Multivalued spatial discretization of dynamical systems *

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Introduction

Suppose that a mapping $f: \Omega \mapsto \Omega$, on a compact metric space Ω, generates a discrete dynamical system $\{f^n(x) : n = 0, 1, 2, \ldots\}$ with chaotic behavior. For brevity we will refer to the system f. Standard computer models of this system are dynamical systems ψ defined on a finite subset $\mathbf{L}$ of Ω. The set of all trajectories of an individual system ψ can differ dramatically from that of the original system even for a fine discretization. For instance, the mapping $f(x) = 2x \,(\bmod 1)$ is chaotic with a unique absolutely continuous invariant measure and cycles of all periods. Yet every N–digital binary discretization ψ_N, defined as the restriction of f to the set $\{i/2^N : i = 1, \ldots, 2^{N-1}\}$ is asymptotically trivial, $\psi_N^k \equiv 0$, $k \geq N$. Such effects are an inevitable consequence of discretization in the sense that there always exists some discretization which collapses a given system f onto a given f–invariant set, in particular onto a fixed point or cycle [6].

Degenerate, collapsing behavior such as this can be eliminated by instead modelling with systems φ which can be regarded as either stochastic or multivalued perturbations of the original system f. However, the choice of an appropriate model system φ introduces a conundrum which frequently arises in the theory of ill–posed problems. If the perturbation that is introduced is too large, then the behaviour of the system φ, while not degenerate can differ markedly from f. On the other hand, if the perturbation is not strong enough, collapsing effects will not be avoided. Consequently, questions about the robustness of systems to various levels of stochastic or multivalued perturbation are very important.

Fundamental theoretical results concerning stochastic perturbations can be found in [11, 1, 2]. Approaches using multivalued systems are much less investigated. However, this approach seems to be efficient in investigating systems with fast changing and discontinuous characteristics. Often the most adequate mathematical descriptions of

*This research has been supported by the Australian Research Council Grant A 89132609.

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such systems include multivalued operators. For this reason we will consider dynamical systems generated by multivalued mappings $f: \ \Omega \mapsto 2^{\Omega}$.

1 Consistency of multivalued discretizations

Throughout, (Ω, ρ) denotes a compact metric space, $\mathcal{B}$ the totality of Borel subsets of Ω and $\mathcal{O}_\varepsilon(X)$ the ε-neighborhood of X for any $X \in \mathcal{B}$. Denote by $\text{Sep}(y, X) = \inf_{x\in X} \rho(y, x)$ the *separation* of an element $x \in \Omega$ from the set $X \subseteq \Omega$. and denote by $\text{Sep}(Y, X)$, $Y, X \subseteq \Omega$, the *Hausdorff separation* of Y from X: $\text{Sep}(Y, X) = \sup_{y\in Y} \text{Sep}(y, X)$. The same symbol Sep will be used for the Hausdorff separation between sets belonging to the Cartesian product $\Omega \times \ldots \times \Omega$ with the metric

$$\rho_N((x_0, x_1, \ldots x_N), (y_0, y_1, \ldots y_N)) = \max_{0\le n\le N} \rho(x_n, y_n).$$

Below, a dynamical system is generated by a Borel mapping $f: \ \Omega \mapsto \mathcal{B}$. That is, for any $X \in \mathcal{B}$, $\{x \in \Omega : f(x) \bigcap X \neq \emptyset\} \in \mathcal{B}$. By $\text{Gr}(f)$ is meant the graph of f. Suppose that $\mathbf{L}$ is a finite subset of Ω and consider a map $\varphi: \ \mathbf{L} \mapsto 2^{\mathbf{L}}$ with the graph $\text{Gr}(\varphi) \subseteq \mathbf{L} \times \mathbf{L} \subseteq \Omega \times \Omega$ as a discretization of the system f. An estimate of the accuracy of such a discretization φ is provided by the two quantities

$$d(\varphi, f) = \text{Sep}(\text{Gr}(\varphi), \text{Gr}(f)) \qquad \text{and} \qquad d(f, \varphi) = \text{Sep}(\text{Gr}(f), \text{Gr}(\varphi)). \tag{1}$$

Note that d is not a metric.

