On convergence of some integral transforms.

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ABSTRACT.

We sketch some results concerning extensions of the Carleson-Hunt Theorem to higher dimensional Euclidean spaces and to some symmetric spaces.

ONE DIMENSIONAL BACKGROUND.

We begin by recalling the results of Carleson and Hunt in the setting of the real line.

Suppose that f is in a Lebesgue space $L^p(\mathbf{R})$ for some $1 < p \leq 2$ and that its Fourier transform is $\mathcal{F}f$. Define the partial sums of its inverse Fourier transform by

$$S_R f(x) = \int_{-R}^{R} \mathcal{F}f(y) e^{ixy}\, dy,$$

for all $x \in \mathbf{R}$ and $R > 0$. The convergence behaviour of $\{S_R f : R > 0\}$ is controlled by the maximal function

$$\mathcal{S}_1^* f(x) = \sup_{R>0} |S_R f(x)|, \qquad \forall x \in \mathbf{R}.$$

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CARLESON-HUNT THEOREM.

(a) If $1 < p \leq 2$ then there is a constant $C_p > 0$ such that

$$\|\mathcal{S}_1^* f\|_p \leq C_p \|f\|_p, \qquad \forall f \in L^p(\mathbf{R}).$$

(b) If $f \in L^p(\mathbf{R})$ and $1 < p \leq 2$ then $\lim_{R\to\infty} S_R f(x) = f(x)$ for almost every $x \in \mathbf{R}$.

Part (b) follows from part (a) since $\lim_{R\to\infty} S_R f(x) = f(x)$ for all x when $f \in C_c^\infty(\mathbf{R})$ and $C_c^\infty(\mathbf{R})$ is dense in $L^p(\mathbf{R})$. See [dG] for the details of this argument.

The original result of Carleson [C] was concerned with Fourier series. See the paper of Kenig and Tomas [KT] for the transition from the case of Fourier series on the circle to Fourier transforms on the line.

RADIAL FUNCTIONS ON EUCLIDEAN SPACES.

For each $n \geq 1$, let $\mathcal{F}_n$ denote the Fourier transform acting on functions on $\mathbf{R}^n$, normalized as in Stein's book [S]. If $R > 0$ then m_R will denote the characteristic function of the ball of radius R in $\mathbf{R}^n$. We are interested in the convergence behaviour of the spherical partial sums

$$S_R f(x) = \mathcal{F}_n^{-1} (m_R \mathcal{F}_n f)(x),$$

for $R > 0$. Such a summation process is natural from the geometric point of view since it corresponds to the spectral resolution of the Laplacian on $\mathbf{R}^n$. Define the maximal function

$$\mathcal{S}_n^* f(x) = \sup_{R>0} |S_R f(x)|.$$

This controls the convergence of $\{S_R f(x) : R > 0\}$.

The Carleson-Hunt Theorem cannot be extended to higher dimensions when $p \neq 2$, on account of Fefferman's Theorem which states that m_R is not a multiplier for $L^p(\mathbf{R}^n)$ when $p \neq 2$, see [F]. The higher dimensional case when $p = 2$ is still an open problem.

Some progress can be made when a symmetry hypothesis is been added. The group of rotations $SO(n)$ acts on functions on $\mathbf{R}^n$ and we let $L^p(\mathbf{R}^n)^{radial}$ denote the subspace of elements of $L^p(\mathbf{R}^n)$ which are invariant under the action of $SO(n)$. Radial functions depend only on the distance from the origin and we sometimes write $f(|x|)$ in place of $f(x)$ when f is radial, to emphasize this dependence. In 1954 Herz [**H**] had found that the partial summation operators were bounded on $L^p(\mathbf{R}^n)^{radial}$ for $2n/(n+1) < p < 2n/(n-1)$. That is, m_R is a multiplier of these spaces for each $R > 0$. Kanjin [**Ka**] and Prestini [**P**] extended the Carleson-Hunt Theorem to these same spaces.

