WATER NON-WAVES

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This paper is about free-surface problems of the type that usually produce water waves. That is, it is about boundary-value problems for Laplace's equation in a water-occupied domain, with a free surface between water and air, which is influenced by gravity g, and therefore on which a quadratically nonlinear constant-pressure condition holds.

A large class of such problems is such that the free surface is plane when undisturbed: for example, the plane surface of a calm sea. A small disturbance can then produce small waves, which are asymptotically sinusoidal and linear. A large disturbance can produce large waves, which can be periodic but non-sinusoidal – nonlinear Stokes waves.

But waves are not essential, whether linear or nonlinear. For example, consider the effect on a stream U in a two-dimensional flow, of a disturbance created by a small symmetrical over-pressure $P(x)$, as could be caused by blowing air on the water surface over a finite segment $-\ell < x < \ell$, as in Figure 1. This could be a model of a (rather wide!) hovercraft.

This pressure disturbance $P(x)$ certainly deforms the free surface, and in general creates a wave trailing behind it, as $x \to +\infty$. The linearised theory of water waves predicts (Vanden-Broeck and Tuck [27]) that this trailing wave has amplitude proportional to

$$A = \int_{-\ell}^{\ell} P(x) \cos(\kappa x) \, dx$$

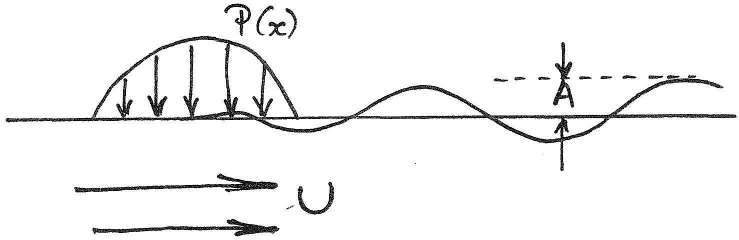


Figure 1

where $\kappa = g/U^2$. It is possible for the wave amplitude A to vanish for some choices of $P(x)$ and κ. For example, if $P(x)$ is constant, then

$$A = \frac{2P}{\kappa} \sin(\kappa \ell)$$

which vanishes for $\kappa\ell = \pi, 2\pi, \ldots$ (see Lamb [12], p. 404).

That is, an infinitely-wide uniform-pressure "hovercraft" makes no waves if its length 2ℓ and speed U combine to give $\kappa\ell = n\pi$, or if the Froude number

$$F = \frac{U}{\sqrt{g(2\ell)}}$$

takes one of the discrete set of special values

$$F = F_n = 1/\sqrt{2\pi n}, \quad n = 1, 2, \ldots$$

These are quite practical speeds, the highest being $F_1 = 0.40$. A similar set of "waveless" speeds can be computed for most (but not quite all) choices of $P(x)$, see Vanden-Broeck and Tuck [25].

More generally, for most finite-length small two-dimensional disturbances, there will exist some discrete spectrum of Froude numbers $F = F_n$ at which waves are not generated. This applies to two-dimensional (infinitely wide) "ships", planing surfaces, hydrofoils, etc., as well as to hovercraft. These special Froude numbers are "good", in the sense that no waves means no wave drag. This is an interference phenomenon, very roughly due to cancellation of the "bow" wave by a "stern" wave that is 180° out of phase, as sketched in Figure 2. Hence between every pair of "good" speeds there is a "bad" speed, where the two waves reinforce instead of cancelling! In practice, we tend to be most interested in the highest good speed F_1, which is more isolated from its nearest bad speed.

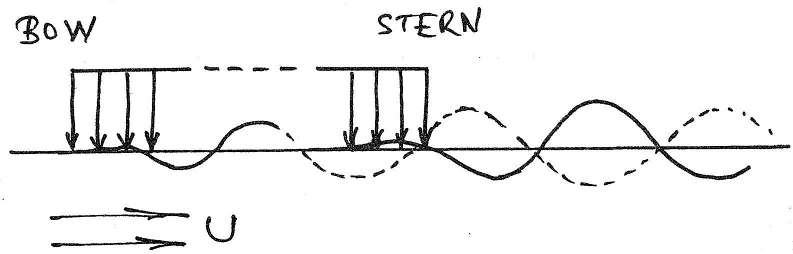


Figure 2

The above is a linearised small-disturbance argument, and there is no actual proof that a similar conclusion holds for cancellation of nonlinear large-amplitude waves – after all, interference is a linear superposition idea. However, numerical experience (Schwartz [19]) suggests that the qualitative conclusion that there exists a discrete spectrum of waveless speeds, still holds in the nonlinear case.

The more practical generalisation is to three-dimensional disturbances, that is to ships or other disturbing agents of finite width. It was long thought that full cancellation of waves in three dimensions was impossible. After all, three-dimensional disturbances send waves in all directions, and how can we organise the shape of the disturber so as to cancel all these waves at the same time? For example, Krein (in [10], p. 355; see also [11]) showed that no surface piercing body of finite volume can have zero wave resistance.

