

INDEFINITE INTEGRALS OF FUNCTIONS FROM $L^p(\mathbb{R})$

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ABSTRACT. In this paper we seek conditions under which the indefinite integrals of a function φ from $L^p(\mathbb{R})$ belong to $L^p(\mathbb{R}) + \mathbb{C}$. We prove that if the spectrum $sp(\varphi)$ of φ is isolated from zero, then it is improperly integrable for $(1 \leq p < \infty)$ and its indefinite integrals belong to $L^p(\mathbb{R}) + \mathbb{C}$. Also, we give applications to the differential equation $u'(x) + \lambda u(x) = \varphi(x)$.

§1. Introduction. Let $\varphi \in L^p(\mathbb{R})$ for some $1 \leq p \leq \infty$ and define

$$(1.1) \quad P\varphi(x) = \int_0^x \varphi(t) dt, \quad x \in \mathbb{R}.$$

We seek conditions under which $P\varphi \in L^p(\mathbb{R}) + \mathbb{C}$. Similar problems have been studied extensively when φ belongs instead to certain classes of functions of almost periodic type. (See [1], [3], [4], [5], [7], [8], [10], [13], [16].) In particular, let $AP(\mathbb{R})$ denote the Banach space of complex-valued almost periodic functions defined on \mathbb{R} . Bohl-Bohr [5, p.58] proved that if $\varphi \in AP(\mathbb{R})$ and $P\varphi$ is bounded then $P\varphi \in AP(\mathbb{R})$. More generally, let X be a Banach space and $AP(\mathbb{R}, X)$ the Banach space of X -valued almost periodic functions on \mathbb{R} . If $\varphi \in AP(\mathbb{R}, X)$ and $P\varphi$ is bounded then $P\varphi$ does not necessarily belong to $AP(\mathbb{R}, X)$. However, Kadets [10] showed that if X does not contain a subspace isomorphic to the Banach space c_0 then again $P\varphi \in AP(\mathbb{R}, X)$.

Now let $C_{ub}(\mathbb{R}, X)$ denote the space of uniformly continuous bounded functions from \mathbb{R} to X , and recall that a function $\varphi \in C_{ub}(\mathbb{R}, X)$ is called ergodic if there exists $a \in X$ such that $\| \lim_{T \rightarrow \infty} \sup_{x \in \mathbb{R}} \frac{1}{2T} \int_{-T}^T [\varphi(t+x) - a] dt \| = 0$. Also let Λ be a closed translation invariant subspace of $C_{ub}(\mathbb{R}, X)$. Basit [1] recently proved that if $\varphi \in \Lambda$ and if $P\varphi$ is

bounded and ergodic, then $P\varphi \in \Lambda$. This result is not true for $\Lambda = L^1(\mathbb{R}) + \mathbb{C}$. Indeed, consider the function defined by

$$(1.2) \quad \varphi(x) = x \text{ for } |x| \leq 1 \text{ and } \varphi(x) = \frac{\text{sign}(x)}{x^2} \text{ for } |x| > 1.$$

Then $\varphi \in L^1(\mathbb{R})$ and $P\varphi \in C_0(\mathbb{R}) + \mathbb{C}$. In particular $P\varphi$ is bounded and ergodic, yet $P\varphi \notin L^1(\mathbb{R}) + \mathbb{C}$.

In this paper we consider functions $\varphi \in L^p(\mathbb{R})$ and replace assumptions concerning $P\varphi$ by conditions on the spectrum of φ in order to conclude $P\varphi \in L^p(\mathbb{R}) + \mathbb{C}$. Spectra are defined in section 2 and the main result appears in section 3. In section 4 we discuss derivatives in place of indefinite integrals, and in section 5 we provide an application to differential equations.