The mapping $\varphi: \ \mathbf{L} \mapsto 2^{\mathbf{L}}$ will be called an *α–consistent discretization of f* if at least one of the following two conditions hold:

C1. There exists a cover $X(\xi)$, $\xi \in \mathbf{L}$ of Ω with $X(\xi) \subseteq \mathcal{O}_\alpha(\xi)$, $\xi \in \mathbf{L}$ and

$$f(X(\xi)) \subseteq X(\varphi(\xi)).$$

C2. There exist nonempty sets $\Xi(x)$, $x \in \Omega$ of $\mathbf{L}$ with $\Xi(x) \subseteq \mathcal{O}_\alpha(x)$ and

$$\varphi(\Xi(x)) \supseteq \Xi(f(x)).$$

Consider some simple examples of α–consistent discretizations. For a given subset $\mathbf{L}$ of Ω the number $h(\mathbf{L}) = \sup_x \inf_\xi \{\rho(\xi, x) : \xi \in \mathbf{L}, \ x \in \Omega\}$ will be called *the step* of $\mathbf{L}$.

Example 1. *For any $\xi \in \mathbf{L}$ denote by $X(\xi)$ the subset of Ω defined by*

$$X(\xi) = \left\{x : \ \rho(x, \xi) = \min_{\eta\in\mathbf{L}} \rho(x, \eta)\right\}$$

and for any subset $\Omega_* \subseteq \Omega$ denote $\Xi(\Omega_*) = \{\xi : \ X(\xi) \bigcap \Omega_* \neq \emptyset\}$. Then the multivalued mapping $\varphi: \ \mathbf{L} \mapsto 2^{\mathbf{L}}$ defined by $\varphi(\xi) = \Xi(f(X(\xi))$, $\xi \in \mathbf{L}$, will be an $h(\mathbf{L})$–consistent discretization of f with $d(\varphi, f) \le h(\mathbf{L})$. In this example the condition C1 *is fulfilled.*

Example 2. *Denote for any $\alpha > h(\mathbf{L})$, $A_\alpha = \mathcal{O}_\alpha(Gr(f)\bigcap \mathbf{L}\times\mathbf{L})$ and consider the multivalued mapping $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ with the graph $\mathcal{A}_\alpha$. Then this map is a $h(\mathbf{L})$-consistent discretization of f with $d(\varphi, f) \le \alpha$. In this example the condition* C1 *holds. with the same cover as in the previous example.*

Example 3. *Let ψ be a given mapping $\mathbf{L} \mapsto 2^{\mathbf{L}}$ with $d(f,\psi) \le \alpha$. Then the multivalued mapping $\varphi(\xi) = \mathcal{O}_{2\alpha}(\psi(\xi))\bigcap \mathbf{L}$ will be an α–consistent discretization of f with $d(\varphi, f) \le 3\alpha$. In this example the condition* C2 *is fulfilled with $\Xi(x) = \mathcal{O}_\alpha(x)\bigcap \mathbf{L}$.*

Denote by $\mathrm{C}(\varphi, f)$ the greatest lower bound of those α for which φ is an α–consistent discretization of f.

2 Approximation of individual trajectories

A sequence

$$\mathbf{x} = x_0, x_1, \ldots, x_N, \quad \text{with} \quad x_{n+1} \in f(x_n) \tag{2}$$

is called a *trajectory* of the system f. Consider a map $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ with the graph $\mathrm{Gr}(\varphi) \subseteq \mathbf{L}\times\mathbf{L} \subseteq \Omega\times\Omega$ as a discretization of the system f. A sequence

$$\boldsymbol{\xi} = \xi_0, \xi_1, \ldots, \xi_N, \qquad \text{with} \quad \xi_{n+1} \in \varphi(\xi_n) \ \text{ and } \ \xi_n \in \mathbf{L} \tag{3}$$

is called a *trajectory* of φ.