However, a fully submerged body which is finned or ray-fish like, as sketched in Figure 3a, has recently been shown to be waveless (Tuck [20]). The plan form of the slender lifting surface which acts as a fin can be chosen so that the fin makes waves which are in all directions of propagation exactly equal and opposite to those of the spheroidal body to which it is attached. Figure 3b shows the resulting zero value of the wave resistance at a design Froude number $F = 0.5$, the dashed curve showing the resistance of the spheroid alone, which has its maximum at this design speed.

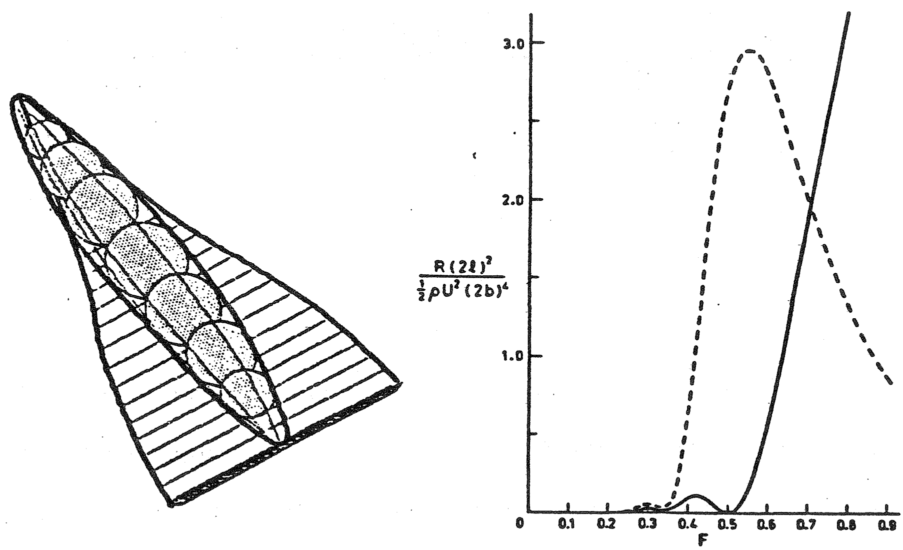


Figure 3

For the remainder of this paper, however, I shall return to the two-dimensional case. This is not quite as restrictive as it might seem, especially for barges or ships like supertankers (see Figure 4), or for hydrofoils of large aspect ratio. In particular, we can be quite interested in the nearly two-dimensional flow near the centerplane of the extreme bow of a bluff body like a supertanker. Our model then is a semi-infinite disturbance, the free surface lying to the left and a solid body to the right. The destructive cancellation discussed earlier can also occur for some such semi-infinite disturbances, but not quite so easily. We need somehow to provide two distinct sets of waves that are out of phase and can cancel each other, and the bulbous bow provides such a possibility, as we shall see.

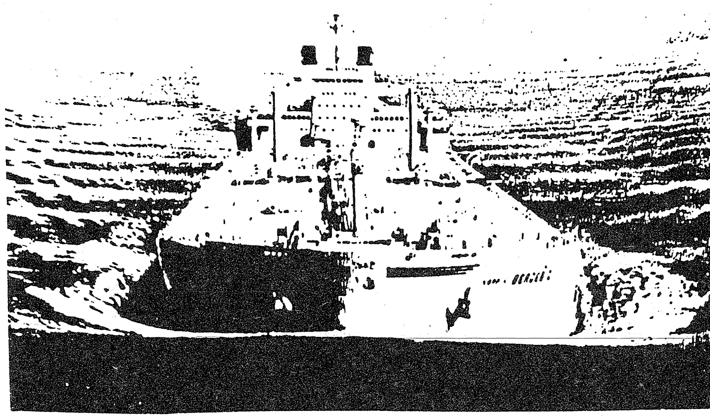


Figure 4

I want for a moment to do some mathematics. Let me set up a general framework for study of steady two-dimensional nonlinear free surface problems. We use the complex potential $f(z) = \phi + i\psi$, with $z = x + iy$, and the complex velocity

$$f'(z) = U \exp(\tau - i\theta)$$

The "logarithmic hodograph" variable so defined, namely

$$\Omega = \tau - i\theta = \log[f'(z)/U]$$

has the physical interpretation that its real part gives the velocity magnitude

$$q = Ue^{\tau}$$

and its imaginary part the flow direction, namely the angle θ between streamlines and the $x-$axis. We solve the problem inversely, i.e. use f as independent variable, and seek $\Omega = \Omega(f)$ in the lower-half f-plane, i.e. seek $\tau = \tau(\phi, \psi), \theta = \theta(\phi, \psi)$, each satisfying Laplace's equation with respect to ϕ, ψ in $\psi < 0$.

Now the free surface is the streamline $\psi = 0$, and the immediate advantage of the inverse approach is clear, in that the free surface is a known boundary in the (ϕ, ψ) plane, whereas its shape in the physical (x, y) plane is unknown, see Figure 5.

