\partial t} - \frac{XY_{\mp}}{2Y} ,$$

thus

$$Y_+(\psi) = \int_{\psi_+}^{\psi} \frac{X(s)}{2Y(s)} e^{\int_s^{\psi} h_+(u) du} ds \quad \begin{cases} > 0, & \psi > \psi_+ , \\ < 0, & \psi < \psi_+ . \end{cases} \quad (2.3.6)$$

□

Lemma 2.3.1 implies, that in the vacuum the topologies S^3 and \mathbb{P}^3 are excluded:

Proposition 2.3.1 *Let $({}^3\Sigma, g_{ij}, K_{ij})$ be $SO(3)$ invariant Cauchy data, ${}^3\Sigma$ compact, connected, orientable. Then ${}^3\Sigma \approx S^1 \times S^2$ or $\mathbb{P}^3 \# \mathbb{P}^3$; moreover, ∇Y is timelike, where Y is the area function (cf. (2.3.3)).*

Proof: Suppose first, that ${}^3\Sigma \approx S^3$. By construction of the coordinate system (2.3.3) we must have

$$\begin{aligned} Y(t=0, \psi=0) &= 0 , \quad \frac{\partial Y}{\partial \psi}(t=0, \psi=0) > 0 , \quad \frac{\partial Y}{\partial t}(t=0, \psi=0) = 0 , \\ Y(t=0, \psi=1) &= 0 , \quad \frac{\partial Y}{\partial \psi}(t=0, \psi=1) < 0 , \quad \frac{\partial Y}{\partial t}(t=0, \psi=1) = 0 \end{aligned} \quad (2.3.7)$$

(cf. e.g. Appendix C to [32] for a detailed description of functions and tensors invariant under rotations; in that reference rotations around a single axis are considered, the results generalize to the full rotation group in a rather straightforward manner). (2.3.7) implies that

$$Y_{\pm}(0, 0) > 0, \quad Y_{\pm}(0, 1) < 0, \quad (2.3.8)$$

therefore there exist ψ_{\pm} such that

$$Y_{\pm}(0, \psi_{\pm}) = 0,$$

which makes (2.3.8) inconsistent with (2.3.4), thus on S^3 no solutions of the vacuum constraints exist. Since a solution of the vacuum constraints on \mathbb{P}^3 can be pulled-back to S^3 via the covering map, no such solutions exist on \mathbb{P}^3 either. It thus remains to show, that on $S^1 \times S^2$ and $\mathbb{P}^3 \# \mathbb{P}^3$ the vector field ∇Y must be timelike. Let us first note that the map $\Phi : [0, 1] \times S^2 \rightarrow [0, 1] \times S^2$ defined by

$$\Phi(\psi, \omega) = \begin{cases} (2\psi, \omega), & \psi \in [0, \frac{1}{2}], \\ (2(1 - \psi), R\omega), & \psi \in (\frac{1}{2}, 1], \end{cases}$$

where R is the antipodal map, extends to a double covering map $\tilde{\Phi} : S^1 \times S^2 \rightarrow \mathbb{P}^3 \# \mathbb{P}^3$. Since $S^1 \times S^2$ is itself covered by $\mathbb{R} \times S^2$ it is sufficient to show timelikeness of ∇Y for periodic solutions (with period 1) of the constraint equations on $\mathbb{R} \times S^2$. Suppose that there exists a point ψ_+ such that $Y_+(\psi_+) = 0$, or there exists a point ψ_- such that $Y_-(\psi_-) = 0$, or both, then from (2.3.6) it follows

$$Y_{\pm}(\psi_{\pm} + 1) = \int_{\psi_{\pm}}^{\psi_{\pm}+1} \frac{X(s)}{2Y(s)} e^{\int_s^{\psi_{\pm}+1} h_{\pm}(u) du} ds > 0, \quad (2.3.9)$$

which contradicts periodicity of Y_{\pm} , thus neither Y_+ nor Y_- can change signs. Let ψ_0 be a local extremum of Y , then $\frac{\partial Y}{\partial \psi}(t = 0, \psi_0) = 0$, and we have

$$(Y_+ Y_-)(t = 0, \psi_0) = -F^2 \left(\frac{\partial Y}{\partial t} \right)^2 \leq 0,$$

which implies

$$(Y_+ Y_-)(t = 0, \psi) < 0 \implies g^{\mu\nu} Y_{,\mu} Y_{,\nu} < 0.$$

One can now proceed as in the Appendix B to [66] to conclude that ${}^3\Sigma$ can be deformed in 4M in such a way that $\frac{\partial A}{\partial \psi} = 0$, a reparametrization of the ψ 's leads to $\frac{\partial f}{\partial \psi} = 0$, A and f as in (2.3.2), and thus in both $S^1 \times S^2$ and $\mathbb{P}^3 \# \mathbb{P}^3$ cases one obtains a two-parameter family of metrics on 4M (parametrized by the area A of the group orbits and the length f of closed geodesics orthogonal to the orbits⁴) which are, locally, isometric to the metric under the horizon of the Schwarzschild–Kruskal–Szekeres manifold.

It may be of some interest to describe a larger family of space-times, with $S^1 \times S^2$ spatial topology, the metric of which is *locally* isometric to the “ $r < 2m$ Schwarzschild metric”, but on which no *global* action of $SO(3)$ by isometries exists. Consider the metric (2.3.2) with $\frac{\partial A}{\partial \psi} = \frac{\partial f}{\partial \psi} = 0$ on the manifold (2.3.1) in which the relation “ \sim ” identifies $\{0\} \times S^2$ with $\{1\} \times S^2$ via a rotation $\omega \in SO(3)$ of S^2 :

$$(0, p) \sim (1, \omega p), \quad \omega \in SO(3).$$

In this way one obtains a five-parameter family of smooth metrics, parametrized by⁵ $(f, A, \omega) \in (0, \infty) \times (0, \infty) \times SO(3)$. The four-dimensional metric is, again, locally isometric to the “ $r < 2m$ Schwarzschild” metric with an appropriately chosen m .

2.4 Spatially compact Bianchi I space-times.

In this Section we shall discuss the space of maximally developed, globally hyperbolic, smooth Hausdorff vacuum metrics with compact $U(1) \times U(1) \times U(1)$ symmetric Cauchy surfaces ${}^3\Sigma$. $G = U(1) \times U(1) \times U(1)$ symmetry implies that ${}^3\Sigma$ is a three dimensional torus T^3 (cf. e.g. [45]) and the action of G is transitive on ${}^3\Sigma$, choosing group coordinates on ${}^3\Sigma$ and the geodesic distance on geodesics normal to ${}^3\Sigma$ as a time coordinate in 4M

⁴Strictly speaking, f is the length of such geodesics when ${}^3\Sigma = S^1 \times S^2$, and half of the length when ${}^3\Sigma = \mathbb{P}^3 \# \mathbb{P}^3$.

⁵It is easily seen that two such metrics with $\omega_1 \neq \omega_2$ will not be isometric: a geodesic orthogonal to a sphere $S_{\psi_0}^2 \equiv \{\psi = \psi_0\}$ and passing through a point $p \in S_{\psi_0}^2$ will again intersect $S_{\psi_0}^2$ at ωp .

we have

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j, \quad g_{ij} = g_{ij}(t). \quad (2.4.1)$$

Let us momentarily forget about the identifications $x^i \equiv x^i + 2\pi$ and consider the metric (2.4.1) on \mathbb{R}^3 . At any chosen time $t = t_o$ there exists a matrix $L_j^i \in SL(3, \mathbb{R})$ such that

$$g_{ij}(t_o)L_k^i L_l^j = \delta_{kl}. \quad (2.4.2)$$

Since the right hand side of (2.4.2) is invariant under $SO(3)$ the matrix L_j^i is not uniquely defined, and we may choose $\omega_j^i \in SO(3)$ so that

$$P_{ij}(t_o)L_k^i L_l^j = \text{diag}(P_1, P_1, P_3); \quad (2.4.3)$$

by an abuse of notation we have used the symbol L_j^i to denote $L_k^i \omega_j^k$. Let

$$y^i = (L^{-1})_j^i x^j; \quad (2.4.4)$$

at $t = t_o$ we have

$$ds^2 = -dt^2 + \sum dy^i dy^i, \\ P_{ij}dy^i dy^j = P_1(dy^1)^2 + P_2(dy^2)^2 + P_3(dy^3)^2. \quad (2.4.5)$$

From 1) the existence of diagonal solutions of the equations of motion, namely the flat metrics on \mathbb{R}^4 or the Kasner metrics [79] on $(0, \infty) \times \mathbb{R}^3$; 2) uniqueness up to coordinate transformation of solutions of vacuum Einstein equations; it follows that all solutions can be diagonalized for all t and are either Minkowski space-time or the Kasner metrics⁶

$$ds^2 = -dt^2 + t^{2p_1}(dy^1)^2 + t^{2p_2}(dy^2)^2 + t^{2p_3}(dy^3)^2, \\ t \in (0, \infty), \quad y^i \in \mathbb{R}^3, \quad \sum p_i^2 = \sum p_i = 1. \quad (2.4.6)$$

Returning to T^3 , the equations (2.4.4), (2.4.6) and (2.4.1) give

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j, \quad \partial_\mu g_{ij} = 0, \quad (2.4.7)$$

⁶This simple argument is due to V.Moncrief.

in the T^3 — Minkowski case, or

$$ds^2 = -dt^2 + \sum t^{2p_i} (\omega^i)^2,$$

$$\omega^i \equiv L_j^i dx^j, \quad x^i \in [0, 2\pi]_{\text{mod } 2\pi}, \quad (2.4.8)$$

in the T^3 — Kasner case. All coordinate transformations preserving (2.4.7) can be shown to be of the form

$$t \rightarrow t + \tau, \quad x^i \rightarrow Z_j^i x^j + x_o^i, \quad Z_j^i \in SL(3, Z) \quad (\det Z_j^i = 1, Z_j^i \in Z),$$

$$\partial_\mu \tau = \partial_\mu x_o^i = 0, \quad (2.4.9)$$

while those preserving (2.4) are

$$t \rightarrow t, \quad x^i \rightarrow Z_j^i x^j + x_o^i, \quad Z_j^i \in SL(3, Z), \quad \partial_\mu x_o^i = 0. \quad (2.4.10)$$

Let $Diff_o(^4M)$ be the path-connected component of the identity of the diffeomorphism group of 4M . Elementary singular homology considerations show that the diffeomorphisms (2.4.9) — (2.4.10) are in $Diff_o(^4M)$ if and only if $Z_j^i = \delta_j^i$. It follows that the set of maximally developed globally hyperbolic Hausdorff Bianchi I space-times with spatially compact Cauchy surfaces divided by $Diff_o$ can be given the structure of a manifold which is the union of three disconnected pieces:

- a six dimensional manifold of metrics (2.4.7) (parametrized by g_{ij}),
- a ten dimensional manifold of metrics (2.4) (one parameter for the p_i 's, nine for the L_j^i 's), and
- a ten dimensional manifold of metrics which differ from (2.4) by a change of time orientation, $t \rightarrow -t$.

The subset of extendible metrics of the above type consists of six disconnected nine dimensional submanifolds, which consist of Kasner metrics with one of the parameters p_i equal to 1, thus a generic maximal Bianchi I space-time is inextendible.

Chapter 3

$U(1) \times U(1)$ stability of the $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ Kasner metrics.

In this Chapter we present the proof of Theorem 1.5.2, namely that the singularity of $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ (or permutation thereof) Kasner metrics is stable under $U(1) \times U(1)$ symmetric perturbations. The problem reduces to establishing *a priori* estimates for a Lorentzian harmonic-type map from two-dimensional Minkowski space-time to the unit two-dimensional hyperboloid of constant negative curvature. We shall start by analyzing the harmonic map equations, the geometric interpretation of the estimates proved below will be given in Section 3.5.

3.1 Introduction — notation

Let M be a Riemannian manifold with scalar product $\langle \cdot, \cdot \rangle$. Let $t_0 < 0$, let $x(t, \theta) : [t_0, 0) \times S^1 \rightarrow M$ satisfy

$$\frac{DX_t}{Dt} - \frac{DX_\theta}{D\theta} = -\frac{X_t}{t} \quad (3.1.1)$$

where $X_t \equiv \frac{\partial x(t, \theta)}{\partial t}$, $X_\theta \equiv \frac{\partial x(t, \theta)}{\partial \theta}$, D denotes the Levi-Civita connection of $\langle \cdot, \cdot \rangle$,

$$\frac{D}{D\theta} \equiv D_{X_\theta} \equiv X_\theta^A D_A, \quad \frac{D}{Dt} \equiv D_{X_t} \equiv X_t^A D_A.$$

$\langle Y \rangle$ or $|Y|$ will be used to denote $\sqrt{\langle Y, Y \rangle}$; we shall identify S^1 with $\{\theta \in [0, 2\pi]_{\text{mod } 2\pi}\}$. We shall throughout use the notation

$$X^{(k)} = \left(\frac{D}{D\theta}\right)^{(k-1)} X_\theta,$$

we shall always assume $X_t(t_0, \cdot), X_\theta(t_0, \cdot) \in H_1(S^1)$. $K(\cdot, \cdot)$ denotes the curvature tensor of $\langle \cdot, \cdot \rangle$ defined by

$$K(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z,$$

We use $|K|$ to denote an upper bound on the sectional curvatures, and $|DK|$, etc. to denote the Riemannian norm of the tensor DK , etc. The matrix $\eta_{\mu\nu}$ will denote the two dimensional Minkowski metric, $\eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\theta^2$. For $|t| < \pi$ it will often be convenient to use the coordinates

$$u = t + \theta, \quad v = t - \theta,$$

so that

$$\frac{\partial}{\partial u} = u^\mu \partial_\mu = \frac{1}{2} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} \right), \quad \frac{\partial}{\partial v} = v^\mu \partial_\mu = \frac{1}{2} \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \theta} \right)$$

$$\eta_{uv} = -\frac{1}{2}, \quad \eta^{uv} = -2.$$

3.2 Pointwise Estimates

In this Section we shall prove some rough pointwise estimates as $t \rightarrow 0$ for solutions of (3.1.1). The ideas of the proofs are inspired by some unpublished work by V. Moncrief; the author is grateful to V. Moncrief for making his results available prior to publication.

Lemma 3.2.1 $X^{(k)}$ satisfies the equation

$$\left[\left(\frac{D}{Dt} \right)^2 - \left(\frac{D}{D\theta} \right)^2 \right] X^{(k)} = -\frac{1}{t} \frac{DX^{(k)}}{Dt} + N^{(k)}. \quad (3.2.1)$$

If for all multi-indices $0 \leq |\alpha| \leq k-1$ we have

$$|D^\alpha X_t| \leq C_1 |t|^{-|\alpha|-1}, \quad |D^\alpha X_\theta| \leq C_1 |t|^{-|\alpha|-1},$$

and if

$$|K| + |DK| + \dots |D^{|\alpha|-1} K| \leq C_2 ,$$

then

$$1. \quad |N^{(k)}| \leq F(C_1, C_2, k) |t|^{-k-2} .$$

2. We also have the estimate

$$|N^{(k)}| \leq \sum_{i=1}^k F_1(C_1, C_2, i, k) |X^{(i)}| |t|^{i-k-2} \quad (3.2.2)$$

where F, F_1 are some constants depending upon the arguments listed.

