

**THE FRÉCHET DIFFERENTIABILITY
OF CONVEX FUNCTIONS ON $C(S)$**

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INTRODUCTION

In this paper we continue our study of the differentiability of convex functions with domain in a topological linear space: here we are particularly concerned with Fréchet differentiability and the space $C(S)$, where S is an arbitrary topological space. The maximum functions, defined by $m_A = \sup_{t \in A} x(t)$ ($A \subset S$ compact), are good test functions: we show that the Fréchet differentiability points of m_A are precisely those functions which attain their maximum on A at a single isolated point of A ; m_A is Fréchet differentiable on a dense subset of $C(S)$ if and only if the set of isolated points of A is dense in A . Relationships between the seminorms p_A and the corresponding maximum functions are given. The set of Fréchet differentiability points of m_A , and hence of p_A , is open; this generalises the well known result that for compact S the sup norm on $C(S)$ is Fréchet differentiable on an open set. In a topological section, connections between the structure of thin and dispersed sets lead into an exploration of the topology of S for Asplund $C(S)$ spaces. In particular, we prove that $C(\mathbb{Q})$ is an Asplund space.

Background

For S compact, $C(S)$ is a Banach space and the sup norm topology coincides with the topology of compact convergence. Čoban and Kenderov study the Gateaux differentiability of the maximum functions m_A on the Banach space $C(S)$ and give a condition on S for the set of points of Gateaux differentiability of the norm to be dense, but contain no G_δ subset [1, 3.3]. Namioka and Phelps prove that for compact S , $C(S)$ is Asplund if and only if S is dispersed [7, 18]. This result is generalised to $C(S)$ for completely regular S in [3, 3.8]: $C(S)$ is Asplund if and only if every compact subset of S is dispersed. It is a consequence of a generalisation of Mazur's Theorem [10, 2.1] that $C(S)$ is Weak Asplund whenever S is σ -compact and metrisable, for example $C(\mathbb{R})$ is Weak Asplund.

Our first results are about the Fréchet differentiability of the maximum functions: it is easy to show that m_A is Gateaux differentiable at $x \in C(S)$ if and only if x attains its maximum on A at only one point; the equivalent condition for Fréchet differentiability is that x attains a maximum at one isolated point of A . We then illustrate how results for a defining family of seminorms can be retrieved from the maximum functions. Properties of *thin* and dispersed sets are explored in a topological interlude which highlights the structure of S for Asplund $C(S)$ spaces. In Section 4 direct proofs are given of some of the theorems of [3], which are there deduced from Banach space theory. Section 5 is devoted to examples; techniques of [10] and [3] are used and in particular we exhibit our favourite Asplund space, $C(\mathbb{Q})$.

Preliminaries

Spaces are assumed Hausdorff; the term *function* is used for a *real valued* map.

Let U be an open subset of a topological linear space X and let \mathcal{M} be a *bornology* on

X , that is, a class of bounded subsets containing all singletons. A real valued function f on U is \mathcal{M} -differentiable at $x \in U$ whenever there exists $u \in X^*$ such that, for all $M \in \mathcal{M}$, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $y \in M$, for all $t : |t| \in (0, \delta)$,

$$\left| \frac{f(x + ty) - f(x)}{t} - u(y) \right| < \epsilon.$$

The map u is uniquely determined by f and x and is denoted by $f'(x)$.

If \mathcal{M} is the class of all *bounded (singleton)* subsets of X then f is *Fréchet (Gateaux) differentiable at x* .

A function f defined on an open convex subset D of a topological linear space X is said to be *convex* whenever, for all $x, y \in D$, for all $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

A continuous convex function f is Gateaux differentiable at $x \in D$ if and only if for all $y \in X$,

$$\phi(y) = \lim_{t \rightarrow 0} \frac{f(x + ty) - f(x)}{t}$$

exists; the continuity and linearity of ϕ are consequences of the continuity and convexity of f .

For S a topological space, $C(S)$ denotes the set of all real valued continuous functions on S with the topology of compact convergence, that is, the topology generated by the seminorms

$$p_A(x) = \max\{|x(t)| : t \in A\},$$

where A ranges over all nonempty compact subsets of S .

The *maximum function* $m_A : C(S) \rightarrow \mathbb{R}$, defined for each nonempty compact subset A of S by

$$m_A(x) = \max\{x(t) : t \in A\},$$

is continuous and convex on $C(S)$.