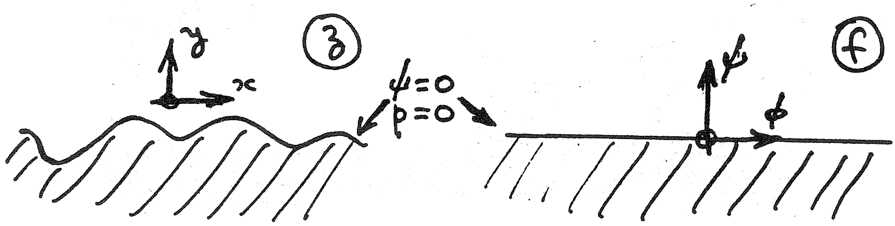


Figure 5

The second free-surface boundary condition is that the pressure is constant, and hence by Bernoulli's equation

$$gy + \frac{1}{2}q^2 = \text{constant}$$

We can differentiate the above pressure condition tangentially, i.e. with respect to ϕ, obtaining

$$g\frac{\partial y}{\partial \phi} + \frac{1}{2}U^2\frac{\partial}{\partial \phi}e^{2\tau} = 0$$

But $\partial y/\partial \phi$ is the imaginary part of dz/df, namely $e^{-\tau}\sin\theta$. Hence we find

$$e^{3\tau}\tau_\phi = -\kappa \sin\theta$$

This boundary condition is attributed to Rudzki's 1898 work by Wehausen and Laitone ([28], p. 727). It integrates to

$$\tau(\phi, 0) = \frac{1}{3}\log\left[1 - 3\kappa\int_{-\infty}^{\phi}\sin\theta(\varphi, 0)\,d\varphi\right]$$

assuming that the flow becomes a uniform stream U far upstream, i.e. that $\tau \to 0$ as $\phi \to -\infty$.

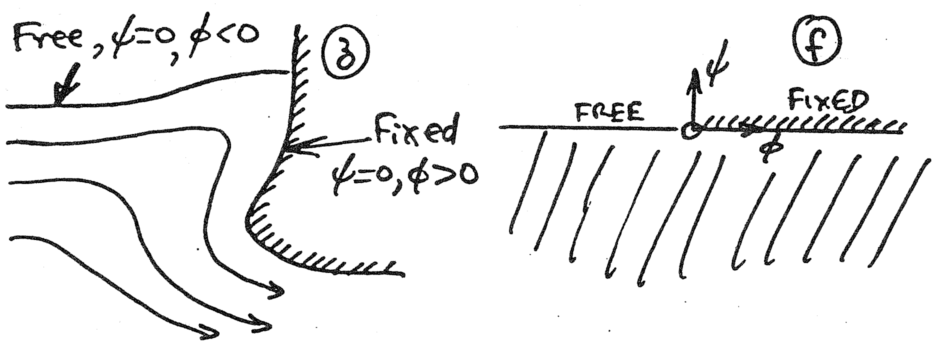


Figure 6

That is, there is an explicit integral relationship between the real and imaginary parts of $\Omega(f)$ on the real axis $f = \phi + i0$. But because $\Omega(f)$ is analytic in the lower half plane (and tends to zero as $\psi \to -\infty$), there is also a Hilbert-transform relationship between these two quantities, namely

$$\tau(\phi, 0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\theta(\varphi, 0)d\varphi}{\varphi - \phi}$$

the integral being of Cauchy principal value form. Use of this Hilbert transform enforces satisfaction of the Laplace equation.

Equating the two integrals for τ gives a nonlinear integral equation for $\theta(\phi, 0)$, namely

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\theta(\varphi,0)d\varphi}{\varphi-\phi}=\frac{1}{3}\log\left[1-3\kappa\int_{-\infty}^{\phi}\sin\theta(\varphi,0)\,d\varphi\right]$$

Numerical solution of this equation solves the original boundary-value problem.