§2. Spectra. Following Reiter[14,p.83] we call a function $w \in L_{loc}^\infty(\mathbb{R})$ a weight function on \mathbb{R} if

$$(2.1) \quad w(x) \geq 1 \text{ for all } x \in \mathbb{R}, \text{ and}$$

$$(2.2) \quad w(x+y) \leq w(x)w(y) \text{ for all } x, y \in \mathbb{R}.$$

A weight function is symmetric if

$$(2.3) \quad w(x) = w(-x) \text{ for all } x \in \mathbb{R}.$$

An important additional condition satisfied by many weights is the Beurling condition

$$(2.4) \quad \sum_{m=1}^{\infty} \frac{\log w(mx)}{m^2} < \infty \text{ for all } x \in \mathbb{R}.$$

Given a weight w on \mathbb{R} we define

$$(2.5) \quad L_w^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : \|f\|_{1,w} = \int_{\mathbb{R}} |f(x)|w(x) dx < \infty\}.$$

Then $L_w^1(\mathbb{R})$ is a subalgebra of $L^1(\mathbb{R})$ which is a Banach algebra under the norm $\|\cdot\|_{1,w}$.

The Banach space dual of $L_w^1(\mathbb{R})$ is

$$(2.6) \quad L_w^\infty(\mathbb{R}) = \{\varphi \in L_{loc}^\infty(\mathbb{R}) : \|\varphi\|_{\infty,w} = \text{ess sup}_{x \in \mathbb{R}} \frac{|\varphi(x)|}{w(x)} < \infty\}.$$

If w satisfies the Beurling condition, then the space of Fourier transforms \hat{f} of functions

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$f \in L_w^1(\mathbb{R})$ is a Wiener algebra. See Reiter [14, p.132 and p.19 remark]. In particular :

Lemma 2.1. *Let w be a weight function on \mathbb{R} satisfying the Beurling condition. Given a neighbourhood V of a compact set W in \mathbb{R} , there exists $f \in L_w^1(\mathbb{R})$ such that $\hat{f} = 1$ on W and $\text{supp } \hat{f} \subset V$.*

If w is a symmetric weight function on \mathbb{R} , then for $f \in L_w^1(\mathbb{R})$ and $\varphi \in L_w^\infty(\mathbb{R})$, the convolution

$$(2.7) \quad f * \varphi(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t) dt$$

is defined for almost every $x \in \mathbb{R}$ and $|f * \varphi(x)| \leq w(x)\|f\|_{1,w}\|\varphi\|_{\infty,w}$, a.e. For such w and φ we define a closed ideal of $L_w^1(\mathbb{R})$ by

$$(2.8) \quad I_w(\varphi) = \{f \in L_w^1(\mathbb{R}) : f * \varphi = 0, \text{ (a.e.) } \}$$

and the w -spectrum of φ by

$$(2.9) \quad sp_w(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0 \text{ for all } f \in I_w(\varphi)\}.$$

Since $\hat{f} \in C_0(\mathbb{R})$ for each $f \in L^1(\mathbb{R})$, the w -spectrum $sp_w(\varphi)$ is closed. For a list of further properties, see [2].

We also require a notion of spectrum for functions $\varphi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$. For such φ , $f * \varphi \in L^p(\mathbb{R})$ for all $f \in L^1(\mathbb{R})$ (see [9, corollary 20.14]) and hence we can define

$$(2.10) \quad I(\varphi) = \{f \in L^1(\mathbb{R}) : f * \varphi = 0\}$$

and the spectrum of φ by

$$(2.11) \quad sp(\varphi) = \{\lambda \in \mathbb{R} : \hat{f}(\lambda) = 0 \text{ for all } f \in I(\varphi)\}.$$

Once again $sp(\varphi)$ is a closed subset of \mathbb{R} .