(There is no loss of generality in assuming S to be completely regular: for any topological space X there is a completely regular S such that $C(X)$ is linearly isomorphic to $C(S)$ [4, 3.9].)

1. FRÉCHET DIFFERENTIABILITY OF THE MAXIMUM FUNCTIONS

In this section we see that Fréchet differentiability of the maximum functions is rare. A function $x \in C(S)$ is a Fréchet differentiability point of m_A if and only if x attains its maximum on A at precisely one isolated point of A ; m_A is Fréchet differentiable on a dense subset of $C(S)$ if and only if the set of isolated points of A is dense in A .

For compact S , 1.1 is [1, Proposition 1.2]. No proof is given here since extending to completely regular S requires only trivial changes: for the *if* case the complete regularity of S ensures the existence of a suitable Urysohn function in $C(S)$ and for the *only if* even this is superfluous.

1.1. *The maximum function m_A is Gateaux differentiable at $x \in C(S)$ if and only if x attains its maximum at only one point of A . If x attains its maximum only at $t_0 \in A$, the derivative of m_A at x is the evaluation function $\delta_{t_0} : C(S) \rightarrow \mathbf{R}$ defined by*

$$\delta_{t_0}(y) = y(t_0).$$

The corresponding condition for Fréchet differentiability is that this maximum occurs at an isolated point of A .

1.2. *Suppose S is a completely regular space and A is a nonempty compact subset of S . The function m_A is Fréchet differentiable at $x_0 \in C(S)$ if and only if x_0 has a maximum on A at precisely one isolated point of A .*

Proof. Assume that $x_0 \in C(S)$ attains its maximum on A only at the isolated point t_0 . Then $A \setminus \{t_0\}$ is closed and so compact, hence x_0 attains its maximum on $A \setminus \{t_0\}$ at some point $t_1 \in A \setminus \{t_0\}$, and $x_0(t_1) < x_0(t_0)$. Let $k = x_0(t_0) - x_0(t_1)$.

Let M be a bounded subset of $C(S)$. There exists $c > 0$ such that for all $y \in M$ and for all $t \in A$, $|y(t)| < c$. Let $\delta = \frac{k}{2c}$. For all $y \in M$, for all $\lambda \in (0, \delta)$, for all $t \in A$,

$$\begin{aligned} |(x_0 + \lambda y)(t) - x_0(t)| &= \lambda |y(t)| \\ &< \frac{1}{2}k, \end{aligned} \tag{*}$$

and specifically,

$$(x_0 + \lambda y)(t_0) > x_0(t_0) - \frac{1}{2}k. \tag{**}$$

For all $t \in A \setminus \{t_0\}$, for all $y \in M$, from (*),

$$\begin{aligned} (x_0 + \lambda y)(t) &< x_0(t) + \frac{1}{2}k \\ &\leq x_0(t_1) + \frac{1}{2}k \\ &= x_0(t_0) - \frac{1}{2}k, \end{aligned}$$

(where the last line follows from the definition of k) which with (**) implies that for all $t \in A \setminus \{t_0\}$

$$(x_0 + \lambda y)(t) < (x_0 + \lambda y)(t_0).$$

Hence $(x_0 + \lambda y)$ attains a maximum on A only at t_0 , and

$$m_A(x_0 + \lambda y) = (x_0 + \lambda y)(t_0),$$

so for all $y \in M$, for all $\lambda \in (0, \delta)$,

$$\begin{aligned} \frac{m_A(x_0 + \lambda y) - m_A(x_0)}{\lambda} &= \frac{(x_0 + \lambda y)(t_0) - x_0(t_0)}{\lambda} \\ &= y(t_0) \\ &= \delta_{t_0}(y), \end{aligned}$$

and m_A is Fréchet differentiable at x_0 .

The plan of the proof of the converse goes like this. Assume that m_A is Fréchet differentiable at x_0 ; then x_0 has a maximum on A only at t_0 . Assume that t_0 is not an isolated point, the complete regularity of S , and the fact that t_0 is not isolated, imply the existence of a bounded sequence (y_n) in $C(S)$ and a sequence (k_n) of positive real numbers converging to 0 such that

$$\left| \frac{m_A(x_0 + k_n y_n) - m_A(x_0)}{k_n} - \delta_{t_0}(y_n) \right| \geq 1,$$

which implies that δ_{t_0} , the Gateaux derivative of m_A at x_0 , is not the Fréchet derivative.