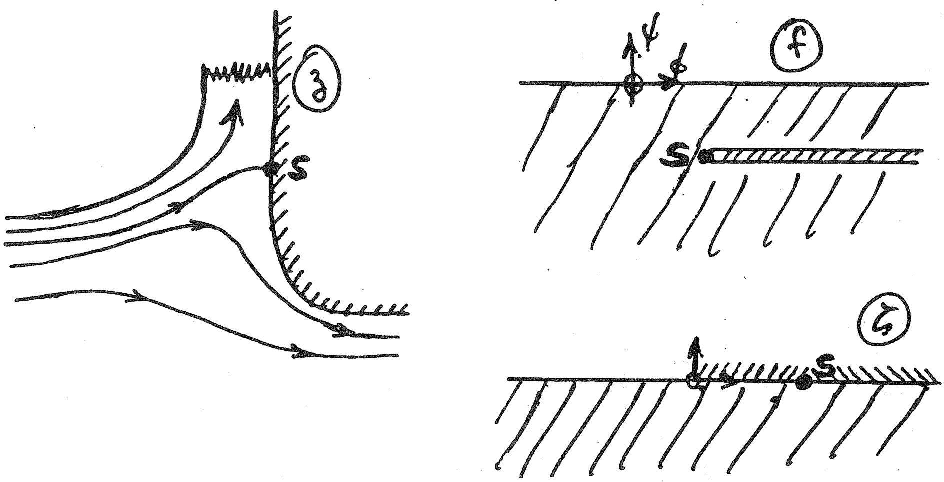


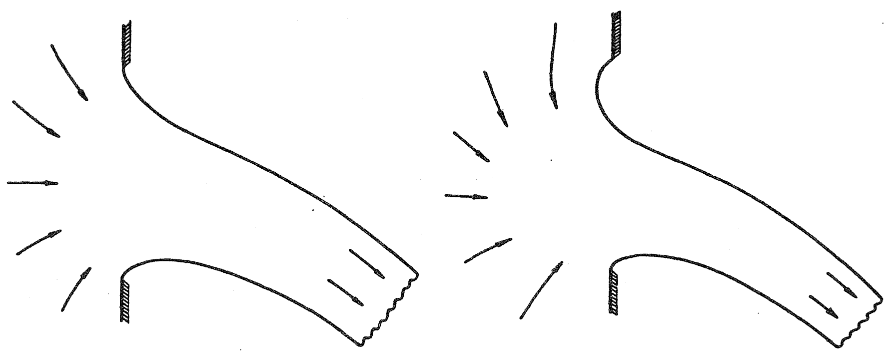
Figure 7

This integral equation has been used in many contexts, not all ship-related. In particular, it can form the basis for study of pure Stokes waves, and perhaps should be called Nekrasov's equation because of the use of a similar equation by Nekrasov in 1922 (see [28], p. 750) to prove existence of such nonlinear waves. A similar equation studied by Conway and Bullock [3] was called the Milne-Thomson integral equation in recognition of a formulation of this class of problem by Milne-Thomson ([15], p. 307).

Only for pure Stokes waves is the Nekrasov integral equation valid as it stands on the whole of the ϕ−axis. Otherwise, either we must take into account a given surface-piercing body over part of that line (Figure 6), or must modify the Hilbert transform relating τ and θ, to take account of submerged disturbances or even just submerged stagnation points that destroy analyticity of $\Omega(f)$ in $\psi < 0$. In many cases, a preliminary conformal mapping takes care of the latter problem, as in Figure 7, where

$$f = \zeta - \log \zeta.$$

The integral equation then has ζ as its independent variable.



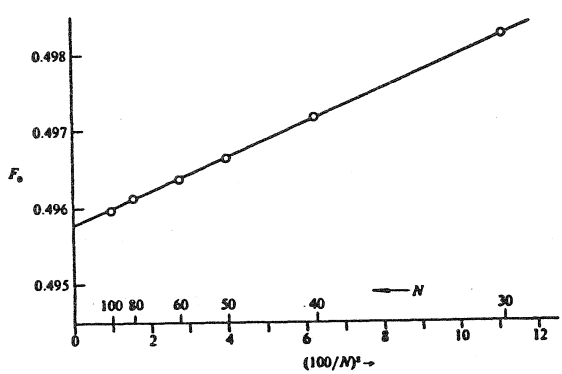


Figure 8

Some of the "modern" uses of the Nekrasov integral equation or its equivalents have been in what can be called "mathematical hydraulics", where the undisturbed free surface is not necessarily plane. For example, the problem in Figure 8a, of efflux from

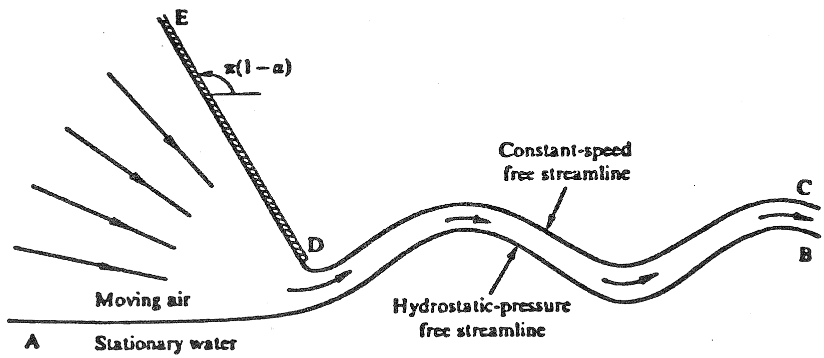


Figure 9

a slit in a wall was solved by Tuck ([20]; see also Goh and Tuck [7]) by direct numerical solution of a similar integral equation. The free surface was discretised into up to $N = 100$ points, and close to 4-figure accuracy obtained, as indicated by the graph in Figure 8c. This problem has the feature that solutions exist only for $F > 0.496$, the number 0.496 being the output quantity tested for accuracy in Figure 8c. The limiting solution at $F = 0.496$ has a stagnation point with a 120° contact angle at the upper detachment point, as shown in Figure 8b.

