THE POINCARÉ–BERTRAND FORMULA FOR THE HILBERT TRANSFORM

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Abstract. The Poincaré–Bertrand formula for the finite Hilbert transform will be proved by applying the properties of Chebyshev polynomial functions. That formulation will then be extended to the Hilbert transform both for the entire real line and the one–dimensional torus.

1. INTRODUCTION AND PRELIMINARIES

Singular integral equations with Cauchy kernel play an important rôle in many fields of physics and engineering, including aerodynamics, elasticity, transport theory and so on, (see, for example, [3], [11] and [14]). In [19, Chapter IV] F.G. Tricomi has demonstrated the usefulness of the Poincaré–Bertrand formula for the finite Hilbert transform in solving those equations.

To be more precise, let H denote the Hilbert transform on the real line $\mathbb{R}$ and let λ denote the Lebesgue measure in $\mathbb{R}$. Let p and q be positive numbers such that $p^{-1} + q^{-1} = 1$. Applying the Sokhotski–Plemelj formula, E.R. Love [9, Corollary] has shown that the identity

$$H(fHg + gHf) = (Hf)(Hg) - fg \tag{1.1}$$

holds λ–almost everywhere for every $f \in \mathscr{L}^p(\lambda)$ and $g \in \mathscr{L}^q(\lambda)$.

The identity (1.1) is known as the Poincaré–Bertrand formula having its origin in [6] and [13]. A brief history for this formula has been written in [9].

By expressing (1.1) in terms of an $\mathscr{L}^1(\lambda)$–valued bilinear map, an alternative proof has been given by R.G. Rooney [16], under a stronger assumption: $p^{-1} + q^{-1} < 1$.

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The principal aim of this note is to present a real analysis proof of the Poincaré–Bertrand formula for the Hilbert transform H, without using the Sokhotski–Plemelj formula; see section 2 (Theorem 2.9). It is essential in the proof there that the identity (1.1) can be rewritten by using an $\mathscr{L}^0(\lambda)$–valued bilinear map.

The main interests of G.H. Hardy [6] and H. Poincaré [13] appear to have been in the finite Hilbert transform rather than H. Accordingly, that formula for the finite Hilbert transform will be established first by considering Chebyshev polynomial functions (Theorem 2.7). This will then be applied to prove Theorem 2.9.

Section 3 provides the Poincaré–Bertrand formula for the Hilbert transform on the one–dimensional torus $\mathbb{T}$ (Theorem 3.4). That formula has been applied to solve those singular integral equations with Hilbert kernel, by D. Elliott in [4].

Let μ be a Radon measure in a non–empty, locally compact Hausdorff space Ω. Those functions which differ only on a μ–null set will be identified.

The linear space of complex valued, μ–measurable functions on Ω is denoted by $\mathscr{L}^0(\mu)$.

Let $1 \leq p < \infty$. Let $\mathscr{L}^p(\mu)$ denote the set of all functions $f \in \mathscr{L}^0(\mu)$ such that $|f|^p$ is μ–integrable. By the Minkowski inequality, the set $\mathscr{L}^p(\mu)$ is a linear subspace of $\mathscr{L}^0(\mu)$, and $\mathscr{L}^p(\mu)$ will be equipped with the usual $\mathscr{L}^p$–seminorm

$$f \mapsto \left[\int_\Omega |f|^p d\mu\right]^{1/p}, \quad f \in \mathscr{L}^p(\mu) .$$

Let $\mathscr{K}(\Omega)$ denote the set of all complex valued, continuous functions on Ω with compact support. Then $\mathscr{K}(\Omega)$ is a dense linear subspace of the seminormed space $\mathscr{L}^p(\mu)$ (cf. [1, Definition 4.3.2]).

The indefinite integral $f\mu$ of a function $f \in \mathscr{L}^1(\mu)$ with respect to the Radon measure μ is the set function defined by

$$(f\mu)(E) = \int_E f \, d\mu$$

for every Borel subset E of Ω. The set function $f\mu$ is also a Radon measure in Ω.