Formally, since x_0 is continuous, for each $n \in \mathbb{N}$,

$$\begin{aligned} U_n &= x_0^{-1} \left[x_0(t_0) + \left(-\frac{1}{2n}, \frac{1}{2n}\right) \right] \\ &= \left\{ t \in S : -\frac{1}{2n} < x_0(t_0) - x_0(t) < \frac{1}{2n} \right\} \end{aligned}$$

is an open neighbourhood of t_0 .

Assume t_0 is not an isolated point of A ; for each $n \in \mathbb{N}$, there exists $s_n \in A \cap U_n$ such that $s_n \neq t_0$ and

$$-\frac{1}{2n} < x_0(t_0) - x_0(s_n) < \frac{1}{2n},$$

but since x_0 has a maximum on A only at t_0 and because $s_n \neq t_0$,

$$0 < x_0(t_0) - x_0(s_n).$$

Let

$$k_n = 2(x_0(t_0) - x_0(s_n))$$

so that $0 < k_n < \frac{1}{n}$. The sequence (k_n) of positive real numbers converges to 0. Since

$$x_0(t_0) - x_0(s_n) = \frac{1}{2}k_n,$$

$$\frac{1}{4}k_n < x_0(t_0) - x_0(s_n) < \frac{3}{4}k_n,$$

so

$$x_0(t_0) - \frac{3}{4}k_n < x_0(s_n) < x_0(t_0) - \frac{1}{4}k_n.$$

Hence for each $n \in \mathbb{N}$, the set D_n defined by

$$D_n = \{t \in S : x_0(t_0) - \frac{3}{4}k_n < x_0(t) < x_0(t_0) - \frac{1}{4}k_n\}$$

is an open neighbourhood of s_n which does not contain t_0 . By complete regularity, for each $n \in \mathbb{N}$ there exists $y_n \in C(S)$ such that

$$\begin{aligned} y_n(t) &\in [0, \frac{3}{2}] && \text{for all } t \in S, \\ y_n(s_n) &= \frac{3}{2}, \\ y_n(t) &= 0 && \text{for all } t \in S \setminus D_n \end{aligned}$$

and in particular $y_n(t_0) = 0$.

Since for any compact set $K \subset S$ and any $n \in \mathbb{N}$,

$$\sup \{y_n(t) : t \in K\} \leq \frac{3}{2},$$

(y_n) is a bounded sequence of points of $C(S)$. Also

$$\begin{aligned} (x_0 + k_n y_n)(s_n) &= x_0(s_n) + \frac{3}{2}k_n \\ &= x_0(t_0) - (x_0(t_0) - x_0(s_n)) + \frac{3}{2}k_n \\ &= x_0(t_0) - \frac{1}{2}k_n + \frac{3}{2}k_n \\ &= m_A(x_0) + k_n \end{aligned}$$

so

$$\begin{aligned} m_A(x_0 + k_n y_n) &\geq (x_0 + k_n y_n)(s_n) \\ &= m_A(x_0) + k_n. \end{aligned}$$

Since for all $n \in \mathbb{N}$, $\delta_{t_0}(y_n) = y_n(t_0) = 0$,

$$\begin{aligned} \frac{m_A(x_0 + k_n y_n) - m_A(x_0)}{k_n} - \delta_{t_0}(y_n) &\geq \frac{k_n}{k_n} - 0 \\ &= 1. \end{aligned}$$

Taking $\epsilon = 1$ and $M = \{y_n\}$ as the bounded set, for each $\delta > 0$ there exist $k_n \in (0, \delta)$ and $y_n \in M$ such that

$$\left| \frac{m_A(x_0 + k_n y_n) - m_A(x_0)}{k_n} - \delta_{t_0}(y_n) \right| \geq \epsilon$$

so δ_{t_0} is not the Fréchet derivative of m_A . ■

A set is said to be *perfect* if it is closed and has no isolated points.

The following corollary is an immediate consequence of 1.2.