Proof: Applying $\left(\frac{D}{D\theta}\right)^k$ to both sides of (3.1.1) one obtains

$$\left[\left(\frac{D}{Dt}\right)^2 - \left(\frac{D}{D\theta}\right)^2\right] X^{(k)} = -\frac{1}{t} \frac{DX^{(k)}}{Dt} + L^{(k)} - \frac{M^{(k)}}{t} ,$$

with

$$\begin{aligned} L^{(k)} &= \left(\frac{D}{Dt}\right)^2 X^{(k)} - \left(\frac{D}{D\theta}\right)^k \frac{DX_t}{Dt} , \\ M^{(k)} &= \left(\frac{D}{D\theta}\right)^k X_t - \frac{D}{Dt} X^{(k)} . \end{aligned}$$

We have the recurrence relations for $k \geq 1$

$$\begin{aligned} L^{(k+1)} &= \frac{DL^{(k)}}{D\theta} + \frac{D}{Dt} (K(X_t, X_\theta) X^{(k)}) + K(X_t, X_\theta) \frac{DX^{(k)}}{Dt} \\ M^{(k+1)} &= \frac{DM^{(k)}}{D\theta} + K(X_\theta, X_t) X^{(k)} \end{aligned}$$

with

$$L^{(0)} = M^{(0)} = M^{(1)} = 0, \quad L^{(1)} = K(X_\theta, X_t) X_t ,$$

and part 1 follows by induction. To prove part 2 one shows by induction that there exists a set of linear operators $A^{(k,i)}$ such that

$$N^{(k)} = \sum_{i=1}^k A^{(k,i)}(X^{(i)})$$

(e.g. $A^{(1,1)}(Y) = K(Y, X_t) X_t$) and the bounds on $|A^{(k,i)}|$ are established by an induction argument. □

Lemma 3.2.2 *Let $T_{\mu\nu}$ be a symmetric traceless tensor, let $j_\mu = T_\mu{}^\nu{}_{,\nu}$. We have*

$$|T_{tt}(t_1, \theta)| \leq \sup_{\psi \in [\theta - t_1 + t_0, \theta + t_1 - t_0]} (|T_{tt}|(t_0, \theta) + |T_{t\theta}|(t_0, \theta)) \quad (3.2.3)$$

$$+ \int_{t_0}^{t_1} \sup_{\psi \in [\theta - t_1 + t, \theta + t_1 - t]} (|j_t|(t, \theta) + |j_\theta|(t, \theta)) dt. \quad (3.2.4)$$

Proof. Let $\bar{T}_{\mu\nu}(\mu, \nu) = T_{\mu\nu}(t = \frac{u+v}{2}, \theta = \frac{u-v}{2})$, let $\bar{T}_{uu} = \bar{T}_{\mu\nu} u^\mu u^\nu$, etc. We have

$$\bar{T}_{uu,v} = -\frac{1}{2} \bar{j}_u, \quad \bar{T}_{vv,u} = -\frac{1}{2} \bar{j}_v, \quad \bar{j}_\mu(u, v) = j_\mu\left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

therefore

$$\begin{aligned} \bar{T}_{uu}(u_1, v_1) &= -\frac{1}{2} \int_{v_1-\lambda}^{v_1} \bar{j}_u(u_1, v) dv + \bar{T}_{uu}(u_1, v_1 - \lambda), \\ \bar{T}_{vv}(u_1, v_1) &= -\frac{1}{2} \int_{u_1-\lambda}^{u_1} \bar{j}_v(u, v_1) du + \bar{T}_{vv}(u_1 - \lambda, v_1), \end{aligned}$$

adding these equations, setting $\lambda = 2(t_1 - t_0)$, one is by elementary manipulations led to

$$\begin{aligned} T_{tt}(t_1, \theta_1) &= -\frac{1}{2} \int_{t_0}^{t_1} \{(j_t + j_\theta)(t, \theta_1 + t_1 - t) + (j_t - j_\theta)(t, \theta_1 - t_1 + t)\} dt \\ &\quad + \frac{1}{2} (T_{tt} + T_{t\theta})(t_0, \theta_1 + t_1 - t_0) + \frac{1}{2} (T_{tt} - T_{t\theta})(t_0, \theta_1 + t_0 - t_1), \end{aligned} \quad (3.2.5)$$

and the result follows. \square

Let us recall Gromwall's lemma:

Lemma 3.2.3 *Let $f, x \in C^1([t_0, 0))$, $y \in C^0([t_0, 0))$, $y \geq 0$, satisfy for $t \in [t_0, 0)$*

$$f(t) \leq x(t) + \int_{t_0}^t y(s) f(s) ds.$$

Then

$$f(t) \leq x(t_0) \exp\left(\int_{t_0}^t y(s) ds\right) + \int_{t_0}^t \frac{dx}{dt}(s) \exp\left\{\int_s^t y(u) du\right\} ds.$$

Proposition 3.2.1 *Let $x(t_0, \theta) \in C^k(S^1)$, $k \geq 1$, $X_t(t_0, \theta) \in C^{k-1}(S^1)$. For all $t \geq t_0$ we have¹*

¹The proof of point a) of Proposition 3.2.1 is a slight variation of an unpublished argument of V. Moncrief.

a)

$$\left(|X_t|^2 + |X_\theta|^2\right)(t, \theta) \leq 2 \left\{ \sup_{\psi \in [\theta-t+t_0, \theta+t-t_0]} \left(|X_t|^2 + |X_\theta|^2\right)(t_0, \psi) \right\} \left(\frac{t_0}{t}\right)^2. \quad (3.2.6)$$

b) If $k \geq 2$, and if

$$|K| + |DK| + \dots + |D^{(k-2)}K| \leq C_2$$

then there exist constants C depending only upon the arguments listed such that, for all $1 \leq |\alpha| \leq k$,

$$|D^\alpha x|(t, \theta) \leq C(|\alpha|, t_0, C_2, \|X_\theta(t_0)\|_{C^{|\alpha|-1}}, \|X_t(t_0)\|_{C^{|\alpha|-1}}) |t|^{-|\alpha|}. \quad (3.2.7)$$

Proof. Let

$$T_{\mu\nu}^{(k)} = |t| \{ \langle X_\mu^{(k)}, X_\nu^{(k)} \rangle - \frac{1}{2} \eta_{\mu\nu} \langle X^{(k)\alpha}, X_\alpha^{(k)} \rangle \}$$

(adding a subscript means differentiation). We have

$$j_\mu^{(k)} = T_{\mu, \nu}^{(k)\nu} = \frac{1}{2} \delta_\mu^0 (\langle X_\theta^{(k)} \rangle^2 - \langle X_t^{(k)} \rangle^2) - |t| \langle X_\mu^{(k)}, N^{(k)} \rangle + \epsilon^{(k)} |t| \langle X^\nu, K(X_\nu, X_\mu) X^{(k)} \rangle, \quad (3.2.8)$$

$\epsilon^{(k)} = 0$ if $k = 0$, $\epsilon^{(k)} = 1$ otherwise. For $k = 0$ it follows

$$j_\theta^{(0)} = 0, \quad |j_t^{(0)}| = \frac{1}{2} |\langle X_t \rangle^2 - \langle X_\theta \rangle^2| \leq \frac{T_{tt}^{(0)}}{|t|}.$$

For $t_0 \leq t \leq t_1$ let

$$\begin{aligned} f^{(k)}(t) &= \sup_{\psi \in [\theta-t_1+t, \theta+t_1-t]} T_{tt}^{(k)}(t, \psi), \\ h^{(k)}(t) &= \sup_{\psi \in [\theta-t_1+t, \theta+t_1-t]} |T_{t\theta}^{(k)}|(t, \psi). \end{aligned}$$

From Lemma 3.2.2 we have

$$f^{(0)}(t) \leq f^{(0)}(t_0) + h^{(0)}(t_0) + \int_{t_0}^t \frac{f^{(0)}(s)}{|s|} ds,$$

so that Gromwall's lemma with $x(t) = f^{(0)}(t_0) + h^{(0)}(t_0)$, $y(t) = \frac{1}{|t|}$, gives

$$\begin{aligned} \forall t_0 \leq t_1 < 0 \quad |f^{(0)}(t_1)| &\leq (f^{(0)}(t_0) + h^{(0)}(t_0))|t_0| \\ &\leq 2f^{(0)}(t_0)|t_0| \end{aligned} \quad (3.2.9)$$

which is equation (3.2.6). To obtain part (b) we shall proceed by induction, suppose therefore that (3.2.7) holds for $|\alpha| \leq k-1$. (3.2.8) and Lemma 3.2.1 yield

$$\begin{aligned} |j_t^{(k)}| &\leq \frac{T_{tt}^{(k)}}{|t|} + |t||X_t^{(k)}||N_k| + C_2|t||X_t||X_\theta|^2|X^{(k)}| \\ &\leq \frac{T_{tt}^{(k)}}{|t|} + \{(2)^{-\frac{1}{2}}|X_t^{(k)}|\}\{2^{\frac{1}{2}}F|t|^{-k-1}\} + C|t|^{-k-2} \\ &\leq \frac{3}{2} \frac{T_{tt}^{(k)}}{|t|} + \tilde{F}^2|t|^{-2k-2}, \\ |j_\theta^{(k)}| &\leq \frac{T_{tt}^{(k)}}{2|t|} + \tilde{F}^2|t|^{-2k-2}, \end{aligned}$$

with some constant $C = C(C_1, C_2)$, $\tilde{F}^2 = F^2 + C|t_0|^k$, and Lemma 3.2.2 gives

$$f^{(k)}(t) \leq f^{(k)}(t_0) + h^{(k)}(t_0) + \int_{t_0}^t \left(\frac{2f^{(k)}(s)}{|s|} + 2\tilde{F}^2 s^{-2k-2} \right) ds \alpha.$$

From Gromwall's Lemma one obtains

$$f^{(k)}(t) \leq (f^{(k)}(t_0) + h^{(k)}(t_0)) \frac{t_0^2}{t^2} + \frac{2\tilde{F}^2}{2k-1} (|t|^{-2k-1} - |t_0|^{-2k+1} t^{-2}), \quad (3.2.10)$$

so that the result for all the derivatives of the form

$$\left(\frac{D}{D\theta} \right)^{i-1} X_\theta, \quad \frac{D}{Dt} \left(\frac{D}{D\theta} \right)^{i-2} X_\theta$$

follows. The estimates for the remaining derivatives can be obtained by e.g. commuting all the t derivatives to the left, and then using (3.2.1) to replace pairs of t -derivatives by pairs of θ derivatives. \square

Remark. If $|X_\theta| \leq C|t|^{\lambda-1}$ for some $\lambda > 0$, then a simple modification of the above proof gives

$$|D^\alpha X_\theta| \leq C|t|^{\lambda-|\alpha|-1}. \quad (3.2.11)$$

It is tempting to conjecture that this is a sharp estimate: (3.2.11) is indeed the best one can expect, since this behaviour is displayed by the maps considered in Appendix B, with any $0 \leq \lambda < 1$.

3.3 Integral “Decay” Estimates

Proposition 3.3.1 *Let $x \in C^i([t_0, 0) \times S^1)$ and let $X_\theta(t_0, \cdot)$, $X_t(t_0, \cdot) \in H_i(S^1)$, $i \geq 1$. There exist constants depending only upon the arguments listed such that*

$$1. \quad \forall \quad 1 \leq |\alpha| \leq i + 1,$$

$$g^{(\alpha)}(t) \equiv \oint d\theta |t|^{2|\alpha|} |D^\alpha x|^2 \leq C \left(|\alpha|, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)}, t_0 \right). \quad (3.3.1)$$

2. *If at least one differentiation is a θ differentiation we have*

$$\lim_{t \rightarrow 0} g^{(\alpha)}(t) = 0. \quad (3.3.2)$$

3. *If at least one differentiation is a θ differentiation then $\frac{g^{(\alpha)}(t)}{|t|} \in L^1([t_0, 0])$ and*

$$\int_{t_0}^0 \frac{g^{(\alpha)}(s)}{|s|} ds \leq C' \left(|\alpha|, t_0, \|X_\theta(t_0)\|_{H_{|\alpha|-1}(S^1)}, \|X_t(t_0)\|_{H_{|\alpha|-1}(S^1)} \right). \quad (3.3.3)$$

4. *$g^{(t)}(s)$ tends to a limit as s goes to zero.*

Remark: The results above are close to being sharp, because, as shown in Appendix B (cf. Proposition B.1.1), for any $\epsilon \in [0, 1)$ there exist solutions of (3.1.1) such that we have $|X_\theta| \approx Ct^{\epsilon-1}$, $|X_{\theta\theta}| \approx Ct^{\epsilon-2}$, etc.

Proof. Let

$$\begin{aligned} T_{\mu\nu}^{(k)} &= |t|^{2k+2} \{ \langle X_\mu^{(k)}, X_\nu^{(k)} \rangle - \tfrac{1}{2} \eta_{\mu\nu} \langle X^{(k)\alpha}, X_\alpha^{(k)} \rangle \}, \\ e^{(k)}(t) &= \oint d\theta T_{tt}^{(k)}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{J}_t^{(k)} = \mathcal{T}_t^{(k)\nu}{}_{,\nu} &= |t|^{2k+1} \{k \langle X_t^{(k)} \rangle^2 + (k+1) \langle X_\theta^{(k)} \rangle^2\} - |t|^{2k+2} \langle X_t^{(k)}, N^{(k)} \rangle \\ &+ \epsilon^{(k)} |t|^{2k+2} \langle X_\theta, K(X_\theta, X_t) X^{(k)} \rangle, \end{aligned} \quad (3.3.4)$$

$\epsilon^{(k)} = 0$ for $k = 0$, $\epsilon^{(k)} = 1$ otherwise, so that for $t_1 > t_0$

$$e^{(k)}(t_1) = e^{(k)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta \mathcal{J}_t^{(k)} \quad (3.3.5)$$

which for $k = 0$ reads

$$e^{(0)}(t_1) = e^{(0)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta |t| \langle X_\theta \rangle^2 \quad (3.3.6)$$

so that $e^{(0)}(t)$ is strictly decreasing and therefore tends to a limit, $e^{(0)}(0)$, which gives (3.3.1) for $\alpha = \theta$ or $\alpha = t$. (3.3.6) and Lebesgue monotone convergence theorem imply $\oint d\theta |t| \langle X_\theta \rangle^2 \in L^1([t_0, 0])$. To show (3.3.3) for higher derivatives we shall proceed by induction, suppose therefore that for $1 \leq k \leq i-1$

$$\int_{t_0}^0 \frac{e^{(k)}(t)}{|t|} dt < \infty. \quad (3.3.7)$$

Part 2 of Lemma 3.2.1 gives

$$\begin{aligned} ||t|^{2k+2} \langle X_t^{(k)}, N^{(k)} \rangle| &\leq (|t|^{k+\frac{1}{2}} |X_t^{(k)}|) (|t|^{k+\frac{3}{2}} |N^{(k)}|) \\ &\leq \frac{1}{2} |t|^{2k+1} |X_t^{(k)}|^2 + C \sum_{i=1}^k |t|^{2i-1} |X^{(i)}|^2 \end{aligned} \quad (3.3.8)$$

for some constant C . From Proposition 3.2.1, point 2, we have $\epsilon^{(k)} |X_\theta| |K(X_\theta, X_t) X^{(k)}| \leq \tilde{C} |t|^{-k-3}$ for some constant \tilde{C} , so that from (3.3.5) we have

$$\begin{aligned} e^{(k)}(t_1) &\leq e^{(k)}(t_0) - \int_{t_0}^{t_1} dt \oint d\theta |t|^{2k+1} \{ (k - \frac{1}{2}) \langle X_t^{(k)} \rangle^2 + (k+1) \langle X_\theta^{(k)} \rangle^2 \} \\ &+ C \sum_{i=1}^k \int_{t_0}^{t_1} dt \oint d\theta |t|^{2i-1} |X^{(i)}|^2 + 2\pi \tilde{C} (t_1 - t_0). \end{aligned} \quad (3.3.9)$$

By hypothesis the integrand of the last integral at the right-hand side of (3.3.9) is in $L^1([t_0, 0])$; therefore by Lebesgue monotone convergence theorem

$$\int_{t_0}^0 dt \oint d\theta |t|^{2k+1} (\langle X_t^{(k)} \rangle^2 + \langle X_\theta^{(k)} \rangle^2) < \infty. \quad (3.3.10)$$

(3.3.10) implies that there exists a sequence $t_i \rightarrow 0$ such that $e^{(k)}(t_i) \rightarrow 0$, and from (3.3.5) we have

$$e^{(k)}(t) = e^{(k)}(t_i) - \int_t^{t_i} dt \oint d\theta \mathcal{J}_t^{(k)}. \quad (3.3.11)$$

(3.3.4), (3.3.7), (3.3.8) and (3.3.10) imply that $\mathcal{J}_t^{(k)}$ is in $L^1([t_0, 0] \times S^1)$ so that we may pass to the limit $t_i \rightarrow 0$ to obtain

$$e^{(k)}(t) = - \int_t^0 dt \oint d\theta \mathcal{J}_t^{(k)}. \quad (3.3.12)$$

(3.3.12) shows in particular that $\lim_{t \rightarrow 0} e^{(k)}(t) = 0$, which is (3.3.2). Finally let $h(t) = \oint |t|^2 |X_\theta|^2(t, \theta)$. For $t_2 > t_1$ we have

$$h(t_2) - h(t_1) = \int_{t_1}^{t_2} k(t) dt ,$$

$$k(t) = -\frac{2h(t)}{|t|} + 2 \oint d\theta t^2 \langle X_\theta, X_{\theta t} \rangle ,$$

by what has been said $k(t) \in L^1([t_0, 0])$ and an argument along the lines of the proof of (3.3.12) shows that (3.3.2) holds for $\alpha = \theta$. The estimate (3.3.3) follows from (3.3.10) by commuting pairs of t derivatives with pairs of θ derivatives using equation (3.1.1). \square

3.4 Pointwise “Decay” Estimates

For $t_0 \leq t \leq 0$ let $C_{t_0}^t$ denote the solid truncated light cone

$$C_{t_0}^t = \{(s, \theta), t_0 \leq s \leq t, s \leq \theta \leq -s\}.$$

Let $B(t)$ denote the “space ball”, $B(t) = \{(s, \theta) : s = t, t \leq \theta \leq -t\}$. Let $R_{t_0}^t, L_{t_0}^t$ be the right and left truncated light-rays from $(0, 0)$, cf. Figure 3.4.1:

$$L_{t_0}^t = \{(s, \theta) : t_0 \leq s \leq t, \theta = s\}$$

$$R_{t_0}^t = \{(s, \theta) : t_0 \leq s \leq t, \theta = -s\}$$

By proposition 3.2.1 we can define

$$v_0 = \sup_{C_{t_0}^0} |t| |X_t| < \infty. \quad (3.4.1)$$

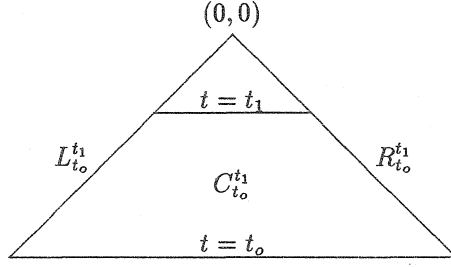


Figure 3.4.1: Truncated light cone.

Lemma 3.4.1 *Let*

$$\|X_t^{(k)}(t_0)\|_{H_k(S^1)} + \|X_\theta^{(k)}(t_0)\|_{H_k(S^1)} \leq M.$$

There exist t_1 independent constants $C_1(M, t_0)$, $C_1(k, M, t_0)$, $C_2(k, M, t_0)$ such that for all $t_0 \leq t_1 < 0$, $k \geq 1$, we have

a)

$$\int_{C_{t_0}^{t_1}} |t|^2 \{4\langle X_{\theta\theta} \rangle^2 + \langle X_{\theta t} \rangle^2\} \leq 6 \int_{C_{t_0}^{t_1}} (|K|v_0^2)^2 |X_\theta|^2 + C,$$

b)

$$\int_{C_{t_0}^{t_1}} |t|^{2k} \{\langle X_\theta^{(k)} \rangle^2 + \langle X_t^{(k)} \rangle^2\} \leq (|K|v_0^2)^2 C_1(k) \int_{C_{t_0}^{t_1}} |X_\theta|^2 + C_2(k),$$

and $|K|$ is defined by

$$|K| = \sup_{p \in M} |K|(p) = \sup_{\substack{p \in M \\ A, B, C, D \in T_p M}} \frac{|\langle K(A, B)C, D \rangle|}{|A||B||C||D|},$$

where K is the curvature tensor of $(M, \langle \cdot, \cdot \rangle)$.