Lemma 1. *Let $\mathbf{x} = x_0, x_1, \ldots, x_N$, $x_n \in \Omega$ be a trajectory of the system f. Then there exists a trajectory $\boldsymbol{\xi} = \xi_0, \xi_1, \ldots, \xi_N$ of the system φ satisfying $\rho(\xi_n, x_n) \le \mathrm{C}(\varphi, f)$, $n = 0, 1, \ldots N$.*

Proof: First of all, construct for any $\delta > 0$ a trajectory $\boldsymbol{\xi}(\delta) = \xi_0(\delta), \xi_1(\delta), \ldots, \xi_N(\delta)$ of φ such that

$$\rho(\xi_n(\delta), x_n) \le \mathrm{C}(\varphi, f) + \delta. \tag{4}$$

By definition of $\mathrm{C}(\varphi, f)$ the mapping φ is an α–consistent discretization of f with

$$\alpha = \mathrm{C}(\varphi, f) + \delta. \tag{5}$$

First, consider the case when the condition C1 is fulfilled. That is, there exists a cover $X(\xi)$ satisfying

$$f(X(\xi)) \subseteq X(\varphi(\xi)), \quad X(\xi) \subseteq \mathcal{O}_\alpha(\xi) \tag{6}$$

for all $\xi \in \mathbf{L}$. Chose an element $\xi_0(\delta)$ that satisfies $x_0 \in X(\xi_0(\delta))$. Such an element exists, because $X(\xi)$, $\xi \in \mathbf{L}$ is a cover. Now inductively construct elements $\xi_n(\delta)$, $n = 1, 2, \ldots, N$ satisfying $x_n \in X(\xi_n)$. Suppose that element $\xi_{n-1}(\delta)$ exists. By the first inclusion of (6), $x_n \in f(X(\xi_{n-1}(\delta))) \subseteq X(\varphi(\xi_{n-1}(\delta))) = \bigcup_{\eta\in\varphi(\xi_{n-1}(\delta))} X(\eta)$. In particular, there exists $\eta \in \varphi(\xi_{n-1})$ such that $x_n \in X(\eta)$. Define $\xi_n(\delta) = \eta$. The sequence $\boldsymbol{\xi}(\delta)$ constructed in this way satisfies (4) by (5) and the second inclusion of (6). Therefore a trajectory $\boldsymbol{\xi}$ exists if for $\alpha = \mathrm{C}(\varphi, f) + \delta$ the condition C1 is fulfilled.

Now suppose that the condition C2 holds. That is, there exist nonempty sets $\Xi(x)$, $x \in \Omega$ satisfying $x \in \Omega$

$$\varphi(\Xi(x)) \supseteq \Xi(f(x)), \quad \Xi(x) \subseteq \mathcal{O}_\alpha(x). \tag{7}$$

Choose an element $\xi_N(\delta)$ such that $\xi_N(\delta) \in \Xi(x_N)$. This exists because the sets $\Xi(x)$, $x \in \Omega$ are not empty. Construct elements $\xi_n(\delta)$, $n = 0, 1, 2, \ldots, N-1$ satisfying $\xi_n(\delta) \in \Xi(x_n)$ by backward induction as follows. Let $\xi_{n+1}(\delta)$ exists. By the first inclusion of (7), $\xi_{n+1}(\delta) \in \Xi(f(x_n)) \subseteq \varphi(\Xi(x_n)) = \bigcup_{\eta \in \Xi(x_n)} \varphi(\eta)$. In particular, there exist $\eta \in \Xi(x_n)$ such that $\xi_{n+1}(\delta) \in \varphi(\eta)$. Define $\xi_n(\delta) = \eta$. The constructed sequence $\boldsymbol{\xi}(\delta)$ satisfies (4) by the second inclusion of (7) and (5). Therefore, a trajectory $\boldsymbol{\xi}$ exists for $\varepsilon = \mathrm{C}(\varphi, f) + \delta$ whenever the condition C2 is fulfilled.