1.3 Corollary. *Suppose S is completely regular and A a nonempty compact subset of S ; m_A is nowhere Fréchet differentiable if and only if A is perfect.*

1.4. *If S is completely regular, and A is a nonempty compact subset of S , the set of points of Fréchet differentiability of m_A is an open subset of $C(S)$.*

Proof. Let A be a nonempty compact set in S , let D be the set of points of Fréchet differentiability of m_A , and let $x \in D$. From 1.2, x has a maximum only at an isolated point $t_0 \in A$. Let t_1 be any point where x attains a maximum on the compact set $A \setminus \{t_0\}$.

Let $\epsilon = x(t_0) - x(t_1)$; then $\epsilon > 0$. Define $V \subset C(S)$ by

$$V = \{y \in C(S) : \text{for all } t \in A, |y(t)| < \frac{1}{2}\epsilon\};$$

then $x + V$ is a neighbourhood of x in $C(S)$. If $y \in x + V$, then

$$y(t_0) > x(t_0) - \frac{1}{2}\epsilon$$

and for all $t \in A \setminus \{t_0\}$

$$y(t) < x(t) + \frac{1}{2}\epsilon$$

$$\leq x(t_1) + \frac{1}{2}\epsilon$$

$$= x(t_0) - \epsilon + \frac{1}{2}\epsilon$$

$$= x(t_0) - \frac{1}{2}\epsilon.$$

So y attains its maximum on A only at t_0 . From 1.2, m_A is Fréchet differentiable at y so each point of $x + V$ is in D . Since each point of D has a neighbourhood contained in D , D is open. ■

1.5. *Let S be completely regular and A a compact subset of S . The maximum function m_A is Fréchet differentiable on a dense subset of $C(S)$ if and only if the set of isolated points of A is dense in A .*

Proof. Assume that m_A is Fréchet differentiable on a dense subset of $C(S)$, let $t_0 \in A$, and let U be a neighbourhood of t_0 in S . Then since S is completely regular, there exists $x \in C(S)$ such that

$$\begin{aligned} x(t) &\in [0, 1] && \text{for all } t \in S, \\ x(t_0) &= 1 \\ x(t) &= 0 && \text{for all } t \in S \setminus U. \end{aligned}$$

Define $V \subset C(S)$ by

$$V = \{y \in C(S) : \text{for all } t \in A, |y(t)| < \frac{1}{2}\}.$$

The set $x + V$ is a neighbourhood of x ; by hypothesis there exists a Fréchet differentiability point z of m_A in $x + V$. From 1.2, z has a maximum on A only at an isolated point, say t_1 . Now

$$z(t_0) > x(t_0) - \frac{1}{2} = \frac{1}{2}$$

and if $t \in (S \setminus U) \cap A$ then

$$\begin{aligned} z(t) &< x(t) + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

It follows that z does not attain its maximum on A in $A \cap (S \setminus U)$, hence t_1 is in U .

Thus each neighbourhood of t_0 contains an isolated point of A as required.

Conversely, let A be a compact subset of S and let the set of isolated points of A be dense in A . Let $x \in C(S)$ and let N be a neighbourhood of x in $C(S)$. Then there exist $\epsilon > 0$ and a compact subset K of S such that if

$$V = \{y \in C(S) : \text{for all } t \in K, |y(t)| < \epsilon\}$$

then $x + V \subset N$. In particular, if $h \in C(S)$ satisfies the condition

$$\text{for all } t \in S, |h(t)| < \epsilon,$$

then $x + h \in N$.

Assume that x attains a maximum on A at t_0 ; then

$$x^{-1}[(m_A(x) - \frac{1}{2}\epsilon, m_A(x) + \frac{1}{2}\epsilon)]$$

is a neighbourhood of t_0 . By hypothesis, there exists an isolated point, t_1 , of A (which could be t_0) contained in

$$x^{-1}[(m_A(x) - \frac{1}{2}\epsilon, m_A(x) + \frac{1}{2}\epsilon)],$$

that is

$$m_A(x) - \frac{1}{2}\epsilon < x(t_1) < m_A(x) + \frac{1}{2}\epsilon.$$

Since t_1 is isolated in A , there is a neighbourhood U in S of t_1 which contains no other point of $U \cap A$. By complete regularity there exists $h \in C(S)$ such that:

$$h(t) \in [0, \frac{3\epsilon}{4}] \quad \text{for all } t \in S,$$

$$h(t_1) = \frac{3\epsilon}{4},$$

$$h(t) = 0 \quad \text{for all } t \in S \setminus U,$$

so in particular, $x + h \in N$.