A similar numerical solution was used by Grundy and Tuck (1987) for a problem that does have waves, produced by an air jet stream blowing on stationary water, as in Figure 9.

A problem involving a submerged sink was solved by Tuck and Vanden-Broeck [23], using a truncated series method, but integral equation methods have also been used on this class of problems, e.g. by Forbes and Hocking [5]. The flow in Figure 10a has a cusped free surface and exists only for the unique Froude number $F = 3.553$.

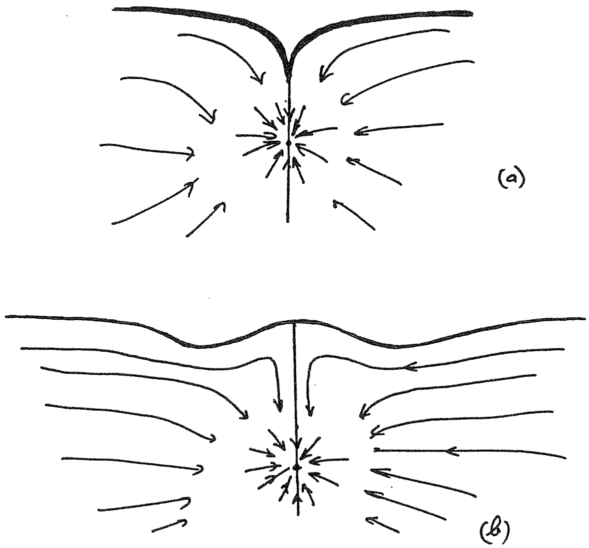


Figure 10

Recent work (e.g. Hocking and Forbes [9] has concentrated on flows with a stagnation point on the free surface as in Figure 10b, which appear to exist in a range $F < F_0$, where F_0 is about 1.4.

Another class of hydraulic flows is flow over weirs or under sluice gates, as illustrated in Figure 11. Series methods were used by Dias, Keller and Vanden-Broeck [4] and integral equation methods by Goh [5].

Returning to the ship context, suppose we assume that the streamline $\psi = 0$ consists of two parts, namely the free surface in $\phi < 0$, and a given body specified by a given function $\theta(\phi, 0) = \Theta(\phi)$ in $\phi > 0$, as sketched in Figure 6. Then the Nekrasov integral equation becomes

$$\frac{1}{\pi}\int_{-\infty}^{0} \frac{\theta(\varphi,0)d\varphi}{\varphi-\phi} + \frac{1}{\pi}\int_{0}^{+\infty} \frac{\Theta(\varphi)d\varphi}{\varphi-\phi} = \frac{1}{3}\log\left[1 - 3\kappa\int_{-\infty}^{\phi} \sin\theta(\varphi,0)\,d\varphi\right]$$

with the second integral a known function, and this is to be solved for $\theta(\phi, 0)$ in $\phi < 0$.

Vanden-Broeck [25] in effect proved that there is no such solution when $\Theta(\phi)$ is a

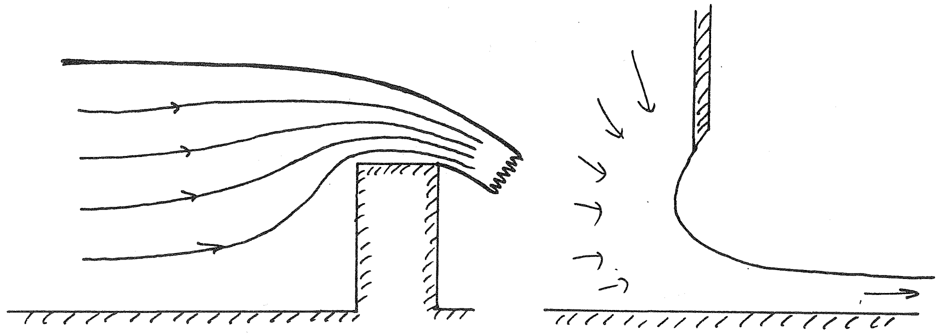


Figure 11

step function

$$\Theta = \begin{cases} -\pi/2, & 0 < \phi < 1 \\ 0, & \phi > 1 \end{cases}$$

corresponding to a simple bluff bow as in Figure 12a; so the flow sketched in Figure 12a cannot occur. On the other hand, there are other choices for $\Theta(\phi)$, where the solution does exist. Madurasinghe and Tuck [14] found solutions with tangential contact between body and free surface as in Figure 12b, and Madurasinghe ([13]; see also Tuck and Vanden-Broeck [23]) found solutions as in Figure 12c with a stagnation point at attachment. The latter class of bodies tends to be bulbous.