KANJIN-PRESTINI THEOREM. *For each $n \geq 1$ and*

$$2n/(n+1) < p < 2n/(n-1)$$

there is a constant $C_{n,p} > 0$ such that

$$\|\mathcal{S}_n^* f\|_p \leq C_{n,p}\|f\|_2, \qquad \forall f \in L^p(\mathbf{R}^n)^{radial}.$$

This result is sharp. If $n > 1$ and $p = 2n/(n+1)$ Kanjin has shown that there is an $f \in L^p(\mathbf{R}^n)^{radial}$ with compact support and $\{S_R f(x) : R > 0\}$ divergent for almost every $x \in \mathbf{R}^n$.

OTHER SYMMETRIES.

One way of thinking of a radial function is to view it as an example of a function whose orbit under the action of $SO(n)$ spans a finite dimensional subspace. In [**MP1**] we considered products of radial functions and spherical harmonics. For each $n \geq 1$ and $k \geq 1$, let $\mathcal{H}_{n,k}$ denote the space of polynomials which are harmonic and homogeneous of degree k on $\mathbf{R}^n$. The elements of $SO(n)$ act as endomorphisms on each space $\mathcal{H}_{n,k}$. The Bochner-Hecke identity tells us how the Fourier transform acts on products of spherical harmonics and radial functions. See page 72 of [**S**] for the proof of this result.

BOCHNER-HECKE THEOREM. *Suppose that f is a radial function on $\mathbf{R}^n$ and $Y_k \in \mathcal{H}_{n,k}$ is such that $f.Y_k \in L^2(\mathbf{R}^n)$. Then*

$$\mathcal{F}_n\,(f.Y_k)\,(x) = i^n \mathcal{F}_{n+2k}(f)(|x|).Y_k(x), \qquad \text{a.e. } x \in \mathbf{R}^n.$$

When f is a radial function on $\mathbf{R}^n$ and $Y_k \in \mathcal{H}_{n,k}$ is such that $f.Y_k \in L^2(\mathbf{R}^n)$ then f can be treated as an element of $L^2(\mathbf{R}^{n+2k})^{radial}$. Furthermore, observe that

$$S_R\,(f.Y_k)\,(x) = \mathcal{F}_n^{-1}\,(m_R \mathcal{F}_{n+2k}(f) Y_k)\,(x).$$

COROLLARY. *Suppose that f is a radial function on $\mathbf{R}^n$ and $Y_k \in \mathcal{H}_{n,k}$ is such that $f.Y_k \in L^2(\mathbf{R}^n)$. Then there are constants $c_{n,k} > 0$ and $\gamma_{n,k} > 0$ for which*

$$\mathcal{S}_n^*\,(f.Y_k)\,(x) = c_{n,k} \mathcal{S}_{n+2k}^*(f)(|x|)\,|Y_k(x)| \quad \textit{and} \quad \|\mathcal{S}_n^*\,(f.Y_k)\,\|_2 \leq \gamma_{n,k} \|f.Y_k\|_2.$$

We can define certain closed subspaces of $L^2(\mathbf{R}^n)$ in terms of products of radial functions and spherical harmonics. For each $k \geq 0$, let $L^2(\mathbf{R}^n)_k$ be the linear span of products $f.Y_k \in L^2(\mathbf{R}^n)$ such that f is radial and $Y_k \in \mathcal{H}_{n,k}$. Then each $L^2(\mathbf{R}^n)_k$ is a closed subspace of $L^2(\mathbf{R}^n)$ and the algebraic direct sum $\oplus_{k=0}^{\infty} L^2(\mathbf{R}^n)_k$ is dense in $L^2(\mathbf{R}^n)$. An element $f \in L^2(\mathbf{R}^n)$ has the property that its orbit under the action of $SO(n)$ spans a finite dimensional subspace if and only if it is in $\oplus_{k=0}^{\infty} L^2(\mathbf{R}^n)_k$. The following result is in [**MP1**].

THEOREM 1. *Fix $n \geq 1$ and $N \geq 0$. There is a constant $c_{n,N} > 0$ such that*

$$\|\mathcal{S}_n^*(f)\|_2 \leq c_{n,N}\|f\|_2, \qquad \forall f \in \oplus_{k=0}^{N} L^2(\mathbf{R}^n)_k.$$

COROLLARY. *If $f \in L^2(\mathbf{R}^n)$ has the property that its orbit under the action of $SO(n)$ spans a finite dimensional subspace then $\{S_R f(x) \,:\, R > 0\}$ converges for almost every $x \in \mathbf{R}^n$.*

See [**R**] for other applications of the Bochner-Hecke identity in dealing with maximal operators.