Since t_1 is the only point of A in U , if $t \in A \setminus \{t_1\}$,

$$(x + h)(t) = x(t) \leq m_A(x).$$

However,

$$\begin{aligned}(x+h)(t_1) &= x(t_1) + h(t_1) \\ &> m_A(x) - \frac{1}{2}\epsilon + \frac{3}{4}\epsilon \\ &= m_A(x) + \frac{1}{4}\epsilon.\end{aligned}$$

Hence $x+h$ has a maximum on A only at an isolated point t_1 of A , from 1.2, m_A is Fréchet differentiable at $x+h$, that is, at a point of N . Hence m_A is Fréchet differentiable on a dense subset of $C(S)$. ■

2. THE DIFFERENTIABILITY OF p_A AND m_A

Some of the differentiability properties of the family of seminorms which generate the topology of compact convergence can be deduced from the differentiability properties of the maximum functions. However, easy examples show that the differentiability of either p_A or m_A at a point does not imply the differentiability of the other at that point.

2.1. *Let S be a topological space, A a nonempty compact subset of S and \mathcal{M} a bornology on $C(S)$. The seminorm p_A is \mathcal{M} differentiable on a dense subset of $C(S)$ if and only if the maximum function m_A is \mathcal{M} differentiable on a dense subset of $C(S)$.*

If m_A is \mathcal{M} differentiable on a dense open (dense G_δ , open, G_δ) subset of $C(S)$ then so is p_A .

Proof. If A is a singleton, say $\{t_0\}$, then the derivative of m_A is δ_{t_0} (see 1.1); m_A is \mathcal{M} differentiable everywhere and p_A is \mathcal{M} differentiable everywhere except where $p_A(x) = 0$. In this case all the theorem's conclusions hold. Assume that A has more than one point.

Suppose that p_A is \mathcal{M} -differentiable on a dense subset of $C(S)$. Let $x \in C(S)$, let N be a neighbourhood of 0 in $C(S)$ and let the minimum value of x on A be k . There

exist $\epsilon > 0$ and a compact set D containing A such that

$$B = \{x \in C(S) : p_D(x) < \epsilon\} \subset N.$$

If $U = x + \epsilon - k + B$, there exists $z \in U$ such that p_A is \mathcal{M} -differentiable at z , but on U , $p_A = m_A$, so m_A is \mathcal{M} -differentiable at z . Hence m_A is \mathcal{M} -differentiable at $z + k - \epsilon \in x + N$.

To see the converse define D_+ , D_- and D_0 by:

$$D_+ = \{x \in C(S) : m_A(x) > m_A(-x)\}$$

$$D_- = \{x \in C(S) : m_A(x) < m_A(-x)\}$$

$$D_0 = \{x \in C(S) : m_A(x) = m_A(-x)\}$$

and let $D = D_+ \cup D_-$. Then, by continuity of the functions m_A and $x \mapsto m_A(-x)$, D_+ and D_- are open in $C(S)$, and D is dense in $C(S)$.

If G_m is the set of \mathcal{M} differentiability points of m_A , then $-G_m$ is the set of \mathcal{M} differentiability points of $x \mapsto m_A(-x)$. Let G_p be the set of \mathcal{M} differentiability points of p_A .

It is an easy consequence of the definitions that if two functions f and g agree on an open set O then f is \mathcal{M} differentiable at $x \in O$ if and only if g is. So,

$$G_p \cap D_+ = G_m \cap D_+$$

and

$$G_p \cap D_- = (-G_m) \cap D_-.$$

By definition $p_A(x) = m_A(|x|)$; if $x \in D_0$ then $m_A(x) = m_A(-x)$, and $|x|$ has at least two distinct points producing a maximum; from 1.1 x is not even a Gateaux differentiability point of p_A , that is $G_p \cap D_0$ is empty.

Hence

$$G_p = (G_m \cap D_+) \cup ((-G_m) \cap D_-). \quad (*)$$

If G_m is dense in $C(S)$, then, since D_+ is open,

$$\overline{G_m \cap D_+} \supset D_+$$

and

$$\overline{(-G_m) \cap D_-} \supset D_-.$$

So,

$$\overline{G_p} = \overline{G_m \cap D_+} \cup \overline{(-G_m) \cap D_-} \supset D_+ \cup D_-$$

so,

$$\overline{G_p} = \overline{\overline{G_p}} = \overline{D_+ \cup D_-} = C(S),$$

that is, G_p is dense in $C(S)$.