Finally, define the sequence $\boldsymbol{\xi}$ as a pointwise limit point of $\boldsymbol{\xi}(\delta)$ as $\delta \to \infty$ and the proof is complete. □

The sequence $\mathbf{y} = y_0, y_1, \ldots, y_N$ is called an *ε–pseudo–orbit* of the system f if the inequalities $\mathrm{Sep}(y_n, f(y_{n-1})) \leq \varepsilon$, $n = 1, 2, \ldots, N$ hold. Let $\delta(\varepsilon)$ be a positive function such that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and Ω_* be a closed subset of Ω satisfying $f(x) \bigcap \Omega_* \neq \emptyset$ for $x \in \Omega_*$. A system f is called *$\delta(\varepsilon)$–shadowing* on Ω_* if for any $\varepsilon > 0$ and any ε–pseudo–orbit $\mathbf{y} = y_0, y_1, \ldots, y_N$, with $y_n \in \Omega_*$ there corresponds $x_0 \in \Omega_*$ and a trajectory $\mathbf{x} = x_0, f(x_0), \ldots, f^N(x_0)$ satisfying $\rho(x_n, y_n) < \delta(\varepsilon)$. Many systems with chaotic behavior are shadowing for some function $\delta(\cdot)$. The classical examples are hyperbolic diffeomorphisms (see [9]). Another well known example is the class of expanding one-dimensional mappings $f : [0,1] \mapsto [0,1]$ satisfying conditions of the Walters theorem [3]. Still another example is the "β-mapping" $f_\beta : [0,1] \mapsto [0,1]$ defined by

$$f_\beta(x) = \beta x \pmod 1 \tag{8}$$

for real $\beta > 1$. For the convenience, Appendix A gives a simple shadowing criteria which is applicable to the Walters mappings, β–mappings and some multivalued mappings f.

Denote by $\mathrm{Tr}(f)$ the totality of trajectories (2) of f and denote by $\mathrm{Tr}(\varphi)$ the totality of trajectories (3) of φ.

Theorem 1. *(a) For any dynamical system f,* $\mathrm{Sep}(\mathrm{Tr}(f), \mathrm{Tr}(\varphi)) \leq \mathrm{C}(f, \varphi)$.

(b) If the system f is $\delta(\varepsilon)$–shadowing on Ω then for any φ

$$\mathrm{Sep}(\mathrm{Tr}(\varphi), \mathrm{Tr}(f)) \leq \delta(2d(f, \varphi)) + d(\varphi, f). \tag{9}$$

Proof: Assertion (a) follows immediately from Lemma 1. It remains to prove (b). Consider a trajectory $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_N) \in \mathrm{Tr}(\varphi)$. By definition there exist corteges $\mathbf{y} = y_0, \ldots, y_N$ and $\mathbf{z} = z_1, \ldots, z_N$ satisfying $\rho(y_n, \xi_n), \rho(z_{n+1}, \xi_{n+1}), \leq d(\varphi, f)$ for $n = 0, 1, \ldots N-1$ and $z_{n+1} = f(y_n)$. Hence, the sequence $\mathbf{y}$ is a $2d(\varphi, f)$–pseudo–orbit. Thus, there exist a trajectory $\mathbf{x} \in \mathrm{Tr}(f)$ satisfying $\rho(y_n, x_n) \leq \delta(2d(f, \varphi))$. From this inequality and the estimate $\rho(y_n, \xi_n) \leq d(\varphi, f)$ the inequality (9) follows immediately. □

Consequently, the flow of the original system f is closely represented by the flow of the multivalued discretization φ provided that φ is consistent, the system f is shadowing, and the graphs $\mathrm{Gr}(f), \mathrm{Gr}(\varphi)$ are close in the Hausdorff metric.