For any given body, these solutions exist only at specified discrete Froude numbers. One way to view this problem is by a flow reversal argument. That is, since the free-surface nonlinearity is quadratic, the flow direction can be reversed without changing the solution. For any given semi-infinite body, we therefore could be computing the flow behind a representation of a ship's stern instead of a ship's bow. That stern flow will in general possess waves as in Figure 13 of amplitude $A(F)$ far behind the body. For some (but clearly not all) bodies, $A(F)$ will vanish at some Froude numbers $F = F_n$, and

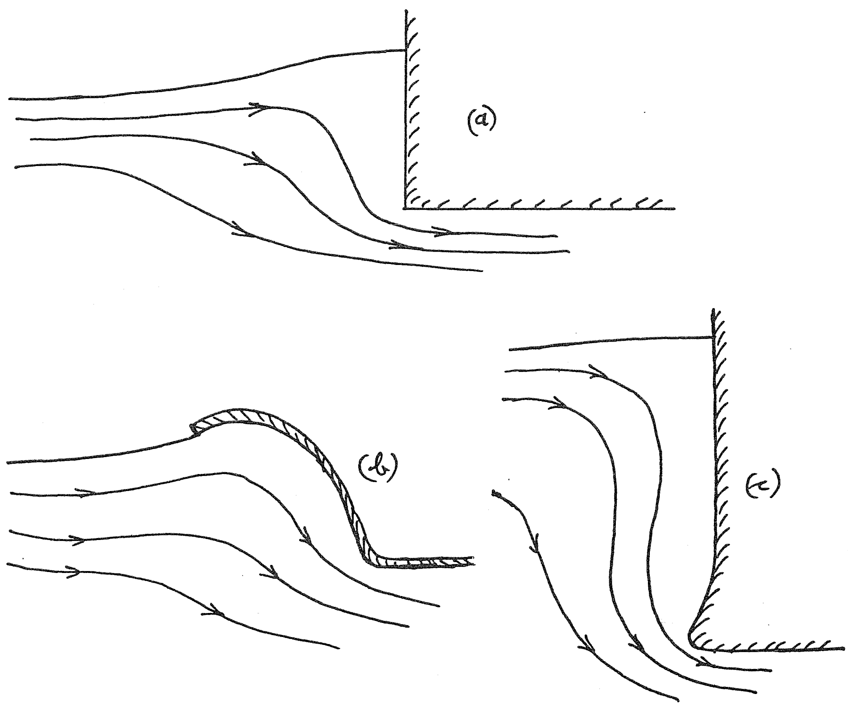


Figure 12

we may expect that then there exists an infinite number of such $F_n, n = 1, 2, \ldots$, with $F_n \to 0$ as $n \to \infty$. (The flow without waves certainly exists at zero Froude number, i.e. at infinite gravity, or with a rigid free surface, for any choice of body shape). Now if we have a stern-like solution without downstream waves, we can reverse its direction to generate a smooth bow flow without upstream waves, as required.

The remaining question is, even given that a smooth bow flow is achievable with one of the above special bow shapes at a specific Froude number $F = F_n$, what happens at other Froude numbers? And what happens to other bow shapes, that do not possess smooth solutions at any Froude number? This is not a question with a certain answer at this time. However, the answer seems to have something to do with splashing, or the formation of bow jets and bow wakes.

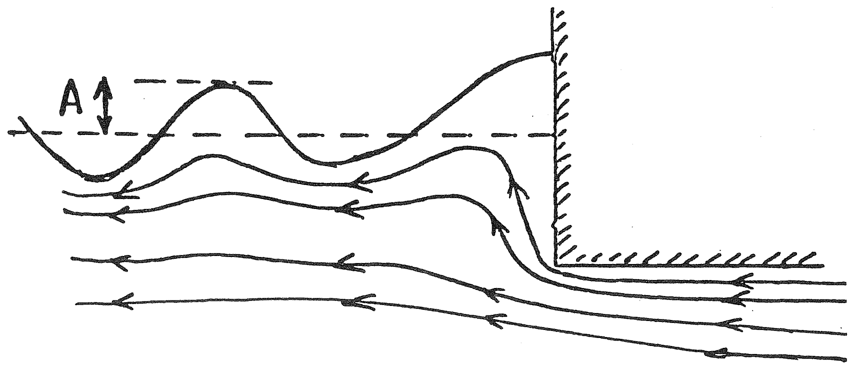


Figure 13

I am at the present moment actively pursuing the numerical and asymptotic solution of the Nekrasov integral equation, on the assumption (already suggested in Vanden-Broeck and Tuck [25]) that the flow topology is as sketched in Figure 14. This flow involves formation of a jet, say of thickness $J = J(F)$. If one could compute this function $J(F)$ for any given body, this would be a considerable achievement. For bodies like the simple right-angle bow, that do not permit any smooth bow flow, one would expect that $J(F)$ never vanishes for any $F > 0$. But if one is working with a bulbous body shape, one would expect that $J(F) = 0$ at some $F = F_n$.