SMOOTHNESS.

An alternative direction in seeking higher dimensional extensions to the Carleson-Hunt Theorem is to assume the function has some smoothness. Let s be a positive real number and let

$$H^s(\mathbf{R}^n) = \left\{ f \in L^2(\mathbf{R}^n) \,:\, \int_{\mathbf{R}^n} |\mathcal{F}_n f(\xi)|^2 (1 + |\xi|)^{2s}\, d\xi < \infty \right\}$$

be the usual Sobolev space. Carbery and Soria [**CS**] have shown that if $g \in \cup_{s>0} H^s(\mathbf{R}^n)$ then $\{S_R g(x) \,:\, R > 0\}$ converges for almost every $x \in \mathbf{R}^n$. (A similar statement for eigenfunction expansions of the Laplace-Beltrami operator of a compact Riemannian manifold was proved in [**M**].)

Combining this with Theorem 1 we see that every finite linear combination of translates of elements of $\oplus_{k=0}^{\infty} L^2(\mathbf{R}^n)_k$ plus an element of $H^s(\mathbf{R}^n)$ with $s > 0$ has almost everywhere convergent spherical partial sums of its inverse Fourier transform.

OTHER SPACES.

In place of Euclidean spaces, we may consider eigenfunction expansions for Laplace-Beltrami operators on other Riemannian manifolds. From the work of Kenig, Mityagin, Stanton, and Tomas [**KST**], we know that if $p \neq 2$ and if the dimension of the manifold is greater than one, then there are L^p functions whose eigenfunction expansions diverge. However, for particular homogeneous spaces where we can add some symmetry hypotheses, we can still get convergence results.

We have some results for the case of K-invariant functions on $SL(2,\mathbf{R})/SO(2)$ and the other noncompact connected rank-one Riemannian symmetric spaces. See [**MP2**] and [**MP3**]. Here we will outline a technique for dealing with bi-invariant elements of $L^2(SL(2,\mathbf{R}))$, similar to the device used in [**MP2**]. This depends on some results of Schindler [**Sc**] on asymptotics of Legendre functions.

PRELIMINARY MATERIAL ABOUT $SL(2,\mathbf{R})$.

Before stating the theorem described in [**MP3**], we present some background material on spherical functions on $SL(2,\mathbf{R})$.

The group $SL(2,\mathbf{R})$ is often the testing ground for questions in analysis on noncompact semisimple real Lie groups. In all that follows G will denote the noncompact semisimple Lie group $SL(2,\mathbf{R})$, consisting of 2×2 matrices with real entries and with determinant equal to 1. Inside G there is the compact subgroup $K = SO(2)$, consisting of all orthogonal matrices in G, so that elements of K are matrices of the form

$$k(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

In addition to the subgroup K, there is the subgroup of diagonal elements of G with positive diagonal entries,

$$A = \left\{ a(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} : \quad s \in \mathbf{R} \right\}.$$

This normalizes the subgroup

$$N = \left\{ n(\xi) = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} : \quad \xi \in \mathbf{R} \right\}.$$

In fact $a(s)n(\xi)a(-s) = n(e^s\xi)$ for all s and ξ in $\mathbf{R}$. As basic references for calculations on G, there is the survey article of T. H. Koornwinder [K].

LEMMA 1 (CARTAN DECOMPOSITION). *Each element $x \in G$ can be decomposed into a product of the form $x = k_1 a k_2$, with k_1, $k_2 \in K$ and $a \in A$. If $x \neq 1$, then there are exactly two elements a, $a' \in A$ such that $x \in KaK$ and $x \in Ka'K$. The elements a, a' are inverses of each other, i.e. $a' = a^{-1}$.*

The decomposition $G = KAK$ is also called the polar decomposition, and is analogous to using polar coordinates in Euclidean space. We can equip K with normalized Lebesgue measure, so that

$$\int_K f(k)\, dk = \frac{1}{2\pi} \int_0^{2\pi} f(k(\theta))\, d\theta$$

for all continuous functions f on K. Similarly, since A is isomorphic with the real line, it can also be equipped with Lebesgue measure. Let μ denote Haar measure on G, normalized according to the following integral formula.