Proof. Let

$$T_{\mu\nu}^{(k)} = |t|^{2k} (\langle X_\mu^{(k)}, X_\nu^{(k)} \rangle - \frac{1}{2} \eta_{\mu\nu} \langle X^{(k)\alpha}, X_\alpha^{(k)} \rangle).$$

We have

$$\begin{aligned}
T_{uu,v}^{(k)} &= -\frac{1}{2} T_{\mu}^{(k)\nu}{}_{,\nu} u^{\mu} \\
&= -\frac{|t|^{2k-1}}{4} \{(2k-1)\langle X_{\theta}^{(k)}, X_t^{(k)} \rangle + (k-1)\langle X_t^{(k)} \rangle^2 + k\langle X_{\theta}^{(k)} \rangle^2\} \\
&\quad + \frac{|t|^{2k}}{2} \langle X_u^{(k)}, N^{(k)} \rangle + \frac{1}{2} \epsilon^{(k)} |t|^{2k} \langle X_{\alpha}, K(X_u, X^{\alpha}) X^{(k)} \rangle, \\
T_{vv,u}^{(k)} &= -\frac{1}{2} T_{\mu}^{(k)\nu}{}_{,\nu} v^{\mu} \\
&= -\frac{|t|^{2k-1}}{4} \{(1-2k)\langle X_{\theta}^{(k)}, X_t^{(k)} \rangle + (k-1)\langle X_t^{(k)} \rangle^2 + k\langle X_{\theta}^{(k)} \rangle^2\} \\
&\quad + \frac{|t|^{2k}}{2} \langle X_v^{(k)}, N^{(k)} \rangle + \frac{1}{2} \epsilon^{(k)} |t|^{2k} \langle X_{\alpha}, K(X_v, X^{\alpha}) X^{(k)} \rangle,
\end{aligned}$$

$\epsilon^{(k)} = 0$ if $k = 0$, $\epsilon^{(k)} = 1$ otherwise, therefore

$$\begin{aligned}
\frac{\partial}{\partial u} (u T_{vv}^{(k)}) + \frac{\partial}{\partial v} (v T_{uu}^{(k)}) &= \frac{|t|^{2k}}{2} \left\{ (k+1) |X_{\theta}^{(k)}|^2 + k |X_t^{(k)}|^2 + \frac{(2k-1)\theta}{|t|} \langle X_{\theta}^{(k)}, X_t^{(k)} \rangle \right\} \\
&\quad - \frac{|t|^{2k+1}}{2} \langle X_t^{(k)} + \frac{\theta}{|t|} X_{\theta}^{(k)}, N^{(k)} \rangle - \frac{1}{2} \epsilon^{(k)} |t|^{2k} \langle X_{\alpha}, K(X_t + \frac{\theta}{|t|} X_{\theta}, X^{\alpha}) X^{(k)} \rangle. \quad (3.4.2)
\end{aligned}$$

Integrating (3.4.2) over $C_{t_0}^{t_1}$ yields, for $k \geq 1$,

$$\begin{aligned}
&\int_{C_{t_0}^{t_1}} |t|^{2k} (3 |X_{\theta}^{(k)}|^2 + |X_t^{(k)}|^2) - 2 \int_{C_{t_0}^{t_1}} |t|^{2k+1} \left\langle X_t^{(k)} + \frac{\theta}{|t|} X_{\theta}^{(k)}, N^{(k)} \right\rangle \\
&\leq |t|^{2k+1} \int_{B(t)} (|X_t^{(k)}|^2 + |X_{\theta}|^2 + \frac{2\theta}{|t|} \langle X_t^{(k)}, X_{\theta}^{(k)} \rangle) d\theta \Big|_{t_0}^{t_1} \\
&\quad + 2\epsilon^{(k)} \int_{C_{t_0}^{t_1}} \left| |t|^{2k} \langle X_{\alpha}, K(X_t + \frac{\theta}{|t|} X_{\theta}, X^{\alpha}) X^{(k)} \rangle \right|, \quad (3.4.3)
\end{aligned}$$

where $f(t)|_{t_0}^{t_1} = f(t_1) - f(t_0)$. For $k = 1$ we have

$$|N^{(1)}| = |K(X_{\theta}, X_t) X_t| \leq |K| |X_t|^2 |X_{\theta}|$$

and straightforward manipulations lead to

$$\int_{C_{t_0}^{t_1}} |t|^2 \{2 |X_{\theta\theta}|^2 + \frac{1}{2} |X_{\theta t}|^2\} \leq 3 \int_{C_{t_0}^{t_1}} (|K| v_0^2)^2 |X_{\theta}|^2 + C,$$

and we have used proposition 3.2.1 to estimate the integrals at the right hand side of (3.4.3) by a constant C . Lemma 3.2.1 part 2 and an induction argument yield similarly

$$\int_{C_{t_0}^{t_1}} |t|^{2k} \{ |X_{\theta}^{(k)}|^2 + |X_t^{(k)}|^2 \} \leq C_1(k) (|K| v_0^2)^2 \int_{C_{t_0}^{t_1}} |X_{\theta}|^2 + C_2(k),$$

which had to be established. \square

Remark. If $k = 0$ we have $N^{(0)} = 0$ and (3.4.2) integrated over $C_{t_0}^{t_1}$ yields

$$\int_{C_{t_0}^{t_1}} (|X_\theta|^2 - \frac{\theta}{|t|} \langle X_\theta, X_t \rangle) \leq C \quad (3.4.4)$$

for some t_1 -independent constant C .

Proposition 3.4.1 *Let $t_0 < 0$, let $x(t_0, \cdot) \in C^k(S^1)$, $X_t(t_0, \cdot) \in C^{k-1}(S^1)$, $k \geq 2$, suppose that either*

$$\int_{C_{t_0}^0} |X_\theta|^2 < \infty \quad (3.4.5)$$

or

$$\int_{C_{t_0}^0} |\langle X_\theta, X_t \rangle| < \infty.$$

Then for all $0 \leq |\alpha| \leq k - 1$

$$\lim_{\substack{(t, \theta) \rightarrow (0, 0) \\ (t, \theta) \in C_{t_0}^0}} t^{|\alpha|+1} |D^\alpha X_\theta| = 0.$$

Proof. (3.4.4) shows that without loss of generality we can suppose that (3.4.5) holds.

Lemma 3.4.1, part a) shows that

$$\int_{C_{t_0}^0} t^2 (|X_{\theta\theta}|^2 + |X_{\theta t}|^2) < \infty,$$

therefore for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $|t| \leq \delta$

$$\int_{C_{2t}^{t/2}} t^2 (|X_{\theta\theta}|^2 + |X_{\theta t}|^2) + \int_{C_{2t}^{t/2}} |X_\theta|^2 \leq \epsilon. \quad (3.4.6)$$

For $0 \leq t \leq \min(|t_0|, \pi/2)$ let $f_t : C_{-2}^{-\frac{1}{2}} \rightarrow \mathbb{R}^+ \cup \{0\}$ be defined by

$$C_{-2}^{-\frac{1}{2}} \ni (s, \theta) \rightarrow f_t(s, \theta) = t |X_\theta|(ts, t\theta).$$

By (3.4.6) and by the pointwise estimates of proposition 3.2.1 for any $p \geq 2$ and for $t \leq \delta$ we have

$$\begin{aligned} \int_{C_{-2}^{-\frac{1}{2}}} |\partial_\mu f_t|^p &\leq C(p) \epsilon, \\ \int_{C_{-2}^{-\frac{1}{2}}} |f_t|^2 &\leq \epsilon, \end{aligned} \quad (3.4.7)$$

so that setting p equal to, say, 3, one obtains by Sobolev inequality

$$\forall (s, \theta) \in C_{-2}^{-\frac{1}{2}} \quad |f_t(s, \theta)| \leq C' \epsilon.$$

so that

$$\forall |\theta| \leq |t| \quad |t| |X_\theta(t, \theta)| \leq C' \epsilon.$$

The higher derivatives estimates follow in a similar way from Lemma 3.4.1, part b). \square

Proposition 3.4.2 *Suppose that $|K|v_0^2 < \frac{1}{3\sqrt{6}}$. Then the conclusion of proposition 3.4.1 holds.*

Proof. Without loss of generality we may suppose $-\pi \leq t_0$. Let

$$C_{t_0}^{t\pm} = \{(s, \theta) \in C_{t_0}^t, \pm\theta \geq 0\}.$$

Integrating $(\frac{d}{d\theta} - \frac{d}{dt})(\theta|X_\theta|^2)$ over $C_{t_0}^{t_1+}$ and $(\frac{d}{d\theta} + \frac{d}{dt})(\theta|X_\theta|^2)$ on $C_{t_0}^{t_1-}$, adding the resulting identities one obtains

$$\int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2 = \int_{C_{t_0}^{t_1}} 2\langle X_\theta, |\theta|X_{\theta t} - \theta X_{\theta\theta} \rangle - \int_{B(t)} |\theta| \langle X_\theta \rangle^2 \Big|_{t_0}^{t_1}, \quad (3.4.8)$$

the estimates of Lemma 3.4.1, part 1, and (3.4.8) give a t_1 -independent bound C_1 for $\int_{B(t)} (\cdot) \Big|_{t_0}^{t_1}$, therefore

$$\begin{aligned} \int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2 &\leq C_1 + 2 \int_{C_{t_0}^{t_1}} |t| |X_\theta| (|X_{\theta t}| + |X_{\theta\theta}|) \\ &\leq C_1 + 2 \left(\int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2 \right)^{\frac{1}{2}} \left\{ \int_{C_{t_0}^{t_1}} |t|^2 |X_{\theta t}|^2 \right\}^{\frac{1}{2}} + \left(\int_{C_{t_0}^{t_1}} |t|^2 |X_{\theta\theta}|^2 \right)^{\frac{1}{2}} \Big\} \\ &\leq C_1 + (\epsilon_1 + \epsilon_2) \int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2 + \frac{1}{\epsilon_1} \int_{C_{t_0}^{t_1}} |t|^2 |X_{\theta t}|^2 + \frac{1}{\epsilon_2} \int_{C_{t_0}^{t_1}} |t|^2 |X_{\theta\theta}|^2 \\ &\leq C_2 + \left[\epsilon_1 + \frac{6(v_0^2|K|)^2}{\epsilon_1} + \epsilon_2 + \frac{3(v_0^2|K|)^2}{2\epsilon_2} \right] \int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2, \end{aligned}$$

where we have used the Schwartz inequality and Lemma 3.4.1, point a). Setting $\epsilon_1 = \sqrt{6} v_0^2 |K|$, $\epsilon_2 = \sqrt{\frac{3}{2}} v_0^2 |K|$ one obtains

$$(1 - 3\sqrt{6} v_0^2 |K|) \int_{C_{t_0}^{t_1}} \langle X_\theta \rangle^2 \leq C_2,$$

so that for $|K|v_0^2 < (3\sqrt{6})^{-1}$

$$\forall t_0 \leq t < 0 \quad \int_{C_{t_0}^t} \langle X_\theta \rangle^2 \leq C_3 = \frac{C_2}{(1 - 3\sqrt{6}v_0^2|K|)} ,$$

and the Lebesgue monotone convergence theorem implies that the hypotheses of proposition 3.4.1 hold. \square

Propositions 3.2.1 and 3.4.2 imply:

Corollary 3.4.1 *Let x be a C^i , $i \geq 2$ solution of (3.1.1) and suppose that*

$$(|X_t|^2 + |X_\theta|^2)(t_0, \theta) < \frac{1}{6^{\frac{3}{2}} t_0^2 |K|} .$$

Then the conclusion of proposition 3.4.1 holds.

3.5 The Stability Theorem.

Let (Σ, g, K) be $U(1) \times U(1)$ symmetric Cauchy data, $\Sigma \approx T^3$, let $X_a = X_a^i \frac{\partial}{\partial x^i}$, $a = 1, 2$ be the Killing vectors generated by the $U(1) \times U(1)$ action. It has been shown in [32] that if one assumes

$$(K_{ij} - g^{kl} K_{kl} g_{ij}) X_a^j = 0 \quad \left(\Longleftrightarrow \quad c_a = \epsilon_{\alpha\beta\gamma\delta} X_1^\alpha X_2^\beta \nabla^\gamma X_a^\delta = 0 \right) , \quad (3.5.1)$$

then there exists a coordinate system $\{t \in (-\infty, 0), \theta, x^a \in [0, 2\pi]_{\text{mod } 2\pi}, a = 1, 2\}$ in which the metric takes the form

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = e^{2B} (-dt^2 + d\theta^2) + \lambda |t| n_{ab} (dx^a + g^a d\theta)(dx^b + g^b d\theta) , \quad (3.5.2)$$

$$\begin{aligned} n_{ab} dx^a dx^b &= (\cosh \rho + \cos \phi \sinh \rho) (dx^1)^2 + 2 \sinh \rho \sin \phi dx^1 dx^2 \\ &\quad + (\cosh \rho - \cos \phi \sinh \rho) (dx^2)^2 , \end{aligned}$$

$$B = B(t, \theta), \quad \rho = \rho(t, \theta), \quad \phi = \phi(t, \theta) ,$$

where λ and g^a are real constants, $\lambda > 0$. For a metric of the form (3.5.2) the dynamical part of Einstein equations reduces (cf. e.g. [32], or [62]) to harmonic-map-type equations

(3.1.1) for a map $x(t, \theta) = (\rho(t, \theta), \phi(t, \theta)) : (-\infty, 0) \times S^1 \rightarrow \mathcal{H}^2$, where $\mathcal{H}^2 \approx \mathbb{R}^2$ is the unit hyperboloid with the metric

$$ds^2 = d\rho^2 + \sinh^2 \rho d\phi^2 .$$

We also have the constraint equations

$$\frac{\partial B}{\partial t} = -\frac{1}{4t} + \frac{t}{4} [|X_t|^2 + |X_\theta|^2] , \quad (3.5.3)$$

$$\frac{\partial B}{\partial \theta} = \frac{t}{2} \langle X_t, X_\theta \rangle . \quad (3.5.4)$$

The main result of this chapter is the following:

Theorem 3.5.1 *Let $\Sigma \approx T^3$ and let (g_o, K_o) be Cauchy data for a Kasner metric with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, or permutation thereof. There exists $\epsilon > 0$ such that for all $U(1) \times U(1)$ symmetric Cauchy data $(g, K) \in C^\infty(\Sigma)$ satisfying (3.5.1), for which*

$$|| (g - g_o, K - K_o) ||_{H_1(\Sigma) \oplus L^2(\Sigma)} < \epsilon ,$$

the maximal globally hyperbolic Hausdorff development (M, γ) of (Σ, g, K) is future inextendible. Moreover on every future inextendible timelike curve in M the curvature scalar $|R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}|$ tends to infinity in finite proper time.

Remark: If one assumes that the Cauchy data are given directly in the form appropriate for a metric of the form (3.5.2), then it is sufficient to assume that $(g, K) \in H_1(\Sigma) \oplus L^2(\Sigma)$. It should be pointed out that the construction which leads from general coordinates to the coordinates of (3.5.2) decreases the degree of differentiability of the components of the metric tensor.