Proposition 2.2. *Let $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$.*

(a) *If $f \in L^1(\mathbb{R})$ then $f * \varphi \in L^p(\mathbb{R})$ and $sp(f * \varphi) \subset \text{supp } \hat{f} \cap sp(\varphi)$.*

(b) *$sp(\varphi) = \emptyset$ if and only if $\varphi = 0$.*

Proof. (a) By [9, corollary 20.14] we have $f * \varphi \in L^p(\mathbb{R})$. Clearly $I(\varphi) \subset I(f * \varphi)$ and so

$sp(f * \varphi) \subset sp(\varphi)$. Finally, suppose $\lambda \in \mathbb{R} \setminus \text{supp } \hat{f}$. By lemma 2.1, with $w = 1$, there exists $g \in L^1(\mathbb{R})$ such that $\hat{g} = 0$ on a neighbourhood of $\text{supp } \hat{f}$ and $\hat{g}(\lambda) = 1$. As $\hat{g}\hat{f} = 0$, so $f * g = 0$ and therefore $g \in I(f * \varphi)$. But $\hat{g}(\lambda) \neq 0$ so $\lambda \notin sp(f * \varphi)$.

(b) If $\varphi = 0$ then $I(\varphi) = L^1(\mathbb{R})$ and $sp(\varphi) = \emptyset$. Conversely, if $sp(\varphi) = \emptyset$ then $f * \varphi = 0$ for all $f \in L^1(\mathbb{R})$. If $p = 1$, $f = \varphi$ gives $\hat{\varphi}^2 = 0$ and hence $\varphi = 0$. If $1 < p \leq \infty$ then $f * \varphi = 0$ for all $f \in C_c(\mathbb{R})$, the space of continuous functions on \mathbb{R} with compact support. Since $C_c(\mathbb{R})$ is dense in $L^q(\mathbb{R})$, where $1/p + 1/q = 1$, and the mapping $f \rightarrow f * \varphi$ is continuous from $L^q(\mathbb{R})$ to $C_0(\mathbb{R})$, we conclude $f * \varphi = 0$ for all $f \in L^q(\mathbb{R})$. So $\int_{-\infty}^{\infty} f(x-t)\varphi(t) dt = 0$ for all $x \in \mathbb{R}$ and $f \in L^q(\mathbb{R})$. Taking $x = 0$ and applying the Hahn-Banach theorem we conclude that $\varphi = 0$.

Let $S(\mathbb{R})$ be the Schwartz space of rapidly decreasing infinitely differentiable complex-valued functions on \mathbb{R} . Let $S'(\mathbb{R})$ be the dual space of tempered distributions. If $\varphi \in L^p(\mathbb{R})$, then $T_\varphi(f) = \int_{\mathbb{R}} f(t)\varphi(t) dt$ for $f \in S(\mathbb{R})$, defines a distribution $T_\varphi \in S'(\mathbb{R})$. So $\hat{T}_\varphi(g) = T_\varphi(\hat{g})$ for $g \in S(\mathbb{R})$, defines the Fourier transform \hat{T}_φ of T_φ . (See [17, p.146-152]).

Proposition 2.3. *Let $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$. Then $sp(\varphi) = \text{supp } \hat{T}_\varphi$.*

The proof is essentially the same as for [2, proposition 4.1]

§3. Indefinite integrals. Let $C_u(\mathbb{R})$ and $C_{ub}(\mathbb{R})$ denote respectively the spaces of uniformly continuous and uniformly continuous bounded functions on \mathbb{R} . To study indefinite integrals, we use the weight

$$(3.1) \quad w(x) = 1 + |x|, \quad x \in \mathbb{R}.$$

It is readily seen that w is a symmetric weight function satisfying condition (2.4).

Proposition 3.1. *If $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$ and w is given by (3.1), then $P\varphi \in C_u(\mathbb{R}) \cap L_w^\infty(\mathbb{R})$. Moreover,*

$$(3.2) \quad sp(\varphi) \subset sp_w(P\varphi) \subset sp(\varphi) \cup \{0\}.$$

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Proof. If $p = 1$, it is well-known that $P\varphi$ is absolutely continuous and hence uniformly continuous. For arbitrary p , and $x, h \in \mathbb{R}$, $|P\varphi(x+h) - P\varphi(x)| = |\int_0^h \varphi(x+t) dt| \leq |h|^{1-1/p} \|\varphi\|_p$ showing that $P\varphi \in C_u(\mathbb{R})$. Moreover, $|P\varphi(x)| = |\int_0^x \varphi(t) dt| \leq |x|^{1-1/p} \|\varphi\|_p$, showing that $P\varphi \in L_w^\infty(\mathbb{R})$. For $p = \infty$, (3.2) is given in [2, proposition 4.4]. The proof of (3.2) for other p is essentially the same.