LEMMA 2 (INTEGRATION FORMULA). *For every compactly supported continuous function f on G,*

$$\int_G f(x)\, d\mu(x) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} f(k(\theta_1)a(s)k(\theta_2)) \sinh(s)\, d\theta_1 d\theta_2 ds.$$

Consider the action of G on the upper half plane $\mathcal{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$. If $g \in G$ is of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and if $z \in \mathcal{H}$, then the action of g on z is

$g \cdot z = (az+b)/(cz+d) \in \mathcal{H}$. In particular, N acts by translation parallel to the real axis ($n(\xi) \cdot z = z + \xi$) and A acts by dilations ($a(s) \cdot z = e^s z$). This shows that the action is transitive. K fixes the point $i \in \mathcal{H}$, which we will treat as the origin. Hence $\mathcal{H}$ can be identified with the homogeneous space G/K, so that $g \cdot i$ is identified with the coset gK in G/K.

If f is a right-K-invariant function on G, then it can be identified with a function, $f^\sharp$ on $\mathcal{H}$ by assigning $f^\sharp(g \cdot i) = f(g)$, for all $g \in G$. Similarly, every function F on $\mathcal{H}$ is equivalent to a right-K-invariant function $F^\flat$ on G, with $F^\flat(g) = F(g \cdot i)$, for all $g \in G$. The set $\mathcal{H}$ can be equipped with the Poincaré metric $ds^2 = dx^2 + dy^2/y^2$, which is invariant under fractional linear transformations $z \mapsto g \cdot z$. The corresponding G-invariant measure on $\mathcal{H}$ is

$$\int_{\mathcal{H}} f(x+iy)\,\frac{dxdy}{y^2}.$$

When $\mathcal{H}$ is equipped the Poincaré metric, it carries the Laplace-Beltrami operator

$$\Delta = y^2\left(\partial^2/\partial x^2 + \partial^2/\partial y^2\right)$$

and this is G-invariant. Clearly Δ acts on smooth right-K-invariant functions on G by forming $\Delta f(g) = \left(\Delta f^\sharp\right)^\flat(g)$.

We now concentrate on analysis of bi-K-invariant functions on G. From the Cartan decomposition, we see that if f satisfies $f(k_1 g k_2) = f(g)$ for all k_1, $k_2 \in K$ and $g \in G$, then f is completely determined by its restriction to $\{a(s) : s \geq 0\}$. In particular,

$$\int_G f(g)\,d\mu(g) = 2\pi \int_0^\infty f(a(s))\sinh(s)\,ds. \tag{1}$$

In addition, if f is bi-K-invariant then $s \mapsto f(a(s))$ is an even function on the real line. Another interpretation of bi-K-invariant functions comes from viewing them as functions on $\mathcal{H}$ with the property that $F(k \cdot z) = F(z)$ for all $k \in K$ and $z \in \mathcal{H}$. In this setting, they are "radial functions" on $\mathcal{H}$, depending only on the distance from i with respect to the Poincaré metric. When thinking in these terms, we then expect that there is an integral transform analogous to the Hankel transform of radial functions and coming from the eigenfunctions of the Laplace-Beltrami operator. The transform which plays this role is called the *spherical transform*. For an introduction to this theory, see the notes of Godement [GD]. The elementary spherical functions are radial eigenfunctions of Δ.

Definition. A continuous function φ on G is said to be an elementary spherical function if it satisfies the following three conditions:

- $\varphi(1) = 1$;
- φ is bi-K-invariant;
- there is a complex number α such that $\Delta\varphi = \alpha\varphi$.

These are given by Legendre functions on $[0, \infty)$. That is,

$$\varphi_\lambda(a(s)) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh(s) + \sinh(s)\cos(\theta))^{-(1/2)+i\lambda}\, d\theta$$

for all nonnegative numbers s and complex λ, with $\Delta\varphi_\lambda = -(\lambda^2 + (1/4))\varphi_\lambda$.