To prove Theorem 3.5.1 we shall need the following:

Proposition 3.5.1 *Let Γ be a future inextendible timelike curve in a vacuum space-time with a metric of the form (3.5.2), then Γ reaches $t = 0$ in finite proper time.*

Proof: Let $\Gamma = \{x^\mu(s)\}$, where s is a proper time along Γ , with $t(s)$ being an increasing function of s . From $\gamma_{\mu\nu}x^\mu x^\nu = -1$ we have

$$e^B \frac{dt}{ds} \geq 1 . \quad (3.5.5)$$

The constraint equation (3.5.3) and Proposition 3.2.1 give

$$\frac{-1 + E_1}{4t} \leq \frac{\partial B}{\partial t} \leq -\frac{1}{4t} ,$$

where

$$E_1 = \sup_{t \geq t_o} t^2[|X_t|^2 + |X_\theta|^2] < \infty$$

which implies, for some constant C ,

$$C^{-1}|t|^{(E_1-1)/4} \leq e^B \leq C|t|^{-1/4} ,$$

so that (3.5.5) implies, for $s_2 \geq s_1$,

$$C|t|^{-1/4} \frac{dt}{ds} \geq 1 \implies s_2 \leq s_1 + 4C(|t(s_1)|^{3/4} - |t(s_2)|^{3/4})/3 ,$$

thus Γ reaches $t = 0$ in finite proper time. \square

Proof of Theorem 3.5.1: Consider the map $x_o(t, \theta) = (\rho_o(t, \theta), \phi_o(t, \theta)) = (0, 0)$; it is easily seen that x_o solves the dynamic equations (3.1.1), integrating (3.5.2) one finds that the corresponding metric is the Kasner metric with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. It follows from Corollary 3.4.1 that for all $x(t_o, \theta)$, $X_t(t_o, \theta)$ satisfying

$$t_o^2[|X_t|^2 + |X_\theta|^2] < 6^{-3/2}$$

we have

$$\lim_{t \rightarrow 0} t|X_\theta| = \lim_{t \rightarrow 0} t^2|X_{\theta\theta}| = \lim_{t \rightarrow 0} t^2|X_{t\theta}| = 0 , \quad (3.5.6)$$

moreover from Proposition 3.2.1, point a) it follows that

$$v(t, \theta) \equiv t|X_t|(t, \theta) \leq 2^{1/2}6^{-3/4} < 1 . \quad (3.5.7)$$

A SHEEP calculation gives

$$R^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}} = \frac{e^{-2B}}{8t^2} \{ (1 - t^2|X_t|^2) A^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}} + B^{\hat{\mu}}{}_{\hat{\nu}\hat{\alpha}\hat{\beta}} \} , \quad (3.5.8)$$

where hats refer to the orthonormal frame

$$e^{\hat{t}} = e^{-B} dt ,$$

$$e^{\hat{\theta}} = e^{-B} d\theta ,$$

$$e^{\hat{1}} = (\lambda|t|)^{1/2} e^{\rho/2} \left[\cos(\phi/2)(dx + g^1 d\theta) + \sin(\phi/2)(dy + g^2 d\theta) \right] ,$$

$$e^{\hat{2}} = (\lambda|t|)^{1/2} e^{-\rho/2} \left[-\sin(\phi/2)(dx + g^1 d\theta) + \cos(\phi/2)(dy + g^2 d\theta) \right] ,$$

with $x \equiv x^1$, $y \equiv x^2$, and where the non-vanishing components of $A^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ are

$$\begin{aligned} A^{\hat{t}}_{\hat{\theta}\hat{t}\hat{\theta}} &= 2, \quad A^{\hat{t}}_{\hat{1}\hat{t}\hat{1}} = v_\rho - 1, \quad A^{\hat{t}}_{\hat{2}\hat{t}\hat{2}} = -v_\rho - 1, \quad A^{\hat{t}}_{\hat{1}\hat{t}\hat{2}} = v_\phi, \\ A^{\hat{\theta}}_{\hat{1}\hat{\theta}\hat{1}} &= -v_\rho - 1, \quad A^{\hat{\theta}}_{\hat{2}\hat{\theta}\hat{2}} = v_\rho - 1, \quad A^{\hat{\theta}}_{\hat{1}\hat{\theta}\hat{2}} = -v_\phi, \quad A^{\hat{1}}_{\hat{2}\hat{1}\hat{2}} = 2, \end{aligned} \quad (3.5.9)$$

$$v_\rho \equiv t\rho_{,t}, \quad v_\phi = t \sinh \rho \phi_{,t},$$

while for $B^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ the following estimations hold

$$|B^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}\hat{\beta}}| \leq C \left[t^2 |X_\theta|^2 (1 + |t||X_t| + |t||X_\theta|) + t^2 |X_t||X_\theta| (1 + |t||X_t|) + t^2 |X_{\theta\theta}| + t^2 |X_{t\theta}| \right] . \quad (3.5.10)$$

(3.5.6) – (3.5.7) allow us to neglect all the terms involving $B^{\hat{\mu}}_{\hat{\nu}\hat{\alpha}\hat{\beta}}$ when calculating $\alpha \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ to obtain

$$\alpha \approx \frac{e^{-4B}}{4t^4} (1 - t^2 |X_t|^2)^2 (3 + t^2 |X_t|^2) .$$

Let

$$\tilde{a} = 2B + \frac{1}{2} \ln |t|, \quad \tilde{a}_o = \sup_{\theta} a(t_o, \theta) . \quad (3.5.11)$$

By equation (3.5.3) the function \tilde{a} is monotonically decreasing, therefore

$$e^{-4B} \geq |t| e^{-2\tilde{a}_o} \quad (3.5.12)$$

which together with (3.5.7) implies that there exists $\epsilon > 0$ such that for $t \geq t_1$, t_1 large enough, we have

$$|\alpha(t, \theta)| > \frac{\epsilon}{|t|^3} .$$

By Proposition 3.5.1 every future inextendible timelike curve reaches $t = 0$ in finite time, and Proposition C.2.4 establishes our claims. \square

The following result proves existence of curvature singularities in polarized Gowdy metrics on T^3 without smallness hypotheses:

Proposition 3.5.2 *Let x be a C^1 solution of (3.1.1) such that $X_\theta(t_0, \cdot)$, $X_t(t_0, \cdot) \in H_1(S^1)$, let*

$$\alpha = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$$

be the quadratic curvature scalar of the associated Gowdy space-time. Suppose that $|t||X_t|(t, \cdot)$ does not converge to 1 in $L^2(S^1)$ as t goes to zero. Then there exists a sequence of points (t_i, θ_i) , $t_i \rightarrow 0$, such that

$$|\alpha(t_i, \theta_i)| > \frac{\epsilon}{|t_i|^3}$$

for some $\epsilon > 0$.

Remarks:

1. Note that a sufficient condition for convergence in $L^2(S^1)$ is pointwise convergence,* so that Proposition 3.5.2 implies in particular that if $|t||X_t|(t, \cdot)$ converges pointwise to something different from 1 as t tends to zero, then there must be a curvature singularity somewhere on the boundary $t = 0$.
2. The proof of Proposition 3.5.2 does not imply existence of a singularity on the whole boundary $t = 0$ since there may be subsets of the set $t = 0$ on which $|t||X_t|(t, \cdot)$ converges to 1. It might happen that the metric is extendible through such subsets — this occurs indeed for some polarized Gowdy metrics [37] [36].

Proof. From the proof of Theorem 3.5.1 one obtains

$$\alpha = \frac{e^{-4B}}{4t^4} \{ (1 - t^2|X_t|^2)^2 (3 + t^2|X_t|^2) + \beta_1 + \beta_2 \}, \quad (3.5.13)$$

$$\beta_1 = \frac{1}{8}(1 - t^2|X_t|^2)B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}A^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}, \quad \beta_2 = \frac{1}{16}B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}B^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}},$$

therefore, by (3.5.12),

$$4 \oint \alpha |t|^3 d\theta \geq e^{-2\tilde{a}_0} \left[3 \oint (1 - |t|^2|X_t|^2)^2 d\theta - \left| \oint \beta_1 d\theta \right| - \left| \oint \beta_2 d\theta \right| \right], \quad (3.5.14)$$

where \tilde{a}_0 has been defined in (3.5.11). Suppose that $1 - |t|^2|X_t|^2$ does not converge to 0 in $L^2(S^1)$, therefore there exists $\epsilon > 0$ and a sequence $t_i \nearrow 0$ such that

$$\oint \left(1 - |t_i|^2|X_\theta|^2(t_i, \theta) \right)^2 d\theta > 8\pi e^{2\tilde{a}_0} \epsilon,$$

Proposition 3.2.1 and (3.5.10) imply,

$$\forall \quad \hat{\mu}, \hat{\nu}, \hat{\alpha}, \hat{\beta} \quad |B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}| \leq C, \quad (3.5.15)$$

which together with proposition 3.3.1, point 2, gives

$$\oint |B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}|^2(t, \theta) d\theta \xrightarrow{t \rightarrow 0}.$$

It follows that there exists $t(\epsilon) < 0$ such that for all $0 > t > t(\epsilon)$ we have

$$\left| \oint \beta_2(t, \theta) d\theta \right| < 8\pi e^{2\tilde{a}_0} \epsilon.$$

From $2xy \leq \delta x^2 + \delta^{-1}y^2$ it follows that

$$\oint 2B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}A^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}} d\theta \leq \sum \oint \delta |A_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}|^2 d\theta + \sum \oint \delta^{-1} |B_{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}|^2 d\theta,$$

which can be made arbitrarily small by an appropriate choice of δ for t large enough, so that for $t > t(\epsilon)$ we can also require

$$\left| \oint \beta_1(t, \theta) d\theta \right| < 8\pi e^{2\tilde{a}_0} \epsilon.$$

It follows that for $t_i > t(\epsilon)$ we have

$$\oint \alpha(t_i, \theta) d\theta > \frac{2\pi\epsilon}{|t_i|^3},$$

therefore there exist points θ_i such that

$$\alpha(\theta_i, t_i) > \frac{\epsilon}{|t_i|^3}.$$

□

Appendix A

On the “hyperboloidal initial data”, and Penrose conditions.

Let us briefly recall the conformal framework introduced by Penrose [104] to describe the behaviour of physical fields at null infinity. Given a, say vacuum, smooth “physical” space-time $(\tilde{M}, \tilde{\gamma})$ one associates to it a smooth “unphysical space-time” (M, γ) and a smooth function Ω on M , such that \tilde{M} is a subset of M and

$$\Omega|_{\tilde{M}} > 0, \quad \gamma_{\mu\nu}|_{\tilde{M}} = \Omega^2 \tilde{\gamma}_{\mu\nu}, \quad (\text{A.0.1})$$

$$\Omega|_{\partial\tilde{M}} = 0, \quad (\text{A.0.2})$$

$$d\Omega(p) \neq 0 \quad \text{for } p \in \partial\tilde{M}, \quad (\text{A.0.3})$$

where $\partial\tilde{M}$ is the boundary of \tilde{M} in M (it should be stressed that in this section a notation *inverse* to that used in 1.6 is used: tilded quantities denote the physical ones, while non-tilded quantities denote the unphysical (conformally rescaled) ones). It is common usage in general relativity to use the symbol \mathcal{I} for $\partial\tilde{M}$, and we shall sometimes do so. If Σ is a hypersurface in M , by \mathcal{I}^+ we shall denote the connected component of \mathcal{I} which intersects the causal future of Σ . The hypothesis of smoothness of (M, γ, Ω) and the fact that $(\tilde{M}, \tilde{\gamma})$ is vacuum imposes several restrictions on various fields; if one defines (cf. [104])

$$P_{\mu\nu} = \frac{1}{2} (R_{\mu\nu} - \frac{1}{6} R \gamma_{\mu\nu}), \quad (\text{A.0.4})$$

with an analogous definition for the tilded quantities, one has

$$0 = \tilde{P}_{\mu\nu} = P_{\mu\nu} - \frac{1}{\Omega} \nabla_\mu \nabla_\nu \Omega + \frac{1}{2\Omega^2} \nabla^\alpha \Omega \nabla_\alpha \Omega \gamma_{\mu\nu}, \quad (\text{A.0.5})$$

where ∇_μ is the covariant derivative of the metric $\gamma_{\mu\nu}$. (A.0.2) and (A.0.5) imply

$$\nabla^\alpha \Omega \nabla_\alpha \Omega|_{\partial\tilde{M}} = 0 \quad (\text{A.0.6})$$

$$(\nabla_\mu \nabla_\nu \Omega - \frac{1}{4} \nabla^\alpha \nabla_\alpha \Omega \gamma_{\mu\nu})|_{\partial\tilde{M}} = 0. \quad (\text{A.0.7})$$

Suppose that $\Sigma \subset M$ is a spacelike hypersurface in (M, γ) , let $\tilde{\Sigma} = \Sigma \cap \tilde{M}$, $\partial\Sigma = \partial\tilde{\Sigma} = \Sigma \cap \partial\tilde{M}$, and let g_{ij}, K_{ij} , respectively $\tilde{g}_{ij}, \tilde{K}_{ij}$, be the induced metric and extrinsic curvature of Σ in (M, γ) , respectively $\tilde{\Sigma}$ in $(\tilde{M}, \tilde{\gamma})$. If we denote by L^{ij} and \tilde{L}^{ij} the traceless part of $K^{ij} = g^{ik} g^{j\ell} K_{k\ell}$, $\tilde{K}^{ij} = \tilde{g}^{ik} \tilde{g}^{j\ell} \tilde{K}_{k\ell}$,

$$\begin{aligned} L^{ij} &= K^{ij} - \frac{1}{3} K g^{ij}, \quad K = g^{ij} K_{ij}, \\ \tilde{L}^{ij} &= \tilde{K}^{ij} - \frac{1}{3} \tilde{K} \tilde{g}^{ij}, \quad \tilde{K} = \tilde{g}^{ij} \tilde{K}_{ij}, \end{aligned} \quad (\text{A.0.8})$$

one finds

$$\begin{aligned} \tilde{L}^{ij} &= \Omega^3 L^{ij}, \quad |\tilde{L}|_{\tilde{g}} = \Omega |L|_g, \\ \tilde{K} &= \Omega K - 3 n^\alpha \Omega_{,\alpha}, \end{aligned} \quad (\text{A.0.9})$$

where n^α is the unit normal to Σ for the metric γ , and $|\cdot|_h$ denotes the tensor norm in a Riemannian metric h . Since n^α is timelike and $\nabla\Omega(p)$ is null for $p \in \partial\tilde{\Sigma}$ we have

$$\tilde{K}|_{\partial\tilde{\Sigma}} = -3n^\alpha \Omega_{,\alpha}|_{\partial\tilde{\Sigma}} \gtrless 0, \quad (\text{A.0.10})$$

because the scalar product of two non-vanishing non-spacelike vectors cannot change sign. From (A.0.2) we also have

$$g_{ij}|_{\tilde{\Sigma}} = \Omega^2 \tilde{g}_{ij},$$

and since $\nabla\Omega$ is null non-vanishing at $\partial\tilde{\Sigma}$ the equations (A.0.9)–(A.0.10) imply

$$D^i \Omega D_i \Omega|_{\partial\tilde{\Sigma}} = \left(\frac{\tilde{K}}{3} \right)^2 \Big|_{\partial\tilde{\Sigma}} > 0, \quad (\text{A.0.11})$$

where D_i is the Riemannian connection of the metric g_{ij} . To summarize, necessary conditions for an initial data set $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ to arise from an “extended initial data set (Σ, g, K) intersecting a smooth \mathcal{I} ” are

C1. There exists a Riemannian manifold (Σ, g) , $g \in C^k(\Sigma)$, such that $\tilde{\Sigma}$ is a submanifold of Σ with smooth boundary $\partial\tilde{\Sigma}$. Moreover there exists a function $\Omega \in C^k(\Sigma)$ such that

$$g_{ij}|_{\tilde{\Sigma}} = \Omega^2 \tilde{g}_{ij}, \quad (\text{A.0.12})$$

$$\Omega|_{\partial\tilde{\Sigma}} = 0, \quad |D\Omega|_g|_{\partial\tilde{\Sigma}} > 0. \quad (\text{A.0.13})$$

C2. The symmetric tensor field \tilde{K}^{ij} satisfies, for some $\tilde{K} \in C^{k-1}(\Sigma)$, $\tilde{L}^{ij} \in C^{k-1}(\Sigma)$,

$$\tilde{K}^{ij} = \tilde{L}^{ij} + \frac{1}{3} \tilde{K} \tilde{g}^{ij}, \quad \tilde{K} = \tilde{g}_{ij} \tilde{K}^{ij}, \quad (\text{A.0.14})$$

$$\tilde{K}|_{\partial\tilde{\Sigma}} \text{ is nowhere vanishing}, \quad (\text{A.0.15})$$

$$|\tilde{L}|_{\tilde{g}}|_{\partial\tilde{\Sigma}} = 0. \quad (\text{A.0.16})$$

If there existed “a lot” of space-times satisfying the Penrose conformal conditions, there should exist “a lot” of initial data satisfying C1–C2. It is therefore natural to ask the question, can one construct such data sets? This involves constructing solutions of the scalar constraint equation,

$$\bar{R}(\tilde{g}) + \tilde{K}^2 - \tilde{K}_{ij} \tilde{K}^{ij} = 0, \quad (\text{A.0.17})$$

where $\bar{R}(\tilde{g})$ is the Ricci scalar of the metric \tilde{g} , and the vector constraint equation,

$$\tilde{D}_i(\tilde{K}^{ij} - \tilde{K} \tilde{g}^{ij}) = 0, \quad (\text{A.0.18})$$

where \tilde{D} is the Riemannian connection of the metric \tilde{g} , under appropriate asymptotic conditions. No general method of producing solutions of (A.0.17)–(A.0.18) is known, unless one assumes

C3.

$$\tilde{D}_i \tilde{K} \equiv 0. \quad (\text{A.0.19})$$

Under (A.0.19) the scalar and the vector constraint equations decouple, and the well known Choquet-Bruhat-Lichnerowicz-York conformal procedure allows one to construct

solutions of (A.0.17)–(A.0.18). An initial data set satisfying C1–C3 will be called a C^k *hyperboloidal initial data set* (smooth if $k = \infty$), while conditions C1–C2 will be called Penrose’s C^k conditions. Without loss of generality we may normalize \tilde{K} so that

$$\tilde{K} = 3, \quad (\text{A.0.20})$$

and (A.0.17)–(A.0.18) can be rewritten as

$$\tilde{R}(\tilde{g}) + 6 = \tilde{L}_{ij} \tilde{L}^{ij} \quad (\text{A.0.21})$$

$$\tilde{D}_i \tilde{L}^{ij} = 0. \quad (\text{A.0.22})$$

To construct solutions of (A.0.21)–(A.0.22) one can proceed as follows: fix a compact Riemannian manifold $(\Sigma, \overset{\circ}{g})$, let $\tilde{\Sigma}$ be an open submanifold of Σ with compact closure and with smooth boundary $\partial\tilde{\Sigma}$, and let $\overset{\circ}{\Omega}$ be any defining function for $\partial\tilde{\Sigma}$ (by definition,

$$\overset{\circ}{\Omega}|_{\partial\tilde{\Sigma}} = 0, \quad |d\overset{\circ}{\Omega}|_{\tilde{g}}|_{\partial\tilde{\Sigma}} > 0,$$

and $\overset{\circ}{\Omega}(p) = 0 \Rightarrow p \in \partial\tilde{\Sigma}$), set

$$\bar{g}_{ij} = \overset{\circ}{\Omega}^{-2} \overset{\circ}{g}_{ij}.$$

Given a smooth traceless symmetric tensor field $\overset{\circ}{L}^{ij}$ on Σ satisfying

$$\bar{D}_i(\overset{\circ}{\Omega}^3 \overset{\circ}{L}^{ij}) = 0 \implies \overset{\circ}{D}_i(\Omega_0^{-2} \overset{\circ}{L}^{ij}) = 0, \quad (\text{A.0.23})$$

where \bar{D} , $\overset{\circ}{D}$ are the Riemannian connections of the metrics \bar{g} , $\overset{\circ}{g}$, it is not too difficult to check that the fields

$$\tilde{g}_{ij} = \phi^4 \bar{g}_{ij},$$

$$\tilde{L}^{ij} = \phi^{-10} \bar{L}^{ij} \equiv \phi^{-10} \overset{\circ}{\Omega}^3 \overset{\circ}{L}^{ij},$$

will satisfy (A.0.21)–(A.0.22) if

$$8\bar{\Delta}\phi - \bar{R}\phi + \lambda\phi^{-7} - 6\phi^5 = 0, \quad (\text{A.0.24})$$

$$\lambda \equiv \bar{g}_{ij} \bar{g}_{k\ell} \bar{L}^{ik} \bar{L}^{j\ell},$$

where $\bar{\Delta} = \bar{D}^i \bar{D}_i$ is the Laplacian of the metric \bar{g}_{ij} . If moreover

$$\phi|_{\partial\tilde{\Sigma}} = 1 \quad (\text{A.0.25})$$

$$|\bar{L}|_{\bar{g}}|_{\partial\tilde{\Sigma}} = 0. \quad (\text{A.0.26})$$

then $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will satisfy C1–C3. In [2] [3] the following is proved:

Theorem A.0.2 *For any smooth $(\tilde{\Sigma}, \overset{\circ}{g}, \overset{\circ}{\Omega}, \overset{\circ}{L})$ as above there exists a solution of (A.0.24)–(A.0.25), moreover:*

1. *For given $(\Sigma, \tilde{\Sigma})$ and for an open dense set (in $C^\infty(\Sigma)$ topology) of $(\overset{\circ}{g}, \overset{\circ}{\Omega}, \overset{\circ}{L})$ the function ϕ^{-2} can be extended to a C^2 function from $\tilde{\Sigma}$ to Σ , but not to a C^3 function on Σ (the third derivatives of any extension of ϕ will logarithmically blow up as one approaches $\partial\tilde{\Sigma}$); in particular for generic (in the above sense) triples $(\overset{\circ}{g}, \overset{\circ}{\Omega}, \overset{\circ}{L})$ the initial data set $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ will display asymptotic behaviour incompatible with Penrose’s C^4 conditions.*
2. *There exists a “large set” of non-generic $(\overset{\circ}{g}, \overset{\circ}{\Omega}, \overset{\circ}{L})$ for which $\Omega \equiv \phi^{-2} \overset{\circ}{\Omega}$ can be smoothly extended from $\tilde{\Sigma}$ to Σ .*

It should be emphasized that in Theorem A.0.2 no hypotheses on the topology of Σ , $\tilde{\Sigma}$ and $\partial\tilde{\Sigma}$ are made, thus the resulting space-time may have a conformal boundary consisting of several connected components of varying topology (recall that *e.g.* some Robinson-Trautman space-times admit a smooth \mathcal{I} the “spatial” topology of which is not a sphere). Let us also note that even considering only those data sets for which $\overset{\circ}{L} = 0$, or for which $\overset{\circ}{L}{}^{ij}$ vanishes on $\partial\tilde{\Sigma}$ to some desired order, point (1) above will still hold in the sense that for generic $(\overset{\circ}{g}, \overset{\circ}{\Omega})$ and $\overset{\circ}{L}$ ’s vanishing to some prescribed order (or even *e.g.* identically vanishing) no C^3 extensions of ϕ from $\tilde{\Sigma}$ to Σ will exist.

To complete the construction of initial data sets one also has to produce solutions of (A.0.23), the standard approach proceeds as follows: Let A^{ij} be a traceless tensor field, let X^i solve the equation

$$\overset{\circ}{D}{}_j(\overset{\circ}{D}{}^i X^j + \overset{\circ}{D}{}^j X^i - \frac{2}{3} \overset{\circ}{D}{}^k X_k \overset{\circ}{g}{}^{ij}) = -\overset{\circ}{D}{}_j A^{ij}. \quad (\text{A.0.27})$$

The tensor field defined by

$$\mathring{L}^{ij} = \mathring{\Omega}^2(A^{ij} + \mathring{D}^i X^k + \mathring{D}^j X^i - \frac{2}{3} \mathring{D}^k X_k \mathring{g}^{ij}) \quad (\text{A.0.28})$$

will satisfy (A.0.23). If we want to obtain \mathring{L}^{ij} 's which do not necessarily vanish to second order at $\partial\tilde{\Sigma}$, we have to admit A^{ij} 's of the form

$$A^{ij} = \mathring{\Omega}^{-2} \mathring{A}^{ij}, \quad (\text{A.0.29})$$

for some smooth \mathring{A}^{ij} , which gives $\mathring{D}^i A^{ij} = \mathring{\Omega}^{-3} Z^i$, for some smooth vector field Z^i , nonidentically vanishing at $\partial\tilde{\Sigma}$ for generic non-vanishing \mathring{A}^{ij} at $\partial\tilde{\Sigma}$; note that X^i should vanish to third order at $\partial\tilde{\Sigma}$ if \mathring{A}^{ij} vanishes to second order there). In [2] the following is established:

Theorem A.0.3 *Let (Σ, \mathring{g}) be a smooth Riemannian manifold, let $\tilde{\Sigma}$ be a smooth submanifold of Σ with smooth boundary $\partial\tilde{\Sigma}$, with a defining function $\mathring{\Omega}$. Consider the problem*

$$\begin{aligned} (\Delta_{\mathring{L}, \mathring{g}} X)^j &\equiv \mathring{D}^i (\mathring{D}^j X^i + \mathring{D}^i X^j - \frac{2}{3} \mathring{D}^k X_k \mathring{g}^{ij})|_{\tilde{\Sigma}} \\ &= \mathring{\Omega}^\alpha Y^j|_{\tilde{\Sigma}}, \end{aligned} \quad (\text{A.0.30})$$

where Y^j is a smooth vector field on Σ and α — a negative integer. There exists a solution of (A.0.30) of the form

$$X = \mathring{\Omega}^{\alpha+2} X_1 + \log \mathring{\Omega} X_{\log} + X_0, \quad (\text{A.0.31})$$

where X_1 , X_{\log} , X_0 are smooth vector fields on Σ . For any $\alpha \neq 0$ there exists an open dense set of Y 's (in a $C^\infty(\Sigma)$ topology) for which $X_{\log}|_{\partial\tilde{\Sigma}} \neq 0$. If $\alpha = -1$ then $X_{\log} = \mathring{\Omega} \hat{X}_{\log}$, for some smooth vector field \hat{X}_{\log} on Σ . If $\alpha = 0$, $X_{\log} \equiv 0$. X is unique in the class of solutions of the form (A.0.31), with smooth X_0 , X_1 and X_{\log} .

It should be pointed out that for generic \mathring{A}^{ij} , the source term in (A.0.27) with A^{ij} given by (A.0.29) will be generic and thus the corresponding solution X will have log terms, consequently \mathring{L}^{ij} given by (A.0.28) will be C^1 but *not* C^2 extendible from $\tilde{\Sigma}$ to Σ . A

similar thing will happen when considering those \mathring{A}^{ij} which vanish to order one at $\partial\tilde{\Sigma}$: generic such solutions will be C^2 but not C^3 extendible. On the other hand if \mathring{A}^{ij} vanishes to order two or higher at the boundary, then the source term in (A.0.27) will be smooth, and so will be the solution X : in this case no log terms occur.

In order to obtain Cauchy data which can be used in Friedrich's stability theorem, Theorem 1.6.1, some more restrictions on $(\tilde{\Sigma}, \tilde{g}, \tilde{K})$ are needed, namely the vanishing at $\partial\tilde{\Sigma}$ of both the tensor $e_{\alpha\beta}$ defined in Theorem 1.6.1 and of the Weyl tensor. In the case of L_{ij} vanishing on $\partial\tilde{\Sigma}$, it turns out that [3] smooth extendability of the function Ω across $\partial\tilde{\Sigma}$ is a necessary condition for $e_{\alpha\beta}|_{\partial\tilde{\Sigma}} = C_{\beta\gamma\delta}^\alpha|_{\partial\tilde{\Sigma}} = 0$. (More precisely, under the conditions $\tilde{K}_{ij}\tilde{g}^{ij} = \text{const}$, $L_{ij}|_{\partial\tilde{\Sigma}} = 0$, one shows [3] that the condition $e_{ij}|_{\partial\tilde{\Sigma}} = 0$ is equivalent to the fact that the extrinsic curvature of $\partial\tilde{\Sigma}$ in Σ is pure trace; this then implies that Ω is smooth up to boundary, and that $C_{\beta\gamma\delta}^\alpha|_{\partial\tilde{\Sigma}} = 0$, moreover Ω_n and Ω_{nn} can be chosen so that $e_{\alpha\beta}|_{\partial\tilde{\Sigma}} = 0$ holds.) Point 1 of Theorem A.0.2 thus shows, that generic data constructed by the conformal method will not be regular enough to be used in Friedrich's existence theorems. In fact the problem here is much more serious than just being one or two degrees of differentiability away from a threshold, because one of the fields used in Friedrich's "conformally regular system" is $d^\alpha_{\beta\gamma\delta} \equiv \Omega^{-1}C^\alpha_{\beta\gamma\delta}$, where $C^\alpha_{\beta\gamma\delta}$ is the Weyl curvature tensor of the space-time metric, evaluated formally from the Cauchy data (g, K) assuming vacuum Einstein equations. Whenever $C^\alpha_{\beta\gamma\delta}(p) \neq 0$ for $p \in \partial\tilde{\Sigma}$, the field $d^\alpha_{\beta\gamma\delta}$ blows up at $\partial\tilde{\Sigma}$ as $1/\Omega$, and is thus not even in $L^1(\Sigma)$. It should be stressed that nevertheless point 2 of Theorem A.0.2 establishes existence of a large class of non-trivial data with asymptotic behaviour compatible with the Penrose-Friedrich conditions.

Appendix B

On a class of $U(1) \times U(1)$ symmetric metrics found by V.Moncrief.

In this Appendix we shall prove that “strong cosmic censorship” holds in a six parameter family of *non-polarized* $U(1) \times U(1)$ symmetric metrics found by Moncrief¹ [93]. Apart from being interesting in their own right, these metrics provide a good testing ground for various *a priori* estimates one can obtain for general $U(1) \times U(1)$ symmetric metrics, cf. Chapter 3.

Throughout this Appendix the letter C denotes a constant the value of which may vary from line to line.

B.1 A harmonic map problem.

Let $x(t, \theta) = (\rho(t, \theta), \phi(t, \theta))$ be a map from two-dimensional Minkowski space to a two dimensional constant mean curvature hyperboloid, set

$$X_t = \frac{\partial x^A}{\partial t} \frac{\partial}{\partial x^A} = \frac{\partial \rho}{\partial t} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial t} \frac{\partial}{\partial \phi}, \quad X_\theta = \frac{\partial x^A}{\partial \theta} \frac{\partial}{\partial x^A} = \frac{\partial \rho}{\partial \theta} \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \phi}.$$

On the hyperboloid one can introduce coordinates in which the metric takes the form

$$ds^2 = d\rho^2 + \sinh^2 \rho d\phi^2.$$

¹A similar class of harmonic maps has been considered independently by Shatah and Tahvildar-Zadeh in [118]; cf. also [63].

The Christoffel symbols are easily calculated to be

$$\Gamma_{\phi\phi}^{\rho} = -\sinh \rho \cosh \rho, \quad \Gamma_{\phi\phi}^{\theta} = \frac{\cosh \rho}{\sinh \rho},$$

so that the Gowdy equations

$$\frac{DX_t}{Dt} - \frac{DX_{\theta}}{D\theta} = -\frac{X_t}{t},$$

where D is the covariant derivative in the target space, $D_t \equiv D_{X_t}$, $D_{\theta} \equiv D_{X_{\theta}}$, take the form

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \theta^2}\right)\phi = -\frac{1}{t} \frac{\partial \phi}{\partial t} - 2 \coth \rho \left(\frac{\partial \rho}{\partial t} \frac{\partial \phi}{\partial t} - \frac{\partial \rho}{\partial \theta} \frac{\partial \phi}{\partial \theta}\right) \quad (\text{B.1.1})$$

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \theta^2}\right)\rho = -\frac{1}{t} \frac{\partial \rho}{\partial t} + \sinh \rho \cosh \rho \left(\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial \theta}\right)^2\right). \quad (\text{B.1.2})$$

It has been observed by V. Moncrief [93] that the ansatz

$$\rho = \rho(t), \quad \phi = n\theta \quad (\text{B.1.3})$$

is compatible with the above equations, which then reduce to a single ordinary differential equation for ρ

$$\frac{d^2 \rho}{d\tau^2} = -n^2 \sinh \rho \cosh \rho e^{-2\tau}, \quad (t = e^{-\tau}). \quad (\text{B.1.4})$$

For given θ_0 the function $\rho(t)$ should be thought of as an affine parameter on the geodesic $\Gamma = \{\theta = \theta_0, \rho \geq 0\} \cup \{\theta = \pi + \theta_0, \rho \geq 0\}$ on the hyperboloid, rather than a radial coordinate constrained to satisfy $\rho \geq 0$, so that a change of sign of ρ means that $\rho(t)$ has crossed the origin along Γ . The following gives a complete description of the behaviour as $t \rightarrow 0$ of solutions of (B.1.4):

Proposition B.1.1 *1. For $\tau_0 > -\infty$ and for every solution $\rho \in C_2([\tau_0, \infty))$ of (B.1.4) there exist constants $0 \leq |v_{\infty}| < 1$ and $\rho_{\infty} \in \mathbb{R}$ such that*

$$|\rho(\tau) - v_{\infty}\tau - \rho_{\infty}| + \left|\frac{d\rho}{d\tau}(\tau) - v_{\infty}\right| \leq C e^{-2(1-|v_{\infty}|)\tau} \quad (\text{B.1.5})$$

for all $\tau_0 \leq \tau < \infty$, for some constant C .

2. For every $0 \leq |v_\infty| < 1$ and every $\rho_\infty \in \mathbb{R}$ there exists a solution of (B.1.4) satisfying (B.1.5). If $\rho_\infty = v_\infty = 0$ then $\rho(\tau) \equiv 0$.

Proof: 1. Let

$$g(\tau) \equiv \left(\frac{d\rho}{d\tau}\right)^2 + \frac{n^2}{2} \cosh^2 \rho e^{-2\tau}.$$

We have

$$\frac{dg(\tau)}{d\tau} = -n^2 \cosh^2 \rho e^{-2\tau},$$

which shows that g is monotonically decreasing, so that $g_\infty = \lim_{\tau \rightarrow \infty} g(\tau)$ exists and we have

$$\begin{aligned} g(\tau) &= g_\infty + n^2 \int_\tau^\infty \cosh^2 \rho(s) e^{-2s} ds \implies \\ &\int_{\tau_0}^\infty \cosh^2 \rho(s) e^{-2s} ds < \infty. \end{aligned} \quad (\text{B.1.6})$$

For $\tau_1 \geq \tau_2$ it follows from (B.1.4) that

$$\begin{aligned} \left| \frac{d\rho}{d\tau}(\tau_1) - \frac{d\rho}{d\tau}(\tau_2) \right| &= n^2 \left| \int_{\tau_2}^{\tau_1} \sinh \rho(s) \cosh \rho(s) e^{-2s} ds \right| \\ &\leq n^2 \int_{\tau_2}^{\tau_1} \cosh^2 \rho(s) e^{-2s} ds, \end{aligned}$$

which together with (B.1.6) implies that $\lim_{\tau \rightarrow \infty} \frac{d\rho}{d\tau}(\tau) = v_\infty$ exists.

Let us first assume $v_\infty = 0$, then for any $\epsilon > 0$ there exists τ_1 such for $\tau > \tau_1$ we have $|\frac{d\rho}{d\tau}| < \epsilon$, which implies that $|\rho| \leq \epsilon\tau + C$, and (B.1.4) gives

$$\left| \frac{d^2\rho}{d\tau^2} \right| \leq C_1 e^{-2(1-\epsilon)\tau} \implies \left| \frac{d\rho}{d\tau} \right| \leq C_2 e^{-2(1-\epsilon)\tau},$$

by integration, and one more integration shows that the limit $\lim_{\tau \rightarrow \infty} \rho(\tau) = \rho_\infty$ exists and we have

$$|\rho - \rho_\infty| \leq C_3 e^{-2(1-\epsilon)\tau} \implies \left| \frac{d^2\rho}{d\tau^2} \right| \leq C_4 e^{-2\tau} \implies |\rho - \rho_\infty| \leq C_5 e^{-2\tau},$$

for some constants $C_1 - C_5$, which had to be established for $v_\infty = 0$.