Proposition 3.2. *If $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$ and $0 \notin sp(\varphi)$, then $P\varphi \in C_{ub}(\mathbb{R})$.*

Proof. Since $0 \notin sp(\varphi)$ there exists a neighbourhood $V = [-\delta, \delta]$ such that $sp_w(\varphi) \cap V = \emptyset$. Let $w(x) = 1 + |x|$. By lemma 2.1 there is a function $h \in L_w^1(\mathbb{R})$ such that $\hat{h} = 1$ for $|\lambda| \leq \delta/4$ and $\hat{h} = 0$ for $|\lambda| \geq \delta/3$. By proposition 2.2, $h * \varphi = 0$. Similarly, by [2, proposition 3.12] and (3.2) $sp_w(h * P\varphi) \subset \text{supp } \hat{h} \cap sp_w(P\varphi) \subset \{0\}$. Since $\frac{d(h * P\varphi)}{dx} = h * \varphi = 0$ for all $x \in \mathbb{R}$, we conclude that $h * P\varphi = c$, a constant. If $\eta = P\varphi - c$ then $0 \notin sp_w(\eta)$. Indeed, $h * \eta = h * P\varphi - h * c = c - c = 0$. Thus $h \in I_w(\eta)$ and $\hat{h}(0) = 1$, showing $0 \notin sp_w(\eta)$. By proposition 3.1, $\eta \in C_u(\mathbb{R})$ and so by [2, theorem 9.5], η is bounded and so is $P\varphi$. This proves that $P\varphi \in C_{ub}(\mathbb{R})$.

Proposition 3.3. *If $\varphi \in L^p(\mathbb{R}) \cap C_u(\mathbb{R})$ where $1 \leq p \leq \infty$, then $\varphi \in C_0(\mathbb{R})$.*

Proof. Assume on the contrary that $\limsup_{t \rightarrow \infty} |\varphi(t)| \geq 3c > 0$. Choose a sequence $\{t_n\} \subset \mathbb{R}$ such that $t_{n+1} > 2 + t_n$ and $|\varphi(t_n)| \geq 2c$ for all $n \in \mathbb{N}$. Since φ is uniformly continuous, there exists $0 < \delta < 1$ such that $|\varphi(t)| \geq c$ whenever $|t_n - t| \leq \delta$ for some $n \in \mathbb{N}$. Hence $\int_{-\infty}^{\infty} |\varphi(t)|^p dt \geq \lim_{n \rightarrow \infty} 2nc^p \delta = \infty$, a contradiction.

Theorem 3.4. *If $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$ and $0 \notin sp(\varphi)$, then $a + P\varphi \in L^p(\mathbb{R})$ for some $a \in \mathbb{C}$. If moreover $p < \infty$ then φ is improperly integrable and $a = -\lim_{|T| \rightarrow \infty} \int_0^T \varphi(t) dt$.*

Proof. For $p = \infty$ the result is contained in proposition 3.2. So assume $1 \leq p < \infty$. Let