The spherical transform of an integrable bi-K-invariant function f on G is

$$\mathcal{F}f(\lambda) = \int_G f(g)\varphi_\lambda(g)\, d\mu(g) = 2\pi\int_0^\infty f(a(s))\varphi_\lambda(a(s))\sinh(s)\, ds.$$

In particular, there is the Plancherel formula for square-integrable bi-K-invariant functions on G. For each $p \geq 1$, let ${}^K L^p(G)^K$ denote the subspace of bi-K-invariant elements of $L^p(G)$. There is a measure ν on $[0, \infty)$ such that $f \mapsto \mathcal{F}f|_{[0,\infty)}$ extends from ${}^K L^1(G)^K \cap L^2(G)$ to be an isometry

$$\mathcal{F} : {}^K L^2(G)^K \longrightarrow L^2([0, \infty), \nu).$$

The density for this measure is $d\nu(\lambda) = \lambda \tanh(\pi\lambda)\, d\lambda/\pi$, and the inversion formula is

$$f(a(s)) = \frac{1}{\pi} \int_0^\infty \mathcal{F}f(\lambda)\varphi_\lambda(a(s))\lambda \tanh(\pi\lambda)\, d\lambda. \tag{2}$$

When f is a smooth, compactly supported bi-K-invariant function, this inversion formula converges absolutely for all $s \geq 0$.

THE PARTIAL SUMS.

In [**MP2**] we proved the following theorem when G/K is a rank one, noncompact, connected Riemannian symmetric space.

THEOREM 2. *Suppose that f is in ${}^K L^2(G)^K$ and that*

$$\mathcal{M}_G f(g) = \sup_{R>1} \left| \int_1^R \mathcal{F}f(\lambda)\varphi_\lambda(g)\, d\nu(\lambda) \right|$$

for all g in G. Then $\mathcal{M}_G f$ is also in $L^2(G)$ and there is a constant $c > 0$ which is independent of f such that

$$\|\mathcal{M}_G f\|_2 \leq c\|f\|_2.$$

In addition,

$$f(g) = \lim_{R\to\infty} \int_0^R \mathcal{F}f(\lambda)\varphi_\lambda(g)\, d\nu(\lambda)$$

for almost every g in G.

We will be dealing with asymptotic expansions involving terms of the form

$$(x, y) \mapsto h(x, y)\exp(ixy)$$

where $x \in [a, b]$, $0 \leq a < b < \infty$, and $y \geq 1$. In addition, h is C^1 on $[a, b] \times [1, \infty)$ and we will have some control on it and its derivatives. We set up the maximal function

$$\mathcal{M}_h f(x) = \sup_{R>1} \left| \int_1^R \mathfrak{F}_1 f(y) h(x, y) \exp(ixy)\, dy \right|$$

for $x \in [a,b]$. The transform $\mathcal{F}_1 f$ of an element $f \in L^p(\mathbf{R})$ is locally integrable and so we can carry out an integration by parts on the integral on the right hand side. That is

$$\begin{aligned}\int_1^R & \mathcal{F}_1 f(y) h(x,y) \exp ixy \, dy \\ &= -\int_1^R \left(\int_1^S \mathcal{F}_1 f(y) \exp(ixy)\, dy \right) \frac{\partial}{\partial S} h(x,S)\, dS \\ &+ h(x,R) \int_1^R \mathcal{F}_1 f(y) \exp(ixy)\, dy.\end{aligned}$$

Taking absolute values of this, we find that it is bounded above by

$$\mathcal{S}_1^* f(x) \left(\int_1^R \left| \frac{\partial}{\partial S} h(x,S) \right| dS + |h(x,R)| \right).$$

This final upper bound will be in $L^p(a,b)$ provided

$$(x,R) \mapsto \left(\int_1^R \left| \frac{\partial}{\partial S} h(x,S) \right| dS + |h(x,R)| \right)$$

is uniformly bounded on $[a,b] \times [1,\infty)$.