If $v_\infty \neq 0$, it follows that $\frac{d\rho}{d\tau}$ has constant sign for $\tau \geq \tau_1$, τ_1 large enough, so that ρ has constant sign for large enough times, and multiplying ρ by -1 if necessary we may

assume that for $\tau > \tau_1$ we have $\rho(\tau) > 0$, and also $v_\infty > 0$. Let us show that $v_\infty < 1$. Equation (B.1.4) implies that $\frac{d\rho}{d\tau}$ is non-increasing, so that if $0 < \frac{d\rho}{d\tau}(\tau_o) < 1$ we are done, let us therefore assume that $\frac{d\rho}{d\tau}(\tau_o) > 1$. Let $\tau_1 \leq \infty$ be such that for $\tau_o \leq \tau < \tau_1$ we have $\frac{d\rho}{d\tau}(\tau) > 1$, with $\frac{d\rho}{d\tau}(\tau_1) = 1$ if $\tau_1 < \infty$. For $\tau \in [\tau_o, \tau_1]$ we have

$$\rho(\tau) - \rho(\tau_o) = \int_{\tau_o}^{\tau} \frac{d\rho}{d\tau}(s) ds \geq \tau - \tau_o,$$

so that

$$\begin{aligned} \frac{d\rho}{d\tau}(\tau) &= \frac{d\rho}{d\tau}(\tau_o) - \int_{\tau_o}^{\tau} \frac{n^2}{4} (e^{2(\rho(\tau)-s)} - e^{-2\rho(s)-2s}) ds \\ &\leq \frac{d\rho}{d\tau}(\tau_o) - \int_{\tau_o}^{\tau} \frac{n^2}{4} e^{2(\rho(\tau_o)-\tau_o)} ds + \frac{n^2}{4} \int_{\tau_o}^{\tau} e^{-2(\rho(\tau_o)+s)} ds \\ &\leq \frac{d\rho}{d\tau}(\tau_o) - \frac{n^2}{4} (\tau - \tau_o) e^{2(\rho(\tau_o)-\tau_o)} + \frac{n^2}{8} e^{-2(\tau_o+\rho(\tau_o))}, \end{aligned}$$

which is smaller than 1 for sufficiently large τ so that $\tau_1 < \infty$, and our claim follows. Define

$$r = \rho(\tau) - v_\infty \tau.$$

The function r satisfies $\lim_{\tau \rightarrow \infty} \frac{dr}{d\tau} = 0$, setting $\lambda = 2(1 - v_\infty) > 0$ one obtains

$$\frac{d^2 r}{d\tau^2} = -\frac{n^2}{4} (e^{2r-\lambda\tau} - e^{-2r-2(1+v_\infty)\tau}). \quad (\text{B.1.7})$$

For $\tau > \tau_1(\epsilon)$, with $\tau_1(\epsilon) > 0$ sufficiently large, we have $|\frac{dr}{d\tau}| \leq \epsilon$ for any $\epsilon > 0$, therefore $|r(\tau)| \leq |r(\tau_1(\epsilon))| + \epsilon\tau$ and for $2\epsilon < \lambda/2$ one gets $\frac{d^2 r}{d\tau^2} = O(e^{-\lambda\tau/2})$; by integration one obtains $|\frac{dr}{d\tau}(\tau_2) - \frac{dr}{d\tau}(\tau)| = O(e^{-\lambda\tau/2})$, for $\tau_2 > \tau$. Passing with τ_1 to ∞ one gets $\frac{dr}{d\tau} = O(e^{-\lambda\tau/2})$, which integrating again yields, for $\tau_2 > \tau$

$$|r(\tau_2) - r(\tau)| = O(e^{-\lambda\tau/2}),$$

it ensues in a simple way that there exists a constant ρ_∞ such that

$$|r(\tau) - \rho_\infty| = O(e^{-\lambda\tau/2}).$$

Coming back to the equation (B.1.7) satisfied by r we have in fact

$$\left| \frac{d^2 r}{d\tau^2} \right| = O(e^{-\lambda\tau}),$$

which gives by a similar argument

$$\left| \frac{dr}{d\tau}(\tau) \right| + |r(\tau) - \rho_\infty(\tau)| \leq C e^{-2(1-v_\infty)\tau} ,$$

so that (B.1.5) follows.

2. Multiplying ρ by -1 if necessary we may assume $v_\infty \geq 0$. Equations (B.1.4) and (B.1.5) are equivalent to the following integral equation for $p(\tau) = \rho(\tau) - v_\infty \tau - \rho_\infty$:

$$p(\tau) = T(p)(\tau)$$

$$T(p)(\tau) = -\frac{n^2}{4} \int_\tau^\infty (s - \tau) (e^{2p(s)+2\rho_\infty} - e^{-2p(s)+2\rho_\infty-4v_\infty s}) e^{-\lambda s} ds ,$$

with $\lambda = 2(1 - v_\infty)$. Let $H = \{p \in C([\tau_1, \infty)), \|p(\tau)\| = \sup_\tau |e^{\lambda\tau/2} p(\tau)| < \infty\}$. One checks without difficulty that there exists $\tau_1(v_\infty, \rho_\infty, n) < \infty$ such that T takes the unit ball of H into itself, and that T is a contraction — the claim follows by the contraction mapping principle. Finally if $v_\infty = \rho_\infty = 0$ then there are no “driving terms” in T so that the contraction property implies $p \equiv 0$. \square

Let us analyze the behaviour of solutions of (B.1.4) as $t \rightarrow \infty$ ($\tau \rightarrow -\infty$):

Proposition B.1.2 *For $t_o > 0$ let $\rho \in C^2([t_o, \infty))$ be a solution of (B.1.4).*

1. *There exists a constant C such that*

$$|\rho| + \left| \frac{d\rho}{dt} \right| \leq C t^{-1/2} . \quad (\text{B.1.8})$$

2. *There exist constants e_∞, C_1 such that*

$$\left| t \left[\left(\frac{d\rho}{dt} \right)^2 + n^2 \sinh^2 \rho \right] - e_\infty \right| \leq C t^{-1} . \quad (\text{B.1.9})$$

If $e_\infty = 0$, then $\rho \equiv 0$.

Remark: The proof below suggests very strongly that we have the expansion

$$\rho = A \cos(nt + \delta) t^{-1/2} + B_1(t) t^{-3/2} + B_2(t) t^{-5/2} + \dots$$

for some constants A, δ and some functions $B_i(t)$ which are polynomials in $\sin(nt)$ and $\cos(nt)$.

Proof: Define

$$\psi(t) = t^{1/2}\rho(t) ;$$

ψ satisfies the equation

$$\frac{d^2\psi}{dt^2} = -\frac{\psi}{4t^2} - \frac{n^2 t^{1/2}}{2} \sinh(2\rho) . \quad (\text{B.1.10})$$

Let us set

$$e(t) = \left(\frac{d\psi}{dt}\right)^2 + \frac{\psi^2}{4t^2} + \frac{n^2 t}{2} (\cosh(2\rho) - 1) ; \quad (\text{B.1.11})$$

we have

$$\frac{de}{dt} = -\frac{\psi^2}{2t^3} - n^2 V(\rho) , \quad (\text{B.1.12})$$

$$V(\rho) = \rho \sinh(2\rho) + 1 - \cosh(2\rho) .$$

We have $V(0) = V'(0) = 0$ and $V''(\rho) = 4\rho \sinh(2\rho) \geq 0$, thus $V(\rho) \geq 0$, so that

$$\frac{de}{dt} \leq 0 , \quad (\text{B.1.13})$$

which shows that

$$\left(\frac{d\psi}{dt}\right)^2 \leq C, \quad t \left(\cosh(2t^{-1/2}\psi) - 1 \right) \leq C \quad (\text{B.1.14})$$

for some constant C . The inequality $\cosh(2\rho) - 1 \geq 2\rho^2$ and the second inequality in (B.1.14) give

$$\psi^2 \leq C , \quad (\text{B.1.15})$$

and (B.1.8) follows. Taylor expanding $\cosh \rho$ to fourth order in (B.1.11) and making use of (B.1.15) one obtains

$$e(t) = \left(\frac{d\psi}{dt}\right)^2 + n^2 \psi^2 + O(t^{-1}) . \quad (\text{B.1.16})$$

From (B.1.13) it follows that e is monotone, therefore the limit $e_\infty = \lim_{t \rightarrow \infty} e(t)$ exists. There exists a constant C_V such that for $\rho \leq 1$ we have $|V(\rho)| \leq C_V \rho^4$, and since ρ tends to zero as t goes to infinity there exists a T such that for $t \geq T$ we have $|V(\rho)| \leq C_V \psi^4 t^{-2} \leq C' t^{-2}$, and integrating (B.1.12) one obtains

$$|e(t) - e_\infty| \leq C t^{-1} , \quad (\text{B.1.17})$$

so that (B.1.16) leads to

$$\left(\frac{d\psi}{dt}\right)^2 + n^2\psi^2 - e_\infty = O(t^{-1}) ;$$

a simple calculation gives

$$t\left[\left(\frac{d\rho}{dt}\right)^2 + n^2 \sinh^2 \rho\right] = e_\infty + O(t^{-1}) ,$$

which proves (B.1.9). Suppose finally that $e_\infty = 0$. The inequality (B.1.17) together with $e_\infty = 0$ implies

$$\left|\frac{d\psi}{dt}(t)\right| + |\psi(t)| \leq Ct^{-1/2} . \quad (\text{B.1.18})$$

Inserting (B.1.18) into (B.1.12) and repeating iteratively the above argument one shows that for any $\ell \in \mathbb{N}$ there exists $C(\ell)$ such that

$$|\psi(t)| + e(t) \leq C(\ell) t^{-\ell} . \quad (\text{B.1.19})$$

The inequalities (B.1.19) with $\ell = 2$ and $V(\rho) \leq C_V \rho^4$ give

$$n^2 V(\rho) \leq C_V n^2 \rho^4 = \frac{C_V n^2 \psi^4}{t^2} \leq \frac{C_V n^2 C(2)^2 \psi^2}{t^3} \leq \frac{\psi^2}{2t^3}$$

for $t \geq t_1 = [2C_V n^2 C(2)^2]^{1/3}$, which leads to

$$\begin{aligned} -\frac{e}{t} &\leq -\frac{\psi^2}{4t^3} = \frac{1}{4} \left[\frac{de}{dt} - \left(\frac{\psi^2}{2t^3} - n^2 V(\rho) \right) \right] \leq \frac{1}{4} \frac{de}{dt} \\ \Rightarrow \quad \frac{de}{dt} &\geq -\frac{4e}{t} , \end{aligned}$$

which implies

$$t_1 \leq t_2 \leq t_3 \quad e(t_2) \leq \frac{e(t_3)t_3^4}{t_2^4} ,$$

passing with t_3 to infinity one obtains $e(t) \equiv 0$, $\psi \equiv \rho \equiv 0$, which had to be established.

□

B.2 Moncrief's space-times.

Let $M = \{t \in (0, \infty), \theta, x^a \in [0, 2\pi]_{\bmod 2\pi}, a = 1, 2\}$. Consider the following Gowdy-type metrics

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = e^{2B}(-dt^2 + d\theta^2) + \lambda t n_{ab} (dx^a + g^a d\theta)(dx^b + g^b d\theta), \quad (\text{B.2.1})$$

$$\begin{aligned} n_{ab} dx^a dx^b &= (\cosh \rho + \cos \phi \sinh \rho) (dx^1)^2 + 2 \sinh \rho \sin \phi dx^1 dx^2 \\ &\quad + (\cosh \rho - \cos \phi \sinh \rho) (dx^2)^2, \end{aligned}$$

$$\rho = \rho(t, \theta), \quad \phi = \phi(t, \theta),$$

where λ and g^a are real constants, $\lambda > 0$. For a metric of the form (B.2.1) the dynamical part of Einstein equations reduces to the equations (B.1.1)–(B.1.2), and assuming Moncrief's ansatz (B.1.3) one finds that $B = B(t)$ (*cf. e.g.* eqs. (2.30) and (2.33) of [32]; the constants c_a appearing in these equations vanish for the metrics (B.2.1)), and

$$\frac{dB}{dt} = -\frac{1}{4t} + \frac{t}{4} \left[\left(\frac{d\rho}{dt} \right)^2 + n^2 \sinh^2 \rho \right]. \quad (\text{B.2.2})$$

In this way we obtain a family of metrics parametrized by six parameters — λ , g^a , $a = 1, 2$, an integration constant B_o for B and two real constants parametrizing solutions of the equation (B.1.4), *e.g.* v_∞ and ρ_∞ given by Proposition B.1.1. (Out of these parameters of course only v_∞ and ρ_∞ are dynamically interesting.) We shall refer to these space-times as *Moncrief's space-times*.

Proposition B.2.1 *All Moncrief's space-times are future causally geodesically complete.*

Proof: If the constant e_∞ given by Proposition B.1.2 vanishes ($\Rightarrow \rho \equiv 0$, *cf.* Proposition B.1.2) one easily checks that the metric can be put in Kasner's form with exponents $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ or permutation thereof (*cf.* Section 2.4 for a description of Kasner metrics), in which case it is easy to show future geodesic completeness, we shall thus consider the case $e_\infty \neq 0$ only. Let $\Gamma(s) = \{x^\mu(s)\}$ be a future inextendible future

directed affinely parametrized causal geodesic. From $\gamma_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -\epsilon$, $\epsilon \in \{0, 1\}$, and from

$$\frac{dp_a}{ds} \equiv \frac{d}{ds} \left(\gamma_{\mu\nu} \frac{dx^\mu}{ds} X_a^\nu \right) = 0 ,$$

where $X_a^\nu \frac{\partial}{\partial x^\nu}$ are the Killing vectors $\frac{\partial}{\partial x^a}$, $a = 1, 2$, one obtains the following equations,

$$e^{2B} \left[\left(\frac{dt}{ds} \right)^2 - \left(\frac{d\theta}{ds} \right)^2 \right] = \epsilon + \bar{g}^{ab} p_a p_b ,$$

$$\frac{dx^a}{ds} = -g^a \frac{d\theta}{ds} + p^a , \quad p^a \equiv \bar{g}^{ab} p_b ,$$

where $\bar{g}^{ab} \equiv (g_{ab})^{-1}$, $g_{ab} = \lambda t n_{ab}$. The t part of the equations satisfied by a geodesic reads

$$\frac{d}{ds} \left(B^2 \frac{dt}{ds} \right) = B^2 e^{-2B} \left[(\epsilon + \bar{g}^{ab} p_a p_b) \frac{dB}{dt} - \frac{1}{2} \frac{\partial g_{ab}}{\partial t} p^a p^b \right] . \quad (\text{B.2.3})$$

Non-spacelikeness of Γ implies that Γ can be parametrized by t , which allows us to rewrite (B.2.3) as

$$\frac{d}{dt} \left(B^4 \left(\frac{dt}{ds} \right)^2 \right) = f , \quad (\text{B.2.4})$$

$$f \equiv 2B^4 e^{-2B} \left[(\epsilon + \bar{g}^{ab} p_a p_b) \frac{dB}{dt} - \frac{1}{2} \frac{\partial g_{ab}}{\partial t} p^a p^b \right] . \quad (\text{B.2.5})$$

From (B.2.2) and Proposition B.1.2, point 2, we have

$$\frac{dB}{dt} = \frac{e_\infty}{4} + O(t^{-1}) \implies B = \frac{e_\infty}{4} t + O(\ln t) ,$$

which together with (B.1.8) shows that f converges exponentially fast to zero as t goes to infinity, therefore there exists a constant C such that

$$B^4 \left(\frac{dt}{ds} \right)^2 \leq C ,$$

which for t large enough gives

$$t^4 \left(\frac{dt}{ds} \right)^2 \leq \left(\frac{8}{e_\infty} \right)^4 C ,$$

and for $s_2 \geq s_1$ one obtains

$$s_2 \geq s_1 + \left[3 \left(\frac{8}{e_\infty} \right)^2 C^{1/2} \right]^{-1} \left(t^3(s_2) - t^3(s_1) \right) ,$$

so that $s_2 \rightarrow \infty$ as $t(s_2) \rightarrow \infty$, which had to be established. \square

Proposition B.2.2 *Let Γ be either a past inextendible timelike curve parametrized by proper time, or an affinely parametrized past inextendible null geodesic, in a Moncrief's space-time. Then*

1. Γ reaches the boundary $t = 0$ in finite proper time or finite affine time, say s_0 , and

2. we have

$$\lim_{s \rightarrow s_0} (R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta})|_{\Gamma}(s) = \infty .$$

Proof: From (B.2.2) and from Proposition B.1.1 it follows that there exists a constant B_0 such that

$$e^B = e^{B_0 t^{(v_\infty^2 - 1)/4}} (1 + O(t^{2(1 - |v_\infty|)})) . \quad (\text{B.2.6})$$

Consider first a timelike curve $\Gamma = \{x^\mu(s)\}$ parametrized by proper time s , with $t(s)$ decreasing as s increases,

$$e^{2B} [(\frac{dt}{ds})^2 - (\frac{d\theta}{ds})^2] - \lambda t n_{ab} \left(\frac{dx^a}{ds} + g^a \frac{d\theta}{ds} \right) \left(\frac{dx^b}{ds} + g^b \frac{d\theta}{ds} \right) = 1 . \quad (\text{B.2.7})$$

Equation (B.2.7) implies

$$e^B \frac{dt}{ds} \leq -1$$

(recall that Γ is past-directed) which together with (B.2.6) gives, for $s_2 \geq s_1$, with $t(s_1)$ — small enough,

$$s_2 \leq s_1 + \frac{8e^{B_0}}{3 + v_\infty^2} \left(t^{(3+v_\infty^2)/4}(s_1) - t^{(3+v_\infty^2)/4}(s_2) \right) ,$$

so that any timelike curve reaches $t = 0$ in finite proper time. To prove the result for null geodesics, some more work is required. In what follows we shall write that

$$f \approx g$$

if

$$\lim_{t \rightarrow 0} \frac{f}{g} = 1 .$$

From (B.2.2) and from Proposition B.1.1 we have

$$\frac{dB}{dt}(t) \approx \frac{v_\infty^2 - 1}{4t}, \quad B(t) \approx \frac{v_\infty^2 - 1}{4} \ln t, \quad (\text{B.2.8})$$

$$ds^2 \approx e^{2B_0} t^{(v_\infty^2 - 1)/2} \left(-dt^2 + d\theta^2 \right) + \lambda t^{1-|v_\infty|} \left\{ \frac{1 + \cos(n\theta)}{2} (dx^1 + g^1 d\theta)^2 \right. \\ \left. + \sin(n\theta) (dx^1 + g^1 d\theta)(dx^2 + g^2 d\theta) + \frac{1 - \cos(n\theta)}{2} (dx^2 + g^2 d\theta)^2 \right\}, \quad (\text{B.2.9})$$

$$\frac{\partial g_{ab}}{\partial t} \equiv \frac{\partial(\lambda t n_{ab})}{\partial t} \approx \frac{1 - |v_\infty|}{t} g_{ab}, \quad (\text{B.2.10})$$

with (B.2.9) holding for $|v_\infty| > 0$, and (B.2.8), (B.2.10) holding for $0 \leq |v_\infty| < 1$. Consider null geodesics such that $(p^1)^2 + (p^2)^2 \neq 0$; the case $p^1 = p^2 = 0$ is analyzed by a similar simpler argument. From (B.2.4)–(B.2.5) and (B.2.10) one obtains t small enough,

$$\frac{d}{dt} \left(B^4 \left(\frac{dt}{ds} \right)^2 \right) \approx -\frac{e^{-2B_0}}{2} \left(\frac{v_\infty^2 - 1}{4} \right)^4 (3 - 2|v_\infty| - v_\infty^2) t^{-(1+v_\infty^2)/2} \ln^4 t \bar{g}^{ab} p_a p_b \leq 0,$$

so that $B^4 \left(\frac{dt}{ds} \right)^2$ increases as $t(s)$ goes to zero for $t(s_1) \leq t_1$, for some t_1 small enough, therefore for $s \geq s_1$

$$B^4 \left(\frac{dt}{ds} \right)^2 \geq C,$$

and an argument similar to the one for timelike curves shows that Γ must reach $t = 0$ in finite affine time.