A linear operator Λ: $\mathscr{L}^1(\mu) \to \mathscr{L}^0(\mu)$ is said to be of weak type (1, 1) if there exists a constant $K > 0$ such that

$$\varepsilon\mu(\{\omega \in \Omega: |\Lambda f(\omega)| > \varepsilon \}) \leq K\left[\int_\Omega |f| \, d\mu)\right]$$

for every $\varepsilon > 0$ and every $f \in \mathscr{L}^1(\mu)$. It is clear that the natural inclusion map from $\mathscr{L}^1(\mu)$ into $\mathscr{L}^0(\mu)$ is of weak type (1, 1).

2. THE POINCARÉ–BERTRAND FORMULA FOR THE HILBERT TRANSFORM ON $\mathbb{R}$.

Let f be a function belonging to the set

(2.1) $$\cup\{\mathscr{L}^p(\lambda) : 1 \leq p < \infty\} ,$$

where λ is the Lebesgue measure in the real line $\mathbb{R}$. Then the Cauchy principal value

$$Hf(t) = \lim_{\varepsilon \downarrow 0} \left[\int_{-\infty}^{t-\varepsilon} + \int_{t+\varepsilon}^{\infty}\right] \frac{f(\tau)}{\pi(\tau - t)} \, d\lambda(\tau)$$

exists for λ–almost every $t \in \mathbb{R}$ and the function Hf thus defined is λ–measurable in $\mathbb{R}$ (see, for example, [2, Theorem 8.1.6]). The resulting linear operator H from the subspace (2.1) of the space $\mathscr{L}^0(\lambda)$ into $\mathscr{L}^0(\lambda)$ will be called the *Hilbert transform* on $\mathbb{R}$.

The restriction of H to the subspace $\mathscr{L}^1(\lambda)$ of (2.1) is denoted by H_1.

LEMMA 2.1. *The linear operator* H_1: $\mathscr{L}^1(\lambda) \to \mathscr{L}^0(\lambda)$ *is of weak type* (1, 1).

Proof. See, for example, [2, Theorem 8.1.5] or [17, Lemma V.2.8]. □

M. Riesz proved the following in [15, VII].

LEMMA 2.2. *Let* $1 < p < \infty$. *Then* $Hf \in \mathscr{L}^p(\lambda)$ *for every* $f \in \mathscr{L}^p(\lambda)$ *and the* $\mathscr{L}^p(\lambda)$*-valued linear operator* $f \mapsto Hf$, $f \in \mathscr{L}^p(\lambda)$, *is continuous.*

Let $n = 1,2,\ldots$. Let λ_n denote the restriction of the Lebesgue measure λ to the open interval $]-n, n[$; that is, $\lambda_n(E) = \lambda(E)$ for every Lebesgue measurable subset E of $]-n, n[$. The space $\mathscr{L}^0(\lambda_n)$ will always be equipped with the topology of convergence in measure. Recall that a sequence of functions $f_m \in \mathscr{L}^0(\lambda_n)$, $m = 1,2,\ldots$, converges to zero in measure if and only if

$$\lim_{m\to\infty} \lambda_n(\{t \in]-n, n[: |f_m(t)| > \varepsilon \}) = 0$$

for every $\varepsilon > 0$. Define a linear surjection $P_n: \mathscr{L}^0(\lambda) \to \mathscr{L}^0(\lambda_n)$ by

$$P_n f(t) = f(t) , \quad t \in]-n, n[,$$

for every $f \in \mathscr{L}^0(\lambda)$.

An immediate consequence of Lemma 2.1 is the following.

LEMMA 2.3. *Let linear operator* $P_n H_1: \mathscr{L}^1(\lambda) \to \mathscr{L}^0(\lambda_n)$ *is continuous for every* $n = 1,2,\ldots$.

The natural extension Jf of a function $f \in \mathscr{L}^1(\lambda_1)$ to $\mathbb{R}$ is the function which coincides with f on $]-1, 1[$ and vanishes outside of $]-1, 1[$; then $Jf \in \mathscr{L}^1(\lambda)$. The resulting linear operator $J: \mathscr{L}^1(\lambda_1) \to \mathscr{L}^1(\lambda)$ is a continuous injection.