LEMMA 3. *Suppose that $[a,b]$ is an interval in $[0,\infty)$ and that h is a C^1 function on $[a,b] \times [1,\infty)$. Furthermore, take $1 < p \leq 2$ and assume that there is a positive constant β such that*

$$\sup_{a \leq x \leq b} \int_1^\infty \left| \frac{\partial}{\partial y} h(x,y) \right| dy \leq \beta \quad \text{and} \quad \sup_{a \leq x \leq b} \sup_{R \geq 1} |h(x,R)| \leq \beta.$$

Then for all $f \in L^p(\mathbf{R})$ the maximal function $\mathcal{M}_h f$ is in $L^p(a,b)$ and

$$\|\mathcal{M}_h f\|_p \leq \beta c_p \|f\|_p.$$

Note. A similar argument works with weighted L^p-spaces which are A_p spaces on account of the result of Hunt and Young [HY]

THE SPHERICAL TRANSFORM CASE.

We saw earlier that the spherical transform $\mathcal{F}$ maps functions in ${}^K L^2(G)^K$ to functions in $L^2([0,\infty),\nu)$, and is an isomorphism. We put

$$k(t,\lambda) = |\lambda \tanh(\pi\lambda)|^{1/2}(\sinh(t))^{1/2}\varphi_\lambda(a(t)), \tag{3}$$

where t and λ are positive real numbers. This method of changing the special functions is used by Gilbert in [G]. Furthermore, starting with a bi-K-invariant function f on G we set up

$$Vf(t) = (\sinh(t))^{1/2} f(a(t)), \quad t > 0.$$

LEMMA 4. *A bi-K-invariant function f on G is in ${}^K L^2(G)^K$ if and only if $Vf \in L^2(0,\infty)$. Furthermore, there is a constant c depending only on the normalization of Haar measure on G such that $\|Vf\|_2 = c\|f\|_2$.*

If F is an integrable function on $(0,\infty)$ with respect to $\sinh(t)^{1/2}\,dt$, we write

$$KF(\lambda) = \int_0^\infty F(t)k(t,\lambda)\,dt.$$

In particular, when f is an element of ${}^K L^1(G)^K$ its spherical transform is

$$\mathcal{F}f(\lambda) = |\lambda \tanh(\pi\lambda)|^{1/2}(KVf)(\lambda)$$

and the partial sums of the inverse are

$$\int_0^R KVf(\lambda)k(t,\lambda)\,d\lambda = (\sinh(t))^{1/2}\int_0^R \mathcal{F}f(\lambda)\varphi_\lambda(a(t))\lambda\tanh(\pi\lambda)\,d\lambda.$$

Therefore, to prove Theorem 2 it suffices to prove that $Vf \in L^2(0,\infty)$ implies that the partial sums

$$\left\{\int_1^R KVf(\lambda)k(t,\lambda)\,d\lambda \;:\; R > 1\right\}$$

converge as $R \to \infty$ for almost every $t > 0$.

SCHINDLER'S ASYMPTOTIC FORMULAE.

Schindler's paper **[Sc]** contains the asymptotic properties of $k(t, \lambda)$ in a form which we can use. The important cases are when $\lambda \geq 1$. The function $k(t, \lambda)$ is the special case of $K^m(x, y)$ in **[Sc]**, with $m = 0$, $\lambda = x$, and $t = y$. Now assume that $\lambda > 1$ and $t > 1$. Hence, we want to first control the convergence away from the origin. For these values of λ and t Schindler has shown that:

$$\begin{aligned} k(t, \lambda) =& \sqrt{2/\pi} \cos(t\lambda - (\pi/4)) + \sqrt{2/\pi} \frac{1}{8\lambda} \coth(t) \sin(t\lambda - (\pi/4)) \\ &+ \cos(t\lambda - (\pi/4)) \left\{ S_1(\lambda) + \frac{T_1(\lambda)}{e^{2t} - 1} \right\} \\ &+ \sin(t\lambda - (\pi/4)) \left\{ S_2(\lambda) + \frac{T_2(\lambda)}{e^{2t} - 1} \right\} \\ &+ \cos(t\lambda - (\pi/4)) D_1(t, \lambda) + \sin(t\lambda - (\pi/4)) D_2(t, \lambda), \end{aligned} \tag{4}$$

where

$$S_j(\lambda),\ T_j(\lambda) = \mathbf{O}(\lambda^{-2}), \quad S_j'(\lambda),\ T_j'(\lambda) = \mathbf{O}(\lambda^{-3}), \tag{5}$$

and

$$D_j(t, \lambda),\ \frac{\partial}{\partial \lambda} D_j(t, \lambda),\ \frac{\partial}{\partial t} D_j(t, \lambda) \ = \mathbf{O}(\lambda^{-2} e^{-4t}). \tag{6}$$