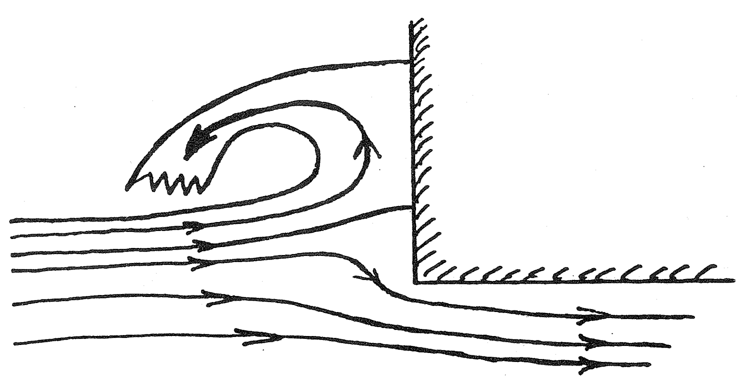


Figure 14

There are considerable conceptual and numerical difficuties with this jet-like problem. In the first place, the jet as sketched in Figure 14 would fall back upon the incident flow. Mathematically, the jet can be forced onto a second Riemann sheet, so passing "through" the incident flow without disturbing it. However, even with this interpretation, no-one has yet been successful in computation of a jet that falls backward from a stagnation point, as sketched in Figure 14, and it is not certain at this time that any such steady irrotational flow exists. There are other possibilities to avoid this difficulty, all along the lines of "catching" the splash before it falls, e.g. as sketched in Figure 15, and it is these models that are being pursued, especially Figure 15b.

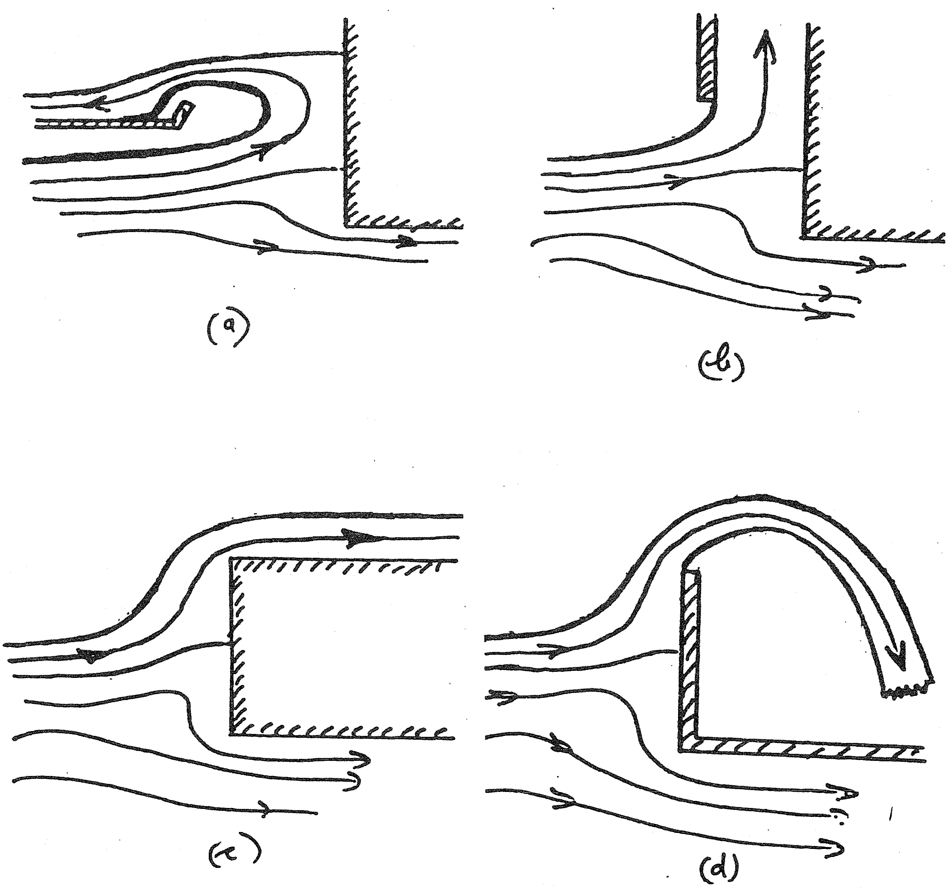


Figure 15

The other, more practical but numerically more daunting possibility, is to build in a model of what might actually happen when the splash hits the incident flow. In practice, there is a "mess" which can be idealised to a region of non-zero vorticity, or a bow wake (Mori [17]), perhaps as sketched in Figure 16. In a truly steady two-dimensional inviscid flow, Batchelor [2] has shown that such a wake must possess constant vorticity ω, and there has been recent progress (Moore et al [16]) in computation of such Batchelor flows in other contexts. If one could devise a program to compute such a flow, the output vorticity $\omega = \omega(F)$ would have a similar character to the quantities $A = A(F)$ or $J = J(F)$, namely would vanish (together with the whole bow wake) for those bulbous shapes allowing smooth flow, at special discrete Froude numbers $F = F_n$. For other bodies, or at other Froude numbers, the information provided from this computation would be of value in estimating drag contributions from the splash (Baba [1]) and other flow properties near the bow.