To analyze the behaviour of the curvature near $t = 0$, with the help of a SHEEP calculation² one finds

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \approx \frac{e^{-4B}}{4t^4} \left[(1 - v_\infty^2)^2 (3 + v_\infty^2) \right],$$

and (B.2.6) gives

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \geq C t^{-(v_\infty^2 + 3)}, \quad (\text{B.2.11})$$

for some constant C , so that the curvature blows up uniformly along all curves as t approaches zero. \square

From Propositions B.2.1, B.2.2, and C.2.4 one obtains

²The author is grateful to D. Singleton for performing this calculation.

Theorem B.2.1 *Let $\Sigma = T^3$, let $X(\Sigma)$ be the space of Cauchy data for Moncrief's metrics. The Theorem-To-Be-Proved holds in $X(\Sigma)$, with $Y(\Sigma) = X(\Sigma)$; more precisely, every maximal Hausdorff developement of a Cauchy data set for a Moncrief metric is globally hyperbolic, therefore unique, and inextendible, in vacuum or otherwise, in the class of Hausdorff Lorentzian manifolds with $C_{loc}^{1,1}$ metrics.*

Appendix C

Maximal developments.

In this Appendix we shall discuss the existence of maximal developments, and we shall also prove some criteria which allow one to decide whether or not a given development is maximal.

C.1 Existence of maximal space-times.

In this section we shall prove the existence of maximal space-times; the reader should note that we are not making any global hyperbolicity hypotheses. The arguments here follow essentially those of [21]. Throughout this section “ W manifold” stands for a connected, paracompact, Hausdorff n -dimensional manifold of differentiability class W such that $W \subset C^1$, where W stands for *e.g.* $C^{k,\alpha}$ or some Sobolev class, etc. The manifold will be said Lorentzian if it is equipped with a metric tensor, perhaps defined only almost everywhere, of a differentiability class adapted to that of W . For example, if $W = C^{k,\alpha}$ then we should have $k \geq 1$ and the metric, defined everywhere, will be of $C^{k-1,\alpha}$ differentiability class. It is useful to keep in mind that W can be a rather complicated space, *e.g.* for the purpose of the Cauchy problem in general relativity an appropriate space W is the set of maps which preserve the condition that the components of the metric tensor $\gamma_{\mu\nu}$ restricted to the hypersurfaces $\Sigma = \{t = \text{const}\}$ are of Sobolev class $H_{loc}^k(\Sigma)$, the time-derivatives of $\gamma_{\mu\nu}$ are in $H_{loc}^{k-1}(\Sigma)$, etc. A W Lorentzian manifold

will be called vacuum if W is such that the equations $R_{\mu\nu} = 0$ can be defined, perhaps in a distributional sense, and if $R_{\mu\nu} = 0$ holds.

Theorem C.1.1 *Let (M, γ) be a W Lorentzian manifold, there exists a W Lorentzian manifold $(\tilde{M}, \tilde{\gamma})$ and an isometric embedding $\Phi : M \rightarrow \tilde{M}$ such that \tilde{M} is inextendible in the class of W Lorentzian manifolds. The same is true if “Lorentzian W manifold” is replaced by “vacuum Lorentzian W manifold” everywhere above.*

Remarks:

1. The C^1 differentiability threshold for M cannot be weakened in the proof below. The author ignores whether or not the C^1 differentiability of M is necessary.
2. The maximal manifolds $(\tilde{M}, \tilde{\gamma})$ need not be unique, and may depend upon W . A non-trivial example of W dependence, with $W = C^{k,\alpha}$, is given by some Robinson–Trautman (RT) space-times, which for $k + \alpha \geq 123$ admit no non-trivial future extensions, while for $k + \alpha < 118$ they admit an infinite number of non-isometric vacuum RT extensions.

Proof: For $\ell \geq n$ let A_ℓ denote the set¹ of subsets of \mathbb{R}^ℓ which are n -dimensional manifolds, set $A_\infty = \cup_{\ell=0}^\infty A_\ell$. By a famous theorem of Whitney [128] every $(C^1, \text{connected, paracompact, Hausdorff})$ manifold can be embedded in \mathbb{R}^ℓ for some ℓ , which shows that every manifold has a representative which is an element of A_∞ ; it follows that without loss of generality a manifold can be defined as an element of A_∞ ; and we shall do so. With this definition the collection of all manifolds is A_∞ , and therefore is a set. It follows from the axioms of set theory that the collection of all C^1 manifolds which are W manifolds forms a set; the same is true of the collection \mathcal{M}_W of W Lorentzian manifolds (recall that a Lorentzian manifold can be identified with a subset of the bundle T_2M , where T_2M is the bundle of 2-covariant tensors on M), and of the collection $\mathcal{M}_{W,vac}$ of W vacuum Lorentzian manifolds. Let (M, γ) be a Lorentzian, respectively

¹cf. e.g. [78][Appendix] for an overview of axiomatic set theory.

a vacuum Lorentzian, W manifold, consider the subset $\mathcal{M}_W(M, \gamma)$ of \mathcal{M}_W , respectively $\mathcal{M}_{W,vac}(M, \gamma)$ of $\mathcal{M}_{W,vac}$, defined as the set of those Lorentzian manifolds $(\tilde{M}, \tilde{\gamma})$ for which there exists an isometric C^1 embedding $\Phi : M \rightarrow \tilde{M}$. On both $\mathcal{M}_W(M, \gamma)$ and $\mathcal{M}_{W,vac}(M, \gamma)$ we can define a partial order \prec as follows: $(\tilde{M}, \tilde{\gamma}) \prec (\tilde{M}_1, \tilde{\gamma}_1)$ if there exists an isometric C^1 embedding $\Phi : \tilde{M} \rightarrow \tilde{M}_1$. If $A \subset \mathcal{M}_W(M, \gamma)$ or $A \subset \mathcal{M}_{W,vac}(M, \gamma)$ is a chain, define $\bar{M} = \left(\bigcup_{(\tilde{M}, \tilde{\gamma}) \in A} \tilde{M} \right) / \sim$, where for $p \in \tilde{M}$ and $q \in \tilde{M}_1$ we set $p \sim q$ iff $q = \Phi(p)$, where $\Phi : \tilde{M} \rightarrow \tilde{M}_1$ is an isometric C^1 embedding. It is not too difficult to show that \bar{M} is a W manifold (Hausdorff, paracompact, connected), a Lorentzian metric $\bar{\gamma}$ can be defined on \bar{M} in an obvious way. Since every element $(\tilde{M}, \tilde{\gamma})$ of A can be embedded in \bar{M} ($\tilde{M} \ni p \rightarrow [p]_{\sim} \in \bar{M}$), it follows that \bar{M} is an upper bound for A . Zorn's Lemma (cf. e.g. [78]) shows that both $\mathcal{M}_W(M, \gamma)$ and $\mathcal{M}_{W,vac}(M, \gamma)$ have maximal elements, which had to be established. \square

C.2 Some maximality criteria.

In this Appendix we will discuss some inextendability criteria. Let us start with some terminology. Recall that we have defined $(\tilde{M}, \tilde{\gamma})$ to be an extension of (M, γ) if there exists an isometric embedding Φ of M into \tilde{M} , and $M \neq \tilde{M}$. We shall often identify M with $\Phi(M)$. If M is a subset of \tilde{M} , by ∂M we always mean the topological boundary of M in \tilde{M} : $\partial M \equiv \{p \in \tilde{M} \setminus M \text{ such that } \exists p_i \in M \text{ with the property that } p_i \rightarrow p\}$. When considering inextendability criteria, it is useful to keep in mind the possibility of allowing extensions in which a weak form of violation of Hausdorffness occurs. Following Clarke [40] we shall say that a (possibly² non-Hausdorff) space-time is a *Hajiček space-time* if there exists no bifurcating causal curves in (M, γ) (more precisely, let $\Gamma_\alpha : [0, b) \rightarrow M$, $\alpha = 1, 2$ be two continuous causal curves such that $\Gamma_1|_{[0, a)} = \Gamma_2|_{[0, a)}$ for some $a < b$, then $\Gamma_1(a) = \Gamma_2(a)$). In this section all geodesics are affinely parametrized; unless specified otherwise we do not impose any particular orientation on the affine parameter. In all the results presented in this section we make essential use of both existence and uniqueness

²If M is Hausdorff, then the Hajiček condition trivially holds.

of solutions of the initial value problem for geodesics; those are guaranteed by $C_{loc}^{1,1}$ differentiability of the metric and by the requirement that the space-times considered satisfy the Hajiček condition. The results here need not to hold in an arbitrary non-Hausdorff space-time, or in space-times with metrics the derivatives of which are not Lipschitz continuous.

Let us recall a criterion which has been used by Misner and Taub [85] to prove inextendability of the Taub–NUT space-time:

Proposition C.2.1 *Let (M, γ) be a Hausdorff space-time with a $C_{loc}^{1,1}$ metric, suppose that for every geodesic segment $\Gamma : [s_0, s_1) \rightarrow M$ which cannot be extended beyond s_1 there exists a compact set K such that $\Gamma \subset K$. Then (M, γ) is inextendible in the class of Hausdorff space-times with $C_{loc}^{1,1}$ metrics.*

Remark: It should be noted that in Taub–NUT space-time there exist inextendible geodesic segments which remain in a compact set.

Proof: Suppose that $(\tilde{M}, \tilde{\gamma})$ is an extension of (M, γ) . For any $p \in \partial M$ there exist some geodesic segment $\Gamma_p : [0, a] \rightarrow \tilde{M}$ such that $\Gamma_p(0) = p$, $\Gamma_p(a) \in M$ (Γ_p — timelike, spacelike or null). Since $p \notin M$, it follows that $\Gamma_p \cap M$ is inextendible in M and is not contained in any compact set K , which leads to a contradiction. \square

In exactly the same way one proves:

Proposition C.2.2 *Let (M, γ) be a Hajiček space-time with a $C_{loc}^{1,1}$ metric, suppose that for every geodesic segment $\Gamma : [s_0, s_1) \rightarrow M$ which cannot be extended beyond s_1 either*

1. *there exists a compact set K such that $\Gamma \subset K$, or*
2. *some polynomial scalar of the curvature tensor is unbounded on Γ as $s \rightarrow s_1$.*

Then (M, γ) is inextendible in the class of Hajiček space-times with $C_{loc}^{1,1}$ metrics.

Note that if γ and $\tilde{\gamma}$ are assumed to be twice differentiable, then point 2 above can be weakened to “some polynomial scalar of the curvature tensor has no finite limit on Γ as $s \rightarrow s_1$ ” (in other words, if the limit exists, it is infinite).

If we have an extension of (M, γ) and a point $p \in \partial M$ such that there exists a timelike curve from M to p , it is easily seen that there necessarily exists a *timelike geodesic* from M to p . Whenever (M, γ) is time-orientable, it is natural to divide extensions in the following classes:

1. there exists a future directed timelike geodesic from M to \tilde{M} ,
2. there exists a past directed timelike geodesic from M to \tilde{M} ,
3. there exist both future and past time directed timelike geodesics from M to \tilde{M} ,
4. there exists no timelike geodesics from M to \tilde{M} .

Let us show that case 4 above cannot occur:

Proposition C.2.3 *Let (M, γ) be a Hajiček space-time with a $C_{loc}^{1,1}$ metric, suppose that $(\tilde{M}, \tilde{\gamma})$ is a Hajiček extension thereof with a $C_{loc}^{1,1}$ metric. Then there necessarily exists*

1. *a timelike geodesic from M to ∂M , and*
2. *a null geodesic from M to ∂M .*

Proof: Suppose there exists $p \in \partial M$ such that there exists no timelike geodesic from M to p , thus $I(p) \cap M = \emptyset$, where $I(p)$ is the union of the future and of the past of p in \tilde{M} . There exists a sequence $p_i \in M$ such that $p_i \rightarrow p$, choose any normal geodesic neighbourhood $\mathcal{O} \subset \tilde{M}$ of p with a local coordinate chart, for i large enough we have $p_i \in \mathcal{O}$. It is easily seen that for i large enough a maximally extended geodesic through p_i with tangent vector $\partial/\partial t$ at p must leave M and enter \tilde{M} , and the result follows because in a Hajiček space-time there exists no bifurcate timelike geodesics. The result for null geodesics is proved in a similar way. \square

We shall say that a time-orientable space-time (M, γ) is *future inextendible* if there exists no extension $(\tilde{M}, \tilde{\gamma})$ of (M, γ) in which a future directed causal curve starting in M enters \tilde{M} ; the notion of *past inextendibility* is defined similarly. Proposition C.2.3 shows that a space-time which is both future and past inextendible is inextendible. This is a useful result, because together with Proposition C.2.2 it reduces the task of proving inextendability of M to an analysis of the behaviour of either null or timelike geodesics in (M, γ) :

Proposition C.2.4 1. In Propositions C.2.1 and C.2.2 “every geodesic segment” can be replaced by “every timelike geodesic segment”.

2. In Propositions C.2.1 and C.2.2 “every geodesic segment” can be replaced by “every null geodesic segment”.

Given a globally hyperbolic space-time (M, γ) , it might be useful in some situations to be able to determine whether (M, γ) is maximal in the class of globally hyperbolic space-times (note that this does not exclude the existence of non-globally-hyperbolic extensions of (M, γ)). We shall say that a globally hyperbolic Lorentzian manifold (M, γ) admits a *crushing future boundary* if there exists a foliation of (M, γ) by spacelike Cauchy hypersurfaces Σ_τ , $\tau \in (\tau_0, \tau_1)$, $-\infty \leq \tau_0 < \tau_1 \leq \infty$, such that

$$\lim_{\tau \rightarrow \tau_1} K_\tau^+ = \infty ,$$

where

$$K_\tau^+ \equiv \inf_{p \in \Sigma_\tau} (g^{ij} K_{ij})(p) ,$$

g^{ij} is the inverse of the metric g_{ij} induced by γ on Σ_τ , and K_{ij} is the extrinsic curvature of Σ_τ . Similarly a *crushing past boundary* is defined by the condition

$$\lim_{\tau \rightarrow \tau_0} K_\tau^- = -\infty , \quad K_\tau^- \equiv \sup_{p \in \Sigma_\tau} (g^{ij} K_{ij})(p) .$$

(Eardley and Smarr [43] have proposed the term “crushing singularity” for the above described behaviour: we find that terminology misleading, because the existence of a

crushing boundary does not imply existence of a singularity in the geometry³, there might exist perfectly smooth extensions of (M, γ) as is seen *e.g.* in some polarized Gowdy metrics [37].) Although it is irrelevant for further purposes, it might be of some interest to note that in spatially compact space-times (M, γ) with crushing boundaries there exist constant mean curvature (CMC) surfaces [7]; if moreover the timelike convergence condition holds ($R_{\mu\nu}X^\mu X^\nu \geq 0$ for all timelike vectors X), then (M, γ) can be foliated by CMC surfaces.