$\alpha_h(x) = P\varphi(x+h) - P\varphi(x) = \int_0^h \varphi(x+t) dt = \chi_{-h} * \varphi(x)$, where χ_{-h} is the characteristic function of the interval $[-h,0]$ if $h \geq 0$, $[0,-h]$ if $h < 0$. As each $\alpha_h \in L^p(\mathbb{R})$, we may define the function $\alpha(h) = \|\alpha_h\|_p$, $h \in \mathbb{R}$. It is easy to verify that α is a continuous function on \mathbb{R} satisfying the property $\alpha(h_1 + h_2) \leq \alpha(h_1) + \alpha(h_2)$ for all $h_1, h_2 \in \mathbb{R}$. Therefore $\omega(h) = 1 + \alpha(h) + \alpha(-h)$, $h \in \mathbb{R}$ defines a symmetric weight function satisfying the Beurling condition (2.4). Choose $\delta > 0$ such that $[-\delta, \delta] \cap sp(\varphi) = \emptyset$. By lemma 2.1, there exists $f \in L^1_w(\mathbb{R})$ such that $\hat{f}(\lambda) = 1$ for $|\lambda| \leq \delta/4$ and $\hat{f}(\lambda) = 0$ for $|\lambda| \geq \delta/3$. By proposition 3.2, $P\varphi \in C_{ub}(\mathbb{R})$ and so $f * P\varphi$ is defined and also belongs to $C_{ub}(\mathbb{R})$. Moreover, $\frac{d(f * P\varphi)}{dx} = f * \varphi = 0$, so $f * P\varphi = -a$ where $a \in \mathbb{C}$.

Next consider $a + P\varphi(x) = \int_{-\infty}^{\infty} [P\varphi(x) - P\varphi(x-t)]f(t) dt = -\int_{-\infty}^{\infty} \alpha_{-t}(x)f(t) dt$. We have $\|\alpha_{-t}\|_p = \alpha(-t) \leq \omega(t)$ and since $f \in L^1_w(\mathbb{R})$, $w|f| \in L^1(\mathbb{R})$. The function $t \rightarrow \psi(t) = \alpha_{-t}f(t) : \mathbb{R} \rightarrow L^p(\mathbb{R})$ is weakly measurable and its range is separable, as $1 \leq p < \infty$. Hence ψ is strongly measurable ([17, p.131]). As the function $t \rightarrow \|\psi(t)\|_p$ is integrable, Bochner's theorem [17, p.133] yields that ψ is Lebesgue-Bochner integrable and its integral is an element of $L^p(\mathbb{R})$. So $a + P\varphi \in L^p(\mathbb{R})$. Finally, by proposition 3.3, $a + P\varphi \in L^p(\mathbb{R}) \cap C_{ub}(\mathbb{R}) \subset C_0(\mathbb{R})$. Hence $\lim_{|T| \rightarrow \infty} \int_0^T \varphi(t) dt = -a$.

Corollary 3.5. *Let $\varphi \in L^p(\mathbb{R})$. Then $f * P\varphi \in L^p(\mathbb{R}) + \mathbb{C}$ for each $f \in L^1(\mathbb{R})$ with $0 \notin \text{supp } \hat{f} = sp(f)$.*

Proof. By theorem 3.4 the function Kf defined by $Kf(x) = \int_{-\infty}^x f(t) dt$, belongs to $L^1(\mathbb{R})$. By [9, corollary 20.14], we conclude $\varphi * Kf \in L^p(\mathbb{R})$. Since $(f * P\varphi)' = (Kf * \varphi)'$, there exists $a \in \mathbb{C}$ such that $f * P\varphi = a + Kf * \varphi \in L^p(\mathbb{R}) + \mathbb{C}$.

Remark 3.6. *Let X be a Banach space and $L^1(\mathbb{R}, X)$ the space of Lebesgue-Bochner integrable X -valued functions on \mathbb{R} . Define $L^p(\mathbb{R}, X)$ similarly for $1 < p \leq \infty$. Then $sp(\varphi)$ for $\varphi \in L^p(X, \mathbb{R})$ is again defined by (2.10) and (2.11). Theorem 3.4 remains true in this*

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more general setting.

§4. Derivatives. In this section we briefly consider derivatives in place of indefinite integrals.