The *finite Hilbert transform* is the linear operator $R: \mathscr{L}^1(\lambda_1) \to \mathscr{L}^0(\lambda_1)$ defined by $R = P_1 H_1 J$. Then R is continuous because so are $P_1 H_1$ and J.

Let $\mathbf{x}$ denote the identity function on $]-1, 1[$, that is

$$\mathbf{x}(t) = t, \quad t \in]-1, 1[.$$

Let $1 < p < \infty$. Let $\alpha \in]-1, p^{-1}[$ and $\beta \in]-1, p^{-1}[$. The function ρ on $]-1, 1[$, defined by

$$\rho = (1 - \mathbf{x})^{\alpha}(1 + \mathbf{x})^{\beta} , \tag{2.2}$$

is λ_1-integrable. The indefinite integral $\rho\lambda_1$, which is a Radon measure in the locally compact space $]-1, 1[$, satisfies $\mathscr{L}^p(\rho\lambda_1) \subset \mathscr{L}^1(\lambda_1)$ by the Hölder inequality. The following result is due to B.V. Khvedelidze [8] (see, for example, [5, Lemma I.4.2] or [10, Theorem II.3.7]).

LEMMA 2.4. *Let* $1 < p < \infty$. *Suppose that* α *and* β *are numbers within the open interval* $]-1, p^{-1}[$ *and that* ρ *is the function defined by* (2.2). *Then*

$$R(\mathscr{L}^p(\rho\lambda_1)) \subset \mathscr{L}^p(\rho\lambda_1)$$

and the restriction of R *to the space* $\mathscr{L}^p(\rho\lambda_1)$ *becomes a continuous linear operator with values in* $\mathscr{L}^p(\rho\lambda_1)$ *itself.*

Suppose that $1 < p < \infty$. Let $q = p/(p-1)$. Let α and β be numbers within $]-1, p^{-1}[$ and let ρ be as in (2.2). Let $B(p;\alpha,\beta)$ denote the bilinear map, whose domain is the product space

$$\mathscr{L}^p(\rho\lambda_1) \times \mathscr{L}^q((1/\rho)\lambda_1) \tag{2.3}$$

and codomain the space $\mathscr{L}^0(\lambda_1)$, defined by

$$B(p;\alpha,\beta)(f,g) = R(fRg + gRf) - (Rf)(Rg) + fg \tag{2.4}$$

for every element (f,g) of the product space (2.3). Then $B(p;\alpha,\beta)$ is continuous, because so are $R: \mathscr{L}^1(\lambda_1) \to \mathscr{L}^0(\lambda_1)$ and the $\mathscr{L}^1(\lambda_1)$-valued bilinear map which assigns the function fg to each element (f,g) of (2.3).

In order to consider the case when $p = 2$, let

$$w = \sqrt{1-x^2}\,.$$

Let $\mathbb{N}_0$ denote the set of all non-negative integers. The Chebyshev polynomial functions of the first kind, T_n, $n \in \mathbb{N}_0$, and of the second kind, U_n, $n \in \mathbb{N}_0$, are defined by

$$T_n(\cos\xi) = \cos n\xi \quad\text{and}\quad U_n(\cos\xi) = \frac{\sin(n\xi + \xi)}{\sin\xi}$$

for every $\xi \in]0, \pi[$, respectively. The addition formulae for the sine and cosine functions lead to the following identities for all $n \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, with the understanding that $U_{-1} = 0$:

(2.5) $$2T_nT_m = T_{n+m} + T_{|n-m|} \ ;$$

(2.6) $$2(1-x^2)U_nU_m = T_{|n-m|} - T_{n+m+2} \ ; \text{ and}$$

(2.7) $$U_{n-1}T_{m+1} + T_nU_m = U_{n+m} \ .$$

LEMMA 2.5. *The linear spans of the sets*

$$\{T_n/w\colon n \in \mathbb{N}_0\} \quad \textit{and} \quad \{U_nw\colon n \in \mathbb{N}_0\}$$

are dense subsets of the seminormed spaces $\mathscr{L}^2(w\lambda_1)$ *and* $\mathscr{L}^2(1/w)\lambda_1)$ *respectively.*