As has been noted by Moncrief [87], the existence of crushing boundaries implies maximality in the class of globally hyperbolic space-times:

Proposition C.2.5 *Let (M, γ) be a globally hyperbolic, spatially compact, Hajiček space-time with a $C_{loc}^{1,1}$ metric.*

1. *Suppose that (M, γ) has a future crushing boundary. Then (M, γ) is future inextendible in the class of globally hyperbolic, Hajiček space-times with $C_{loc}^{1,1}$ metrics.*
2. *The same is true if “future” is replaced by “past” everywhere above.*

Proof: Let $\Sigma_\tau, \tau_0 < \tau < \tau_1$ be a foliation of (M, γ) as in the definition of future crushing boundary, suppose that $(\tilde{M}, \tilde{\gamma})$ is a future extension of (M, γ) , let $p \in \tilde{M} \setminus M$ be such that $M \cap I^-(p; \tilde{M}) \neq \emptyset$, where $I^-(p; \tilde{M})$ is the past of p in \tilde{M} . By global hyperbolicity of $(\tilde{M}, \tilde{\gamma})$ for any $\tau < \tau_1$ there exists a future directed maximizing timelike geodesic $\Gamma_\tau : [0, s_1(\tau)] \rightarrow \tilde{M}$ parametrized by proper time such that $\Gamma_\tau(0) \in \Sigma_\tau$, $\Gamma_\tau(s_1(\tau)) = p$, for some $s_1(\tau) > 0$, where throughout this proof “maximizing” stands for “maximizing in the class of geodesics which start on Σ_τ and have p as endpoint”. Choose some $\tau_0 < \tilde{\tau} < \tau_1$, there exists $\tilde{s} < s_1(\tilde{\tau})$ such that $\Gamma_{\tilde{\tau}}(\tilde{s}) \in \partial M$. Since the Σ_τ ’s are Cauchy surfaces it follows that for $\tau > \tilde{\tau}$ we have $\Sigma_\tau \cap \Gamma_{\tilde{\tau}} \neq \emptyset$, and since the Γ_τ ’s are maximizing geodesics from Σ_τ to p it follows that $s_1(\tau) \geq s_1(\tilde{\tau}) - \tilde{s} > 0$. But by [66][Corollary,

³If the space-time is spatially compact and the timelike convergence condition holds ($R_{\mu\nu}X^\mu X^\nu \geq 0$ for all timelike vectors X) (M, γ) will be, however, geodesically incomplete, *cf. e.g.* [66][Chapter 8, Theorem 4].

Section 6.7] and [66][Proposition 4.4.3] there are no future directed maximizing geodesics of length more then $3/K_\tau^+$ starting at Σ_τ , which gives a contradiction since $3/K_\tau^+ \rightarrow 0$ on Γ as $s \rightarrow s_0$. \square

Appendix D

Some flat metrics with Cauchy horizons.

In this Appendix we present a family of flat $n + 1$ dimensional space-times which have at least two non-isometric extensions across a smooth Cauchy horizon; this example is a generalization of Misner's model for the Taub-NUT space-times [84]. The space-times considered here do not have compact spacelike hypersurfaces, as opposed to Misner's example which, in $n+1$ dimensions, can be given $\mathbb{R} \times S^1 \times T^{n-1}$ topology, where T^ℓ is an ℓ dimensional torus. In view of the renewed interest in flat Lorentzian manifolds [130] [92] [83] it would be useful to understand the global structure of all flat Lorentzian manifolds, at least in 4 dimensions.

For $n \geq 2$ let (t, x, y) , $y = y^1$ if $n = 2$ or $y = (y^1, \dots, y^{n-1})$ otherwise, be coordinates in $n + 1$ dimensional Minkowski space,

$$ds^2 = -dt^2 + dx^2 + dy^2, \quad (\text{D.0.1})$$

with $dy^2 = (dy^1)^2$ if $n = 2$ and $dy^2 = d\bar{y}^2 = (dy^1)^2 + \dots + (dy^{n-1})^2$ otherwise. Let Λ_0 be a fixed Lorentz boost in the $t - x$ plane,

$$\Lambda_0 = \begin{bmatrix} \gamma_0 & \gamma_0\beta & 0 \\ \gamma_0\beta & \gamma_0 & 0 \\ 0 & 0 & id_{\mathbb{R}^{n-1}} \end{bmatrix},$$

$$\gamma_0 = \frac{1}{\sqrt{1 - \beta_0^2}}, \quad |\beta_0| < 1,$$

where $id_{\mathbb{R}^{n-1}}$ is the identity matrix in \mathbb{R}^{n-1} . Let \mathcal{H}_τ denote a hyperboloid in \mathbb{R}^{n+1} ,

$$\mathcal{H}_\tau = \{\tau = t^2 - x^2 - \vec{y}^2\},$$

with \mathcal{H}_0 — the light cone at the origin. Define \mathcal{M}_{Λ_0} as the quotient of the interior $I^+(0)$ of the solid future light cone $J^+(0)$ from the origin by the group generated by Λ_0 . In $I^+(0)$ we can introduce coordinates (τ, β, β_1) as follows:

$$\tau = t^2 - x^2 - y^2 - z^2 > 0,$$

$$t = ch \beta_1 ch \beta + \frac{(\tau - 1)e^{-\beta}}{2ch \beta_1}, \quad x = ch \beta_1 sh \beta - \frac{(\tau - 1)e^{-\beta}}{2ch \beta_1}, \quad (\text{D.0.2})$$

$$y = sh \beta_1 \text{ if } n = 2, \quad \vec{y} = sh \beta_1 \vec{\omega}, \quad \vec{\omega} \in S^{n-1}(1) \text{ if } n > 2, \quad (\text{D.0.3})$$

where $S^k(1)$ is the k -dimensional unit sphere. In terms of (D.0.2) the metric (D.0.1) takes the form

$$\begin{aligned} ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = & (\tau + sh^2 \beta_1) d\beta^2 + \frac{1 + \tau sh^2 \beta_1}{ch^2 \beta_1} d\beta_1^2 - d\tau d\beta \\ & - \frac{sh \beta_1}{ch \beta_1} d\tau d\beta_1 + \frac{2(\tau - 1) sh \beta_1}{ch \beta_1} d\beta_1 d\beta, \end{aligned} \quad (\text{D.0.4})$$

and a term $sh^2 \beta_1 d\vec{\omega}^2$ has to be added to (D.0.4) if $n > 2$. One finds

$$\det \gamma_{\mu\nu} = -\frac{1}{4} ch^2 \beta_1 sh^{2(n-2)} \beta_1 \det h_{AB}^0, \quad (\text{D.0.5})$$

h_{AB}^0 — the round metric on a $n - 1$ dimensional sphere ($\det h_{AB}^0 = 1$ if $n = 2$). Equation (D.0.4) and the definition of \mathcal{M}_{Λ_0} show that (τ, β, β_1) can be used as coordinates on \mathcal{M}_{Λ_0} if β is identified with $\beta + \beta_0$. By (D.0.5) it follows that (D.0.4) gives an analytic extension of the flat metric on \mathcal{M}_{Λ_0} to the larger manifold M ,

$$\tau \in (-\infty, \infty), \quad \beta_1 \in (-\infty, \infty) \text{ for } n = 2, \quad \sinh \beta_1 \vec{\omega} \in \mathbb{R}^{n-1} \text{ for } n > 2, \quad (\text{D.0.6})$$

$$\beta \in [0, \beta_0]_{\text{mod } \beta_0}.$$

Another extension $(\tilde{M}, \tilde{\gamma})$ is obtained by defining coordinates $(\tau, \tilde{\beta}, \beta_1)$, with $\tilde{\beta} \in [0, \beta_0]_{\text{mod } \beta_0}$ and τ, β_1 — as in (D.0.6), by

$$t = ch\beta_1 ch\tilde{\beta} + \frac{(\tau - 1)e^{\tilde{\beta}}}{2ch\beta_1}, \quad x = ch\beta_1 sh\tilde{\beta} + \frac{(\tau - 1)e^{\tilde{\beta}}}{2ch\beta_1}, \quad (\text{D.0.7})$$

y as in (D.0.3), which yields

$$ds^2 = \tilde{\gamma}_{\mu\nu} dx^\mu dx^\nu = (\tau + sh^2\beta_1) d\tilde{\beta}^2 + \frac{(1 + \tau sh^2\beta_1)}{ch^2\beta_1} d\beta_1^2 + d\tau d\tilde{\beta} \quad (\text{D.0.8})$$

$$- \frac{sh\beta_1}{ch\beta_1} d\tau d\beta_1 - \frac{2(\tau - 1)sh\beta_1}{ch\beta_1} d\tilde{\beta} d\beta_1, \quad (\text{D.0.9})$$

$$\det \tilde{\gamma}_{\mu\nu} = -\frac{ch^2\beta_1}{4} sh^{2(n-2)} \beta_1 \det h_{AB}^0.$$

There exists an isometry $\Phi : \{p \in M : \tau(p) > 0\} \rightarrow \tilde{M}$, where Φ is obtained by calculating $(\tau, \tilde{\beta}, \beta_1)$ as a function of (τ, β, β_1) from (D.0.2) and (D.0.7). From the equation

$$e^{\tilde{\beta}} = \frac{e^\beta ch^2\beta_1}{\tau + sh^2\beta_1},$$

it follows that $\tilde{\beta}$ blows up at $\tau = \beta_1 = 0$, thus Φ cannot be extended beyond $\tau = 0$. It should be noted that the map $\Psi : M \rightarrow \tilde{M}$ defined by

$$\tau \rightarrow \tau, \quad \beta \rightarrow -\tilde{\beta}, \quad \beta_1 \rightarrow \beta_1,$$

is a globally defined isometry, thus (M, γ) and $(\tilde{M}, \tilde{\gamma})$ are isometric. (M, γ) and $(\tilde{M}, \tilde{\gamma})$ are, however, *inequivalent* space-times from a Cauchy problem point of view for the following reason: let $i : \Sigma \rightarrow M$, $\tilde{i} : \tilde{\Sigma} \rightarrow \tilde{M}$ be embeddings such that M and \tilde{M} are developments of the Cauchy data set (Σ, g, K) , then there exists no isometry $\Xi : M \rightarrow \tilde{M}$ such that $\Xi \circ i = \tilde{i}$ (cf. [36]).

It should be noted that the space-times described above, restricted to the t - x plane, coincide with the so-called “Misner model” for the Taub–NUT space-time.

Appendix E

On a wave equation with a singular source.

In this Appendix we shall show that the solutions of the problem (1.8.3)–(1.8.4), with $0 \notin \text{supp } f$, $f \in C^\infty(\mathbb{R})$, $\varphi, \psi \in C^\infty(\mathbb{R}^2)$, are smooth on $\mathbb{R}^3 \setminus \text{supp } \rho$. Consider thus the equation

$$\square U = f(t) \delta_0,$$

recall that

$$\Delta \ln r = 2\pi \delta_0, \quad (\text{E.0.1})$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and we also have

$$0 < \alpha \in \mathbb{R}, \quad \Delta r^\alpha \ln r = \alpha(\alpha \ln r + 2) r^{\alpha-2}. \quad (\text{E.0.2})$$

From (E.0.1) one finds that the function $U_1 = U - f(t) \ln r / 2\pi$ satisfies

$$\square U_1 = -\frac{1}{2\pi} \frac{\partial^2 f}{\partial t^2} \ln r,$$

and using (E.0.2) one shows by induction that there exist functions $\varphi_i \in C^\infty(\mathbb{R})$, $0 \notin \text{supp } \varphi_i$, such that for any $k \in \mathbb{N}$ we have

$$\begin{aligned} \square U_k &\equiv \square \left(U - \sum_{i=0}^k \varphi_i(t) r^{2i} \ln r \right) \\ &= \chi_k(t) r^{2k} \ln r + \psi_k \equiv \rho_k, \end{aligned}$$

for some functions $\chi_k \in C^\infty(\mathbb{R})$, $\psi_k \in C^\infty(\mathbb{R}^3)$. For any $\ell \in \mathbb{N}$ we can find k such that $\rho_k \in H_{\ell+2}(\mathbb{R}^3)$; we also have $U_k|_{t=0} = U|_{t=0}$, thus the Cauchy data for U_k are smooth, and by standard theory $U_k \in H_{\ell+2}(\mathbb{R}^3) \subset C_\ell(\mathbb{R}^3)$. This shows that for any ℓ we have $U \in C_\ell(\mathbb{R}^3 \setminus \text{supp } \rho)$, thus $U \in C_\infty(\mathbb{R}^3 \setminus \text{supp } \rho)$, which had to be established. Let us note that the argument presented above provides also an asymptotic expansion for U_ρ in a neighbourhood of $\text{supp } \rho$, which can be used to analyze in detail the nature of the singularities occuring in (M_ρ, γ_ρ) .

Appendix F

Bifurcating geodesics in $C_{loc}^{1,\alpha}$, $0 < \alpha < 1$, metrics.

In this Appendix we present $C_{loc}^{1,\alpha}$, $0 < \alpha < 1$, metrics for which there exist bifurcating spacelike, timelike or null geodesics. From the classical theory of ODE's it is well known that the initial value problem (IVP) for the geodesic equation is uniquely solvable when the metric is $C_{loc}^{1,1}$. Due to the variational character of the geodesic equations one could hope, at least in the strictly Riemannian case, to have uniqueness of the IVP for geodesics under some weaker conditions: the examples of this Appendix show that the requirement of $C_{loc}^{1,1}$ differentiability of the metric cannot be relaxed without introducing some supplementary conditions. The examples presented here have been worked out in collaboration with J.Isenberg, following a suggestion by R.Hamilton¹.

Let us start by noting that if we have a metric which is $C_{loc}^{1,1}$ except possibly at an isolated point x_o in a neighborhood of which

$$\begin{aligned} |g(x)_{ij} - \delta_{ij}| &\leq Cr(x - x_o)^\alpha, \quad |\partial_k g_{ij}(x)| \leq Cr(x - x_o)^{\alpha-1}, \\ |\partial_k \partial_l g_{ij}(x)| &\leq Cr(x - x_o)^{\alpha-2}, \quad 0 < \alpha < 1, \end{aligned} \tag{F.0.1}$$

uniqueness of the initial value problem for geodesics can be established by standard fixed point methods in an appropriately weighted space of functions (recall that existence

¹It has been pointed out to the author by R. Bartnik, that essentially the same example has been presented in [65] for spacelike geodesics in a Riemannian metric.

follows from the classical theorem of Peano [82] which asserts that continuity of the right hand side of the equation

$$\frac{dx}{dt} = f(x, t)$$

is sufficient for existence of solutions; it is only the uniqueness part which necessitates the Lipschitz continuity of f in x). Thus it seems that for nonuniqueness of the IVP for geodesics the set at which the metric is not $C_{loc}^{1,1}$ “must have dimension greater than zero”. For non-unique spacelike or timelike geodesics we show that a singular set of dimension 1 is sufficient, for null geodesics dimension 2 is sufficient and probably also necessary; we have undertaken no attempts to prove this last conjecture.

For $y \in \mathbb{R}$ and for $\epsilon|x|^{1+\alpha} > -1/2$ consider the metric

$$ds^2 = dx^2 + f(x)dy^2, \quad f(x) = \epsilon(1 + \epsilon|x|^{1+\alpha}),$$

$$\epsilon = \pm 1, \quad 0 < \alpha < 1. \quad (\text{F.0.2})$$

The equations for a geodesic are easily found from the constants of motion,

$$f(x)\frac{dy}{ds} = p_y = \text{const}, \quad \left(\frac{dx}{ds}\right)^2 + f(x)\left(\frac{dy}{ds}\right)^2 = \epsilon. \quad (\text{F.0.3})$$

Let $x_o > 0$, consider the geodesic which at $s = 0$ passes through (x_o, y_o) with $(dx/ds)(0) < 0, p_y = 1$; from (F.0.3) we have

$$\frac{dx}{ds} = -\frac{|x|^{(1+\alpha)/2}}{(1 - \epsilon|x|^{1+\alpha})^{1/2}}. \quad (\text{F.0.4})$$

For s such that $x(s) > 0$ x monotonically decreases, which gives

$$|x|^{-(1+\alpha)/2} \frac{dx}{ds} \leq -c_o, \quad c_o^{-1} = \min[1, (1 - \epsilon|x_o|^{1+\alpha})^{1/2}] \Rightarrow$$

$$|x|^{(1-\alpha)/2}(s) \leq x_o^{(1-\alpha)/2} - c_o s, \quad (\text{F.0.5})$$

thus $x(s)$ reaches 0 in finite time, say s_o . (F.0.4) implies that

$$\frac{dx}{ds}(s_o) = 0.$$

For any $s_1 \geq s_o$ we can extend the above geodesic to two different C^1 geodesics as follows:
for $s \in [s_o, s_1]$ set

$$x(s) = 0, \quad y(s) = s$$

and for $s_1 \leq s \leq s_1 + s_o$ set either

$$x(s) = x(s_1 - s + s_o),$$

or

$$x(s) = -x(s_1 - s + s_o).$$

This establishes non-uniqueness of the initial value problem. Uniqueness of the boundary value problem for geodesics for $C_{loc}^{1,\alpha}$ metrics remains an open question². Higher dimensional metrics with bifurcating geodesics can be trivially obtained from (F.0.2) by adding terms $dz_1^2 + dz_2^2 + \cdots + dz_k^2$ to (F.0.2).

Let us finally exhibit a metric on a three dimensional Lorentzian manifold with bifurcating null geodesics. For $|x| < 1, y, z \in \mathbb{R}$ let

$$ds^2 = dx^2 - a(x)dy^2 + b(x)dz^2,$$

$$a(x) = 1 - |x|^{1+\alpha}/2, \quad b(x) = 1 + |x|^{1+\alpha}/2, \quad 0 < \alpha < 1. \quad (\text{F.0.6})$$

Arguments similar to the ones presented above show that the null geodesics satisfying

$$a(x)\frac{dy}{ds} = 1, \quad b(x)\frac{dz}{ds} = 1, \quad \frac{dx}{ds} = -\frac{|x|^{(1+\alpha)/2}}{(a(x)b(x))^{1/2}}, \quad x(0) = x_o > 0,$$

can be C^1 continued in a non-unique way at a finite value s_o of the affine parameter, at which they reach the plane $x = 0$.

²Let us recall that e.g. one has uniqueness of the boundary value problem for causal geodesics in a Lorentzian manifold for metrics which are not $C_{loc}^{1,1}$ but for which the curvature tensor, defined distributionally, is locally bounded, cf. [41].

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