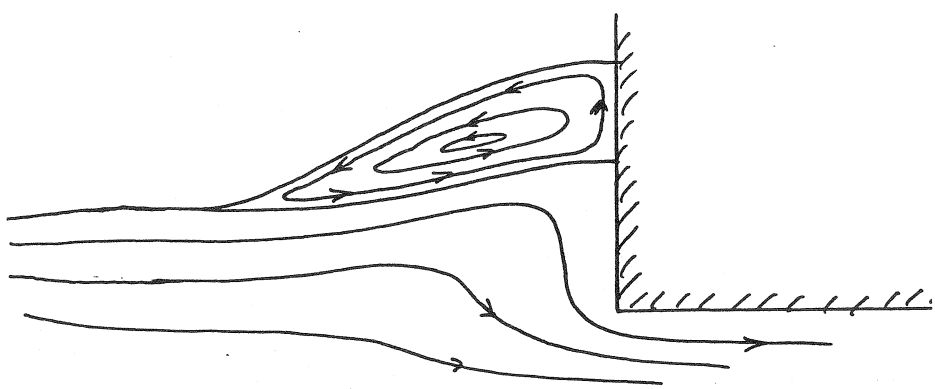


Figure 16

Another conceptual difficulty with this class of splashing flows is associated with its singular character at infinity. The problem is quite like that for planing surfaces (Tuck [22]). For example, suppose we seek an asymptotic solution for large Froude number $F \to \infty$. At first sight, the formal limit is easy, and jet-like solutions to the problem without gravity have been available for some time (e.g. Oertel [18]). For any given body geometry, there is a one-parameter family of flows at $g = 0$, the parameter being the jet thickness J, only one member of which has a flat free surface at infinity at a finite height (the draft D) above the flat bottom of the given body. This special case (examples of which are shown in Figure 17) is such that the jet thickness is exactly equal to the draft, i.e. has $J = D$. Figure 17a is Oertel's [18] solution; Figure 17b gives exact computed streamlines for the $g = 0$ limit of Figure 15d.

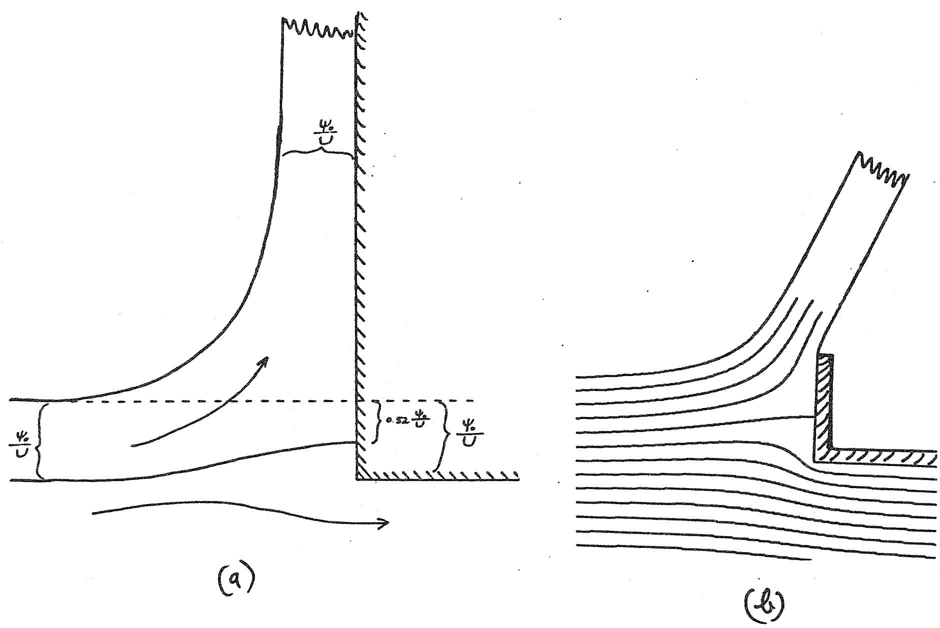


Figure 17

However, it is not certain that this is the appropriate special family member when g is not zero, and indeed one expects then to find $J < D$. The zero-g solutions do not decay sufficiently rapidly at a great distance, and must be interpreted as inner approximations. The outer approximation re-introduces gravity in the far field, albeit in a linearised form, and the asymptotic problem for large F is completed by matching these two approximations together. This has proved very difficult (Wu [29]) in the finite-length planing context with trailing waves, but the semi-infinite wave-less problem of interest here may prove more tractable. The singular-perturbation character of this problem is not just of relevance to large-F asymptotics, but is an indication of difficulties with the numerical solution at arbitrary F, and these difficulties have not yet been resolved.

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