Proof. See, for example, [18, Theorem 3.1.5]. □

LEMMA 2.6. *If* n *and* m *are non-negative integers, then*

(2.8) $$B(2;2^{-1},2^{-1})(T_n/w,\ U_mw) = 0.$$

Proof. It is clear that the identities

(2.9) $$R(T_n/w) = U_{n-1},\ n \in \mathbb{N}_0 \ ; \text{ and}$$

(2.10) $$R(U_nw) = -T_{n+1},\ n \in \mathbb{N}_0 \ ,$$

hold (see, for example, [19, p.174 and pp.180–181]). It follows, from (2.5), (2.6), (2.9) and (2.10), that

(2.11) $$2R((T_n/w)R(U_mw)) = -U_{n+m} - U_{|n-m-1|-1} \ ; \text{ and}$$

(2.12) $$2R((U_mw)R(T_n/w)) = -U_{n+m} + U_{|n-m-1|-1}$$

for all n, m $\in \mathbb{N}_0$. Moreover, by (2.7), (2.9) and (2.10), we obtain

(2.13) $$-R(T_n/w)R(U_mw) + (T_n/w)(U_mw) = U_{n-1}T_{m+1} + T_nU_m = U_{n+m}$$

for all n, m $\in \mathbb{N}_0$. Therefore (2.8) follows from (2.11), (2.12) and (2.13). □

Since $B(2;2^{-1},2^{-1})$ is continuous, it follows from Lemmas 2.5 and 2.6 that

(2.14) $$B(2;2^{-1},2^{-1}) = 0.$$

In other words, the identity

$$B(2;2^{-1},2^{-1})(f,g) = 0$$

holds λ_1–almost everywhere for every element (f,g) of the product space

(2.15) $$\mathscr{L}^2(w\lambda_1) \times \mathscr{L}^2((1/w)\lambda_1) \ .$$

We are now ready to present the Poincaré–Bertrand formula for the finite Hilbert transform.

THEOREM 2.7. *Let* $1 < p < \infty$ *and let* $q = p/(p-1)$. *Suppose that* α *and* β *are numbers within the open interval* $]-1, p^{-1}[$ *and that* ρ *is the function given by* (2.2). *Then*

(2.16) $$B(p;\alpha,\beta) = 0 ;$$

that is, the Poincare–Bertrand formula

$$R(fRg + gRf) = (Rf)(Rg) - fg$$

holds λ_1*-almost everywhere for every* $f \in \mathscr{L}^p(\rho\lambda_1)$ *and* $g \in \mathscr{L}^q((1/\rho)\lambda_1)$.

Proof. Let (φ,ψ) be an element of the subspace

(2.17) $$\mathscr{K}(]-1, 1[) \times \mathscr{K}(]-1, 1[)$$

of the space (2.3). Then (2.14) implies that

$$B(p;\alpha,\beta)(\varphi,\psi) = B(2;2^{-1},2^{-1})(\varphi,\psi) = 0$$

λ_1-almost everywhere because (φ,ψ) belongs to the space (2.15). Consequently the bilinear map $B(p;\alpha,\beta)$ from (2.3) into $\mathscr{L}^0(\lambda_1)$ vanishes on the dense subspace (2.17) of (2.3). Thus (2.16) holds. □

To prove the Poincaré–Bertrand formula for the Hilbert transform on $\mathbb{R}$, let $1 < p < \infty$ and let $q = p/(p-1)$. Define a bilinear map C, from the product space

(2.18) $$\mathscr{L}^p(\lambda) \times \mathscr{L}^q(\lambda)$$

into the space $\mathscr{L}^0(\lambda)$, by

$$C(f,g) = H(fHg + gHf) - (Hf)(Hg) + fg$$

for every member (f,g) of (2.18).