The second half of the theorem follows from (*). ■

2.2. *Let S be a compact space and $C(S)$ the space of continuous functions on S with sup norm topology. The set of points of Fréchet differentiability of the sup norm is an open set in $C(S)$.*

Proof. From 1.4, taking $A = S$, the set of Fréchet differentiability points of m_S is open; from 2.1 the set of Fréchet differentiability points of p_S , which is the sup norm, is open. ■

2.3. *If S is a compact space, the sup norm on $C(S)$ is Fréchet differentiable on a dense set of $C(S)$ if and only if the set of isolated points of S is dense in S .*

Proof. From 1.5 and 2.1, the following are equivalent:

(a) the set of isolated points of S is dense in S ;

- (b) m_S is Fréchet differentiable on a dense set of $C(S)$;
 (c) p_S , the sup norm, is Fréchet differentiable on a dense set of $C(S)$. ■

It was pointed out by Dr D. Yost that since the Stone-Čech compactification of the integers is the closure of its isolated points, as a special case of 2.2 and 2.3 we have the well known result that the natural norm on ℓ_∞ is Fréchet differentiable on a dense open set.

3. THIN AND DISPERSED SETS

In this section we look at features of the topology of S which will give good Fréchet differentiability properties for m_A on $C(S)$.

A subset A of a topological space is *dispersed* if every nonempty subset of A contains a relatively isolated point and *thin* if the set of isolated points of A is dense in A (that is, if A contains only relatively isolated points and cluster points of relatively isolated points). In the literature *dispersed* is synonymous with scattered and *clairsemé* ([6, Ch 1, section 9, VI]). *Thin* is specific to this paper.

3.1. *If S is a dispersed topological space, then all subsets of S are thin, a fortiori S is thin.*

Proof. Let A be a nonempty subset of S , let $x \in A$ and let N be an open neighbourhood of x . The set $N \cap A$ is a nonempty subset of S so by hypothesis contains an isolated point of $N \cap A$ and so, since N is open, of A ; hence x is either an isolated point of A or a cluster point of isolated points of A . ■

The converse is not true in general.

3.2. *Even when S is compact, S may be thin but not dispersed.*

Proof. Consider \mathbb{R}^2 with the usual topology. Define T by

$$T = \{(p/q, 1/q) : p, q \in \mathbb{N}, 0 < p \leq q, \text{ for } p, q \text{ mutually prime}\} \\ \cup \{(x, 0) : 0 \leq x \leq 1\}.$$

Then T is a compact subset of \mathbb{R}^2 ; T is thin, since the points with nonzero second coordinate are isolated points, and the rest are cluster points of isolated points. However T is not dispersed, since

$$\{(x, y) : y = 0\}$$

contains no isolated point. ■

3.3. *A topological space S is dispersed if every closed subset of S is thin.*

Proof. Let W be a nonempty subset of the topological space S ; W is the disjoint union of a dispersed set and a perfect set [6, Section 9, part IV, result 3], that is

$$W = D \cup P \quad \text{and} \quad D \cap P \text{ is empty}$$

where D is dispersed and P is perfect. Since P is closed, it is by hypothesis thin, which, by the definition of perfect is impossible unless it is empty; hence W is dispersed and nonempty, and therefore contains an isolated point. ■

Of course if S is compact, then in 3.3 “closed” can be equivalently replaced by compact. From 3.3 and 3.1 we deduce 3.4.

3.4. *A topological space S is dispersed if and only if every subset of S is thin.*

For a space to be dispersed, it is not sufficient that every *compact* subset be thin.

3.5. *The space of rationals, \mathbb{Q} , is not dispersed but every compact subset is dispersed, and hence thin.*

Proof. It is easy to see that \mathbb{Q} is not dispersed, since \mathbb{Q} itself contains no isolated points. From 3.3 it suffices to show that every compact subset of \mathbb{Q} consists only of

isolated points and cluster points of isolated points. The following proof of this is due to Emeritus Professor Edwin Hewitt.