A point $\xi \in \mathbf{L}$ is called *cyclic* for $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ if there exists a natural number p such that $\xi \in \varphi^p(\xi)$. Recall that an infinite trajectory $\mathbf{x} = x_0, x_1, \ldots$ of f is called *recurrent* for any $\varepsilon > 0$ there exists a natural N such that for any natural M, $\mathrm{Sep}(\mathbf{x}, \mathbf{x}_*) \leq \varepsilon$ where $\mathbf{x}_* = x_M, \ldots, x_{M+N-1}$. In other words, a trajectory is recurrent if it is approximated with a given accuracy by each sufficiently long subtrajectory. From Theorem 1 it follows that

Corollary 1. *(a) Let a trajectory* $\mathbf{x} = x_0, x_1, \ldots$ *be recurrent for* f *and suppose that* φ *is a discretization. Then for any* $\alpha > \mathrm{C}(\varphi, f)$ *there exists a periodic trajectory* $\boldsymbol{\xi}$ *of* φ *satisfying* $\rho(x_n, \xi_n) \leq \alpha, \; n = 0, 1, \ldots$.

(b) If the system f *is* $\delta(\varepsilon)$*-shadowing on* Ω *and* $\boldsymbol{\xi} = \xi_0, \xi_1, \ldots$ *is a periodic trajectory of* φ *then there exists a recurrent trajectory* $\mathbf{x}$ *of* f *with* $\rho(x_n, \xi_n) \leq \delta(2d(f, \varphi)) + d(\varphi, f), \; n = 0, 1, \ldots$.

3 Approximation of invariant measures

Let $\mathcal{P}$ be the set of all Borel probability measures on Ω and endow $\mathcal{P}$ with the *weak topology*. That is, a sequence $\{\mu_n\}$ in $\mathcal{P}$ converges weakly to $\mu \in \mathcal{P}$ if for any real function $\alpha \in C(\Omega)$, $\lim_{n\to\infty} \int_\Omega \alpha \, d\mu_n = \int_\Omega \alpha \, d\mu$. This weak topology is metrizable. The corresponding metric ρ_* can be defined by

$$\rho_*(\mu_1, \mu_2) = \inf\{\varepsilon : \; \mu_1(\mathcal{O}_\varepsilon(X)) \leq \mu_2(X) - \varepsilon \text{ for all } X \in \mathcal{B}\}. \tag{10}$$

The quantity (10) is called the *Prokhorov metric* [8]. A dynamical system f induces an operator f^* on $\mathcal{P}$ by $(f^*\mu)(X) = \mu(f^{-1}(X))$ for $X \in \mathcal{B}$.

Let $\overline{\mathrm{Gr}(f)}$ be the closure of the set $\mathrm{Gr}(f)$. For any $X \in \mathcal{B}$ denote by $\overline{f^{-1}}(X)$ the set

$$\overline{f^{-1}}(X) = \{x \in \Omega : \text{ there exists } y \in \overline{X} \text{ such that } (x, y) \in \overline{\mathrm{Gr}(f)}\}. \tag{11}$$

The measure $\mu \in \mathcal{P}$ is said to be *semi-invariant* for the system f if for any $X \in \mathcal{B}$ $\mu(X) \leq \mu(\overline{f^{-1}}(X))$. For continuous single-valued f a measure μ is semi-invariant if and only if it is invariant. In general, for discontinuous or multivalued f the totality of semi-invariant measures is wider than that of invariant measures.

Example 4. *If* f *is a* β*-mapping then the Dirac measure* δ_1 *concentrated in the point* 1 *is semi-invariant but not invariant.*

Denote by $\mathfrak{S}(f)$ the set of semi-invariant measures of a Borel dynamical system f.