LEMMA 2.8. *Suppose that* $f \in \mathscr{L}^p(\lambda)$ *and* $g \in \mathscr{L}^q(\lambda)$ *are functions vanishing outside some closed interval* $[-m, m]$, $m = 1,2,\ldots$. *Then*

$$P_nC(f,g) = 0$$

λ_n*-almost everywhere for every* $n = 1,2,\ldots$.

Proof. Let n be an integer such that $n \geq m$. Define functions $f_n \in \mathscr{L}^p(\lambda_1)$ and $g_n \in \mathscr{L}^q(\lambda_1)$ by

$$f_n(t) = f(nt) \text{ and } g_n(t) = g(nt), \quad t \in]-1, 1[\ ,$$

respectively. Then

$$(P_n Hf)(t) = (Rf_n)(t/n) \quad \text{and} \quad (P_n Hg)(t) = (Rg_n)(t/n)$$

and hence

$$P_n H(fHg + gHf)(t) = R(f_n Rg_n + g_n Rf_n)(t/n),$$

for λ_n-almost every $t \in]-n, n[$. It then follows from Theorem 2.7 that

$$P_n C(f,g)(t) = B(p;0,0)(f_n,g_n)(t/n) = 0$$

for λ_n-almost every $t \in]-n, n[$.

If n is a positive integer such that $n < m$, then

$$P_n C(f,g) = P_n P_m C(f,g) = 0$$

λ_n-almost everywhere because $P_n = P_n P_m$. □

THEOREM 2.9. *Let* $1 < p < \infty$ *and let* $q = p/(p-1)$. *Then* $C = 0$; *namely, the Poincare–Bertrand formula*

$$H(fHg + gHf) = (Hf)(Hg) - fg$$

holds λ*-almost everywhere for every* $f \in \mathscr{L}^p(\lambda)$ *and* $g \in \mathscr{L}^q(\lambda)$.

Proof. Let $n = 1,2,\ldots$. Then the bilinear map $P_n C$ from the product space (2.18) into the space $\mathscr{L}^0(\lambda_n)$ is continuous by Lemmas 2.2 and 2.3. By Lemma 2.8, the map $P_n C$ vanishes on the dense subspace $\mathscr{K}(\mathbb{R}) \times \mathscr{K}(\mathbb{R})$ of the product space (2.18). Thus $P_n C = 0$. Since n is arbitrary, we obtain $C = 0$. □

3. THE POINCARÉ–BERTRAND FORMULA FOR THE HILBERT TRANSFORM ON $\mathbb{T}$.

Throughout this section, the complex number $\sqrt{-1}$ is denoted by i.

Let ν be the normalized Haar measure in the one-dimensional torus

$$\mathbb{T} = \{e^{it}: t \in]-\pi, \pi]\}.$$

The Lebesgue measure in the interval $]-\pi, \pi]$ will be denoted by λ_π. Then

$$(2\pi)\nu(\{e^{it}: t \in]a, b]\}) = \lambda_\pi(]a, b]) = b - a$$

whenever $-\pi \le a \le b \le \pi$.

Let f be a function belonging to the space $\mathscr{L}^1(\nu)$. Then the Cauchy principal value

$$\mathrm{Sf}(e^{it}) = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \left[\int_{-\pi}^{t-\varepsilon} + \int_{t+\varepsilon}^{\pi} \right] f(e^{i\tau}) \cot \frac{\tau - t}{2} \, d\lambda_{\pi}(\tau)$$

exists for λ_{π}–almost every $t \in \,]-\pi, \pi]$, and the so-defined function Sf is ν–measurable in $\mathbb{T}$ (see, for example, [2, Theorem 9.1.1]).

The $\mathscr{L}^0(\nu)$–valued linear operator S: $f \mapsto \mathrm{Sf}$, $f \in \mathscr{L}^1(\nu)$, is called the *Hilbert transform* on $\mathbb{T}$. The Hilbert transform S is of weak type (1, 1) (see, for example, [2, Proposition 9.1.2]).