Theorem 4.1. *Let $\varphi \in L^1(\mathbb{R})$. If $\varphi' \in L^\infty(\mathbb{R})$ then φ' is improperly integrable, and if $\varphi' \in C_{ub}(\mathbb{R})$ then $\varphi' \in C_0(\mathbb{R})$.*

Proof. Of course $\varphi(x) = \varphi(0) + \int_0^x \varphi'(t) dt$. If $\varphi' \in L^\infty(\mathbb{R})$ then $\varphi \in C_{ub}(\mathbb{R})$ and by proposition 3.3, $\varphi \in C_0(\mathbb{R})$. Hence $\lim_{|x| \rightarrow \infty} \int_0^x \varphi'(t) dt = -\varphi(0)$. If $\varphi' \in C_{ub}(\mathbb{R})$, then $n[\varphi(x + 1/n) - \varphi(x)] = n \int_0^{1/n} \varphi'(t + x) dt = \varphi'(x + \theta/n)$ for some $\theta = \theta(x, n)$, $0 < \theta < 1$. Hence $\lim_{|x| \rightarrow \infty} |\varphi'(x)| = 0$.

Remark 4.2. *It can happen that $\varphi \in L^1(\mathbb{R})$ and $\varphi' \in C_{ub}(\mathbb{R})$ yet $\varphi' \notin L^1(\mathbb{R})$. For example, let $\varphi(x) = \sum_{n=4}^{\infty} n[(x-n)^2 - 1/n]^2 g_n(x)$, where g_n is the characteristic function of the interval $I_n = [n - 1/n^{1/2}, n + 1/n^{1/2}]$. Then $\varphi, \varphi' \in C_0(\mathbb{R})$ with $\varphi \in L^1(\mathbb{R})$ and $\varphi' \notin L^1(\mathbb{R})$.*

§5. Application to a differential equation. Consider the following differential equation.

$$(5.1) \quad u'(x) + \lambda u(x) = \varphi(x), \quad x \in \mathbb{R}.$$

Given $\lambda \in \mathbb{C}$ and $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$, we seek solutions $u \in L^p(\mathbb{R})$. The general solution of (5.1) is

$$(5.2) \quad u(x) = e^{-\lambda x} [c + \int_0^x e^{\lambda t} \varphi(t) dt],$$

where c is a constant. When $\operatorname{Re}(\lambda) \neq 0$ it is easy to see that (5.1) has a unique solution $u \in L^p(\mathbb{R})$ given by

$$(5.3) \quad u(x) = \int_{-\infty}^x e^{-\lambda(x-t)} \varphi(t) dt = g_\lambda * \varphi(x) \text{ if } \operatorname{Re}(\lambda) > 0,$$

$$(5.4) \quad u(x) = - \int_x^{\infty} e^{-\lambda(x-t)} \varphi(t) dt = h_\lambda * \varphi(x) \text{ if } \operatorname{Re}(\lambda) < 0.$$

Here, $g_\lambda(x) = e^{-\lambda x} \chi_+(x)$ and $h_\lambda(x) = -e^{-\lambda x} (1 - \chi_+(x))$ where χ_+ is the characteristic function of the interval $[0, \infty[$. The case $\mathcal{R}e(\lambda) = 0$ is more delicate.

Theorem 5.1. *Suppose $\mathcal{R}e(\lambda) = 0$, $\varphi \in L^p(\mathbb{R})$ and $i\lambda \notin sp(\varphi)$. If $1 \leq p < \infty$ then (5.1) has a unique solution $u \in L^p(\mathbb{R})$ given by*

$$(5.5) \quad u(x) = \lim_{T \rightarrow \infty} \int_{-T}^x e^{-\lambda(x-t)} \varphi(t) dt.$$

If $p = \infty$ then (5.1) has infinitely many solutions $u \in L^\infty(\mathbb{R})$ given by (5.2).

Proof. Let $\psi(x) = e^{\lambda x} \varphi(x)$ and $v(x) = e^{\lambda x} u(x)$. Then u is a solution in $L^p(\mathbb{R})$ of (5.1) if and only if v is a solution in $L^p(\mathbb{R})$ of

$$(5.6) \quad v'(x) = \psi(x), \quad x \in \mathbb{R}.$$

Further, $sp(\psi) = -i\lambda + sp(\varphi)$, so $0 \notin sp(\psi)$. If $1 \leq p < \infty$, then by theorem 3.4, the equation (5.6) has a unique solution $v \in L^p(\mathbb{R})$ given by $v(x) = \lim_{T \rightarrow \infty} \int_{-T}^x \psi(t) dt$. If $p = \infty$, the same theorem shows $v(x) = c + \int_0^x \psi(t) dt$ defines a bounded solution of (5.6) for each constant c .