A measure μ on $\mathbf{L}$ is semi-invariant for a multivalued discretization $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ if for any subset $\mathbf{L}_* \subseteq \mathbf{L}$, $\mu(\mathbf{L}_*) \leq \mu(\varphi^{-1}(\mathbf{L}_*))$. By standard limit constructions it can be shown that

Theorem 2. *Let* $\mathbf{L}_\nu$ *be the sequence of finite subsets of* Ω *and* $\varphi_\nu: \mathbf{L}_\nu \mapsto 2^{\mathbf{L}_\nu}$ *be a sequence of discretizations with* $\lim_{\nu\to\infty} d(\varphi_\nu, f) = 0$. *Then*

$$\lim_{\nu\to\infty} \mathrm{Sep}_*(\mathfrak{S}(\varphi_\nu), \mathfrak{S}(f)) = 0 ,$$

where Sep_* *is the Hausdorff separation with respect to the Prokhorov metric* ρ_*.

It is convenient to introduce a slight modification of the notion of consistency. Denote by f_σ the mapping $x \mapsto \mathcal{O}_\sigma(f(x))$. The mapping $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ will be called a *strong α–consistent discretization of f* if there exists $\sigma > 0$ such that φ is an α–consistent discretization for f_σ. Denote by $\mathrm{C}_*(\varphi, f)$ the greatest lower bound of α for which φ is a strong *alpha*–consistent discretization of f. For the discretizations φ from Examples 1, 2, the inequality $\mathrm{C}_*(\varphi, f) \leq h(\mathbf{L})$ holds; for φ from Example 3 it holds $\mathrm{C}_*(\varphi, f) \leq \alpha$.

Theorem 3. *(a)* $\mathrm{Sep}_*(\mathfrak{S}(f), \mathfrak{S}(\varphi)) \leq \mathrm{C}_*(f, \varphi)$,
(b) If the system f be $\delta(\varepsilon)$–shadowing on Ω then for any φ

$$\mathrm{Sep}_*(\mathfrak{S}(\varphi), \mathfrak{S}(f)) \leq \delta(2d(f, \varphi)) + d(\varphi, f). \tag{12}$$

For any finite subset $X \in \Omega$ denote by μ_X the uniform probability measure

$$\mu_X(x) = \frac{1}{\#(X)}, \qquad x \in X\ ,$$

where $\#(X)$ denotes the cardinality of the set X. If for a given discretization $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$ there exists a cyclic trajectory $\mathbf{y} = y_0, y_1, \ldots, y_p$, that is, $y_p = y_0$, then the measure $\mu_{\mathbf{y}}$ is semi–invariant for φ. If, additionally, the cycle $\mathbf{y}$ is minimal, we will say that $\mu_{\mathbf{y}}$ is *semi–ergodic* for φ

Lemma 2. *Each semi–invariant measure for φ can be represented as a convex combination of semi–ergodic measures.*

The proof is relegated to Appendix B.

A measure $\mu \in \mathcal{P}$ will be called *semi–ergodic* for f if for any $\varepsilon > 0$ there exists a finite set $\mathbf{L} \subseteq \Omega$ and a discretization $\varphi: \mathbf{L} \mapsto 2^{\mathbf{L}}$, $d(\varphi, f) \leq \varepsilon$, with the semi–ergodic measure $\mu_{\mathbf{y}}$ of φ satisfying $\rho_*(\mu_{\mathbf{y}}, \mu) \leq \varepsilon$.

Example 5. *Let $f(x) \equiv x$ and Ω be connected. Then a measure $\mu \in \mathcal{P}$ is semi–ergodic if and only if its support is connected.*

From Lemma 2 it follows that

Lemma 3. *Let μ be a semi–invariant measure for f. Then μ can be approximated to arbitrary accuracy in the Prokhorov metric by a convex combination of semi–ergodic measures.*

Denote by $\mathfrak{E}(f)$ the set of all semi–ergodic measures of a Borel dynamical system f. From Lemmas 2 and 3, Theorem 3 is a particular case of the following assertion:

Theorem 4. *(a)* $\mathrm{Sep}_*(\mathfrak{E}(f), \mathfrak{E}(\varphi)) \leq \mathrm{C}_*(f, \varphi)$,
(b) If the system f be $\delta(\varepsilon)$–shadowing on Ω then for any φ

$$\mathrm{Sep}_*(\mathfrak{E}(\varphi), \mathfrak{E}(f)) \leq \delta(2d(f, \varphi)) + d(\varphi, f).$$

This assertion may, in turn, be proved in much the same way as Theorem 1. We omit the details.