Suppose F is a compact subset of \mathbf{Q} ; then F is compact in \mathbf{R} , so we may suppose that F is an infinite countable compact subset of \mathbf{R} . Then from [6] (see 3.3)

$$F = D \cup P$$

where D is a dispersed set and P a perfect set. Any nonempty perfect set in \mathbf{R} contains a copy of Cantor's ternary set, with cardinality the continuum; but F is countable by hypothesis, so cannot contain a nonempty perfect subset, that is, P is empty. Hence F is dispersed; from 3.1, F is thin. ■

3.6. *If every compact subset of S is thin, then every compact subset of S is dispersed.*

Proof. Suppose A is a compact subset of S and F is a closed subset of A ; then F is compact, so by hypothesis F is thin. Hence every closed subset of A is thin; from 3.3, A is dispersed. ■

4. $C(S)$ AS A DIFFERENTIABILITY SPACE

A topological linear space X is said to be an *Asplund space* (a *Fréchet differentiability space*) whenever every continuous convex function with domain a nonempty open convex subset is Fréchet differentiable on a dense G_δ (dense) subset of that domain. We will use the abbreviations ASP and FDS. If in the above definitions "Gateaux" replaces "Fréchet", the spaces are known as Weak Asplund (WASP) and Gateaux Differentiability Space (GDS). If X is a "bound covering" space (for definition and properties see [3, section 2 and 3.2]), for example a Banach space or $C(S)$ for S completely regular, then ASP and FDS coincide, because the set of points of Fréchet differentiability of a continuous convex function is then a G_δ set.

To begin our study of $C(S)$ as a differentiability space, we have 4.1 and 4.2 immediately from 1.3 and 1.5 respectively.

4.1. *If S is completely regular and contains a nonempty compact perfect set, then $C(S)$ is not FDS.*

4.2. *If S is completely regular and $C(S)$ is FDS then every compact subset of S is thin.*

Namioka and Phelps proved that for S compact, the Banach space $C(S)$ is ASP if and only if S is dispersed [7, 18]. A generalisation of one direction of this theorem, gained from 4.2 and 3.6, is given in 4.3; from which half of their result, 4.4, is an immediate corollary.

4.3. *If $C(S)$ is ASP then every compact subset of S is dispersed.*

4.4 Corollary. *Suppose S is compact and $C(S)$ has the sup norm topology. If $C(S)$ is ASP then S is dispersed.*

Towards the converse of 4.3 we have 4.5.

4.5. *If S is a completely regular space and if every compact subset A of S is thin (or, equivalently, dispersed), then every p_A is Fréchet differentiable on a dense open subset of $C(S)$.*

Proof. From 2.1 it suffices to prove the property for the maximum functions m_A ; the result follows from 1.5. ■

It is tempting to hope that from the result that every p_A is Fréchet differentiable on a dense open set, we can conclude that $C(S)$ is ASP; however ℓ_∞ is not even GDS, yet its natural norm is Fréchet differentiable on a dense open set (see the comment

following 2.3). We do know that if each p_A is Fréchet differentiable everywhere, except perhaps on $\ker p$, then $C(S)$ is ASP [3, 3.5].

The following corollary is now immediate.

4.6 Corollary. *Suppose S is compact and $C(S)$ has the sup norm topology. If S is dispersed then the sup norm on $C(S)$ is Fréchet differentiable on a dense open set.*

Whilst a necessary condition for $C(S)$ to be ASP is given in 4.2, (and it is in fact sufficient), a full characterisation seems not to be available by these direct methods. The full characterisation, proved by different methods in [3, 3.8], is stated in 5.3.

5. EXAMPLES

From 4.1 it is clear that $C(\mathbb{R})$ is not FDS. From 5.2 we conclude that $C(\mathbb{N})$ is ASP which leaves the intermediate problem of the classification of $C(\mathbb{Q})$, posed as an open question in the second author's thesis, to complete the picture.

5.1 ([3, 3.7]). *Suppose S is completely regular. Then $C(S)$ is ASP if and only if for all compact subsets A of S , $C(A)$ is ASP.*

Using Namioka and Phelps result [7, 18] and 3.6, we have 5.2

5.2 ([3, 3.7]). *Suppose S is completely regular. Then $C(S)$ is ASP if and only if every compact subset of S is dispersed (or equivalently thin).*

Thus if S is a discrete space, then $C(S)$ is ASP: in particular $\mathbb{R}^n = C(\{1, 2, 3, \dots, n\})$ and $C(\mathbb{N})$ are ASP.

From 3.5 the compact subsets of \mathbb{Q} are dispersed.

5.3. $C(\mathbb{Q})$ is ASP.

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