Next we consider $0 < t < 1$ and $\lambda > 1$. In this range Schindler has shown that

$$k(t, \lambda) = (t\lambda)^{1/2} J_0(t\lambda) + \phi_1(t) t^{3/2} \lambda^{-1/2} J_1(t\lambda) + F_0(\lambda, t) \tag{7}$$

where

$$F_0(\lambda, t) = (t\lambda)^{1/2} \tilde{k}_0(\lambda) J_0(t\lambda) + \begin{cases} \mathbf{O}\left(\lambda^{-3/2} t^{5/2}\right), & \lambda t \leq 1 \\ \mathbf{O}\left(\lambda^{-2} t^2\right), & \lambda t \geq 1 \end{cases} \tag{8}$$

$\phi_1(t) = \mathbf{O}(1)$ and $\tilde{k}_0(\lambda) = \mathbf{O}(\lambda^{-2})$.

THE L^2-CASE.

The operator K which we have just defined, extends from $L^1 \cap L^2$ to become an isometry of L^2 to L^2. There is also the Plancherel formula for cosine transforms.

LEMMA 5. *There is a constant $c_2 > 0$ such that for every F is in $L^2(0,\infty)$ there is an even function $TF \in L^2(\mathbf{R})$ whose Fourier transform $\mathfrak{F}_1(TF)$ satisfies $\mathfrak{F}_1(TF)|_{(0,\infty)} = KF$ almost everywhere and $\|TF\|_2 = c_2\|F\|_2$.*

Start with a function $f \in {}^K L^2(G)^K$ and consider

$$\int_1^R KVf(\lambda)k(t,\lambda)\,d\lambda = \int_1^R \mathfrak{F}_1(TVf)(\lambda)k(t,\lambda)\,d\lambda. \tag{9}$$

Suppose that $t > 1$, $R > 1$, and expand the right hand side using equation (4) above. We find that

$$\sup_{R>1}\left|\int_1^R KVf(\lambda)k(t,\lambda)\,d\lambda\right|$$

is bounded by a sum of terms of the form:

$$\sup_{R>1}\left|\int_1^R \mathfrak{F}_1(TVf)(\lambda)e^{\pm i\lambda t}\,d\lambda\right| \quad ; \quad \coth(t)\sup_{R>1}\left|\int_1^R \frac{\mathfrak{F}_1(TVf)(\lambda)}{\lambda}e^{\pm i\lambda t}\,d\lambda\right| ;$$

$$\sup_{R>1}\left|\int_1^R \mathfrak{F}_1(TVf)(\lambda)S_j(\lambda)e^{\pm i\lambda t}\,d\lambda\right| \quad ; \quad \frac{1}{e^{2t}-1}\sup_{R>1}\left|\int_1^R \mathfrak{F}_1(TVf)(\lambda)T_j(\lambda)e^{\pm i\lambda t}\,d\lambda\right| ;$$

and

$$\sup_{R>1}\left|\int_1^R \mathfrak{F}_1(TVf)(\lambda)e^{\pm i\lambda t}D_j(t,\lambda)\,d\lambda\right| .$$

Apply Lemma 3 to each of these pieces, using the estimates given in (5) and (6) to verify the hypotheses used there. From this we conclude that if $f \in {}^K L^2(G)^K$ then

$$t \mapsto \sup_{R>1}\left|\int_1^R KVf(\lambda)k(t,\lambda)\,d\lambda\right|, \quad t > 1, \tag{10}$$

is in $L^2(1,\infty)$ and its norm there is dominated by a constant multiple of the $L^2(0,\infty)$ norm of Vf.