Remark 5.2. *If $\varphi \in L^p(\mathbb{R})$ where $1 \leq p \leq \infty$ then $sp(\varphi) \subset \mathbb{R}$. Hence if $\mathcal{R}e(\lambda) \neq 0$, then $i\lambda \notin sp(\varphi)$. On the other hand if $\mathcal{R}e(\lambda) = 0$ and $i\lambda \in sp(\varphi)$ then (5.1) may have no solution $u \in L^p(\mathbb{R})$. For example, if $\lambda = 0$ and φ is a non-zero constant, then (5.1) has no solution $u \in L^\infty(\mathbb{R})$. Again, if $\lambda = 0$ and φ is defined by (1.2), then (5.1) has no solution $u \in L^1(\mathbb{R})$.*

REFERENCES

1. B. Basit, *Some problems concerning different types of vector valued almost periodic function*, Analysis Paper 85, Department of Math., Monash Univ. (1992), 1- 31.
2. B. Basit and A.J. Pryde, *Spectra of unbounded functions on locally compact abelian groups*, Analysis paper 88, Department of Math., Monash Univ. (1993), 1-32.
3. B. Basit and C. Zhang, *Almost periodic type functions, spectral criteria and solutions of differential equations*, Analysis paper 89, Department of Math., Monash Univ. (1993), 1- 35.
4. B. Basit, *On the indefinite integrals of abstract functions*, Analele Stiintifice Ale Univ. "AL.I.CUZA" Din Iasi 29 (3), (1983), 49-54.
5. H. Bohr, *Almost periodic functions*, Chelsea Pub. Company, New York, 1951.

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6. N.Dunford and J.T.Schwartz, *Linear Operators, Part II, Interscience Pub., New York, London, 1963.*
7. J.Favard, *Sur les equations differentielles à coefficients presque-périodiques, Acta Math. 51 (1927), 31-81.*
8. M.Fréchet, *Les fonctions asymptotiquement presque- périodique continue, C.R.Acad.Paris 213 (1941), 520-522.*
9. E.Hewitt and K.A. Ross, *Abstract harmonic analysis I, Springer Verlag, 1979.*
10. M.I.Kadets, *On the integration of almost periodic functions with values in Banach Spaces, Functional Anal. Appl. 3 (1969), 228-230.*
11. Y.Katznelson, *An introduction to Harmonic Analysis, John Wiley & Sons, Inc., 1968.*
12. B.M.Levitan and V.V.Zhikov, *Almost Periodic Functions and Differential Equations, Cambridge Univ. Press, Cambridge, 1982.*
13. B.M.Levitan, *Integration of almost periodic functions with values in Banach spaces, Math. of the USSR, Izvestija 30 (1966).*
14. H.Reiter, *Classical Harmonic Analysis and Locally Compact Groups, Oxford Math. Monographs, Oxford Univ., 1968.*
15. W.Rudin, *Fourier Analysis on Groups, Interscience, New York, London, 1962.*
16. W.M.Ruess and W.H.Summer, *Integration of asymptotically almost periodic functions and weak asymptotic almost periodicity, Diss. Math.(1989).*
17. K.Yosida, *Functional Analysis, Springer Verlag, 1965.*

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The first part of the document discusses the importance of maintaining accurate records of all transactions. It emphasizes that every entry should be supported by a valid receipt or invoice. This ensures transparency and allows for easy verification of the data.

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The second part of the document provides a detailed breakdown of the monthly financial statements. It includes a summary of the income generated from various sources, such as sales and services. It also details the expenses incurred, categorized by department or project.

Finally, the document concludes with a statement of the net profit or loss for the period. It highlights the overall financial health of the organization and provides recommendations for future improvements. The goal is to ensure that the organization remains profitable and sustainable in the long run.

The following table shows the monthly financial performance over the last six months.