LEMMA 3.1. *The linear operator* S: $\mathscr{L}^1(\nu) \to \mathscr{L}^0(\nu)$ *is continuous.*

Proof follows from the fact that the measure ν is finite. □

The proof of the following result which is parallel to Lemma 2.2 can be found, for example, in [2, Proposition 9.1.3].

LEMMA 3.2. *Let* $1 < p < \infty$. *Then* $\mathrm{Sf} \in \mathscr{L}^p(\nu)$ *for every* $f \in \mathscr{L}^p(\nu)$ *and the* $\mathscr{L}^p(\nu)$*–valued linear operator* $f \mapsto \mathrm{Sf}$, $f \in \mathscr{L}^p(\nu)$, *is continuous.*

Let z denote the identity function on $\mathbb{T}$. Let $\operatorname{sgn} m = m/|m|$ for every non–zero integer m and $\operatorname{sgn} 0 = 0$.

The proof of the identity

$$\lim_{\varepsilon \downarrow 0} \left[\int_{-\pi}^{t+\varepsilon} + \int_{t-\varepsilon}^{\pi} \right] e^{im\tau} \cot \frac{\tau}{2} \, d\lambda_{\pi}(\tau) = 2\pi i(\operatorname{sgn} m), \tag{3.1}$$

for every integer m, can be found, for example, in [2, Proposition 9.1.4], while it can be derived also from [10, Lemma II.1.1].

LEMMA 3.3. *If* m *is an integer, then*

$$S(z^m) = i(\operatorname{sgn} m) z^m .$$

Proof. The assertion follows from (3.1), because $e^{im(\tau+t)} = e^{im\tau}e^{imt}$ for all $\tau \in \,]-\pi, \pi]$ and $t \in \,]-\pi, \pi]$ and because $\cot(u + \pi) = \cot u$ whenever $\sin u \neq 0$, $u \in \mathbb{R}$. □

Let $1 < p < \infty$ and let $q = p/(p-1)$. Define a bilinear map D(p), from the product space

$$\mathscr{L}^{p}(\nu) \times \mathscr{L}^{q}(\nu) \tag{3.2}$$

into the space $\mathscr{L}^{0}(\nu)$, by

$$D(p)(f,g) = S(fSg + gSf) - (Sf)(Sg) + fg - \left[\int_{\mathbb{T}} f\, d\nu\right]\left[\int_{\mathbb{T}} g\, d\nu\right]$$

for every element (f,g) of (3.2). Then D(p) is continuous by Lemmas 3.1 and 3.2. It follows from Lemma 3.3 that

$$D(p)(z^{m},z^{n}) = 0, \ (m, n = 0, \pm 1, \pm 2, \dots). \tag{3.3}$$

We now claim that

$$D(2) = 0. \tag{3.4}$$

In fact, it is well known in the theory of Fourier series that the set $\{z^{m}: m = 0, \pm 1, \pm 2, \dots\}$ is dense in the seminormed space $\mathscr{L}^{2}(\nu)$. So, the continuity of D(2) and (3.3) jointly imply (3.4).

The following theorem gives the Poincaré–Bertrand formula for the Hilbert transform S on $\mathbb{T}$.

THEOREM 3.4. *Let* $1 < p < \infty$ *and let* $q = p/(p-1)$. *Then*

$$D(p) = 0 \ ; \tag{3.4}$$

that is, the Poincare–Bertrand formula

$$S(fSg + gSf) = (Sf)(Sg) - fg + \left[\int_{\mathbb{T}} f\, d\nu\right]\left[\int_{\mathbb{T}} g\, d\nu\right]$$

holds ν-almost everywhere for every $f \in \mathscr{L}^{p}(\nu)$ and $g \in \mathscr{L}^{q}(\nu)$.

Proof. Since the linear span of the set $\{z^{m}: m = 0, \pm 1, \pm 2, \dots\}$ is a dense subset of the seminormed spaces $\mathscr{L}^{p}(\nu)$ and $\mathscr{L}^{q}(\nu)$ (see, for example, [7, Theorem II.1.5]) the relationship (3.3) implies the desired identity (3.4). □

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