To handle the case of $0 < t < 1$ we use the same device with expansion (7). That is, our hypothesis is that KVf is in $L^2(0,\infty)$, and so

$$x \mapsto |x|^{-(n-1)/2} KVf(|x|)$$

is in $L^2(\mathbf{R}^n)^{radial}$ for each $n \geq 1$. Let $\mathbf{T}_n(Vf)$ be the element of $L^2(\mathbf{R}^n)^{radial}$ with

$$\mathcal{F}_n(\mathbf{T}_n(Vf))(|x|) = |x|^{-(n-1)/2} KVf(|x|), \quad \text{a.e. } x \in \mathbf{R}^n$$

and $\|\mathbf{T}_n(Vf)\|_2 = c_n \|Vf\|_2$.

The maximal function

$$t \mapsto \sup_{R>1} \left| \int_1^R KVf(\lambda) k(t,\lambda)\, d\lambda \right|, \quad 0 < t < 1, \tag{11}$$

is hence dominated by the sum of:

$$\sup_{R>1} \left| \int_1^R KVf(\lambda)(t\lambda)^{1/2} J_0(t\lambda)\, d\lambda \right| \quad ; \quad |\phi_1(t)| \sup_{R>1} \left| \int_1^R KVf(\lambda) t^{3/2} \lambda^{-1/2} J_1(t\lambda)\, d\lambda \right|$$

and

$$\sup_{R>1} \left| \int_1^R KVf(\lambda) F_0(\lambda, t)\, d\lambda \right|.$$

The first term is dominated by a constant multiple of

$$t^{1/2} \mathcal{S}_2^*(\mathbf{T}_2(Vf))(t\xi),$$

where ξ is an arbitrary unit vector in $\mathbf{R}^2$, and the Kanjin-Prestini Theorem implies that this is in $L^2(0,1)$ as a function of t. Similarly, the second integral is

$$t^{5/2} \int_{1<|y|<R} \frac{\mathcal{F}_4(\mathbf{T}_4(Vf))(|y|)}{|y|} \exp(it\xi \cdot y)\, dy$$

and we can apply the Kanjin-Prestini Theorem again.

In dealing with the cases of the last integral we need to remember that we are assuming $\lambda \geq 1$ and $0 < t < 1$. Therefore, the remainder term is bounded by

$$t^{5/2} \int_1^{1/t} |KVf(\lambda)|\lambda^{-3/2}\, d\lambda + t^2 \int_{1/t}^{R} |KVf(\lambda)|\lambda^{-2}\, d\lambda$$

which is bounded by

$$t^{5/2}\|KVf\|_2 \left(1 - t^2/2\right)^{1/2} + t^2\|KVf\|_2 \left(t^3/3 - 1/(3R^3)\right)^{1/2}$$

and this is square-integrable on [0, 1].

The only term that we still have to treat is

$$\int_1^R KVf(\lambda)(t\lambda)^{1/2}\tilde{k}_0(\lambda) J_0(t\lambda)\, d\lambda$$

and this is handled in the same manner as the second term, using a variant of Lemma 3 for Hankel transforms.

Note. For $p < 2$ and $f \in {}^K L^p(G)^K$ the situation is more complicated since spherical transforms of functions in this space are analytic on a strip surrounding the real axis (see [ST]) and so the partial sums cannot belong to ${}^K L^p(G)^K$. See also the comments by Giulini and Mauceri in [GM]. Despite this, we have the following analogue of the two-dimensional Euclidean case.

THEOREM 3. *If $G = SL(2,\mathbf{R})$, $K = SO(2)$, $4/3 < p \leq 2$, and $f \in {}^K L^2(G)^K$ with then $\mathcal{M}_G f \in L^2(G) + L^p(G)$ and*

$$\|\mathcal{M}_G f\|_{L^2+L^p} \leq c_p \|f\|_p.$$

Notes. The control on the maximal function in Theorem 3, combined with the everywhere convergence of the inverse spherical transforms of test functions, is enough to guarantee the almost everywhere convergence of the inverse spherical transforms of elements of ${}^KL^p(G)^K$ with $4/3 < p \leq 2$. Colzani and Vignati [CV] dealt with the case of $SL(3,\mathbf{R})$ using an equiconvergence argument. At this stage we do not know if Theorem 3 can be extended to L^p spaces of other noncompact rank one symmetric spaces for any $p \neq 2$. Another unsolved problem is whether or not there is an analogue of Theorem 1 for the K-finite elements of $L^2(SL(2,\mathbf{R})/SO(2))$.

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