Appendix A. Shadowing in multivalued systems

Recall that the value $\mathrm{Dist}(Y,X) = \max\{\mathrm{Sep}(Y,X), \mathrm{Sep}(X,Y)\}$ is called the *Hausdorff metric* between closed sets X and Y. We shall also use the value $\mathrm{Dist}(Y,X)$ for arbitrary subsets $Y, X \subseteq \Omega$. Denote $H = \bigcap_{k=1}^{\infty} f^k(\Omega)$. We shall suppose that H is not empty. Denote by f_H the restriction of the system f on H. Let us call the multivalued mapping $f: \Omega \mapsto \Omega$ *q-normal* if H is dense in Ω and for each $x \in H$, $y \in \Omega$, $x \neq y$ from $f^{-1}(y) \neq \emptyset$ it follows that $\mathrm{Dist}(f_H^{-1}(x), f^{-1}(y)) < q\rho(x,y)$. Clearly, for example, the mapping (8) is q–normal for any $q > \beta > 1$.

Lemma 4. *Let f be q–normal with $q < 1$. Then f is δ–shadowing for*

$$\delta(\varepsilon) = \frac{\varepsilon q}{(1-q)}. \tag{13}$$

Proof: Let $\mathrm{y} = y_0, y_1, \ldots, y_N$, be a given ε–pseudo–orbit. We shall construct the trajectory $\mathrm{x} = x_0, x_1, \ldots, x_N$ satisfying $\rho(x_n, y_n) \leq \frac{\varepsilon q}{1-q}$, $n = N, N-1, \ldots, 0$ by induction. By the definition there exist z_n, $n = 1, 2, \ldots, N$ satisfying $z_n \in f(y_{n-1})$, $\rho(z_n, y_n) \leq \varepsilon$. Define x_N as an arbitrary element from H such that $\rho(x_N, y_N) \leq \frac{\varepsilon q}{1-q}$. and suppose that x_n is just defined for a certain $1 \leq n \leq N$ and satisfies

$$x_n \in H, \qquad \rho(x_n, y_n) \leq \frac{\varepsilon q}{1-q}. \tag{14}$$

Then $\rho(x_n, z_n) \leq \varepsilon + \frac{\varepsilon q}{1-q}$. Hence $\mathrm{Dist}(f^{-1}(z_n), f_H^{-1}(x_n)) < \varepsilon(q + \frac{q^2}{1-q}) = \frac{\varepsilon q}{1-q}$. In particular, there exists $x \in f_H^{-1}(x_n)$ satisfying $x \in H$, $\rho(x, y_{n-1}) \leq \frac{\varepsilon q}{1-q}$ which is similar to (14) and is possible to define $x_{n-1} = x$. Thus, the lemma is proven. □

Let us call the system f *correct $\delta(\cdot)$–shadowing* if it has shadowing property not only with respect to finite ε–pseudo–orbit but also with respect to infinite ε–pseudo–orbit of the form $\mathrm{y} = y_0, y_1, \ldots y_n, \ldots$, or of the form $\mathrm{y} = \ldots, y_{-N}, \ldots, y_{-1}, y_0$ or of the form $\mathrm{y} = \ldots, y_{-N}, \ldots, y_{-1}, y_0, y_1, \ldots y_n, \ldots$. Evidently, properties to be shadowing and correct shadowing are equivalent for continuous systems f. When f is discontinuous situation is more complicated. Even the simplest mapping (8) is shadowing but not correct shadowing: for instance, the sequence $0, 0, \ldots$ is an ε–pseudo–orbit any positive ε but it can not be approximated with a proper trajectory with an accuracy less than 1. The system with closed graph is called correctly q–normal if $f(\Omega) = \Omega$ and

$$\mathrm{Dist}(f_H^{-1}(x), f^{-1}(y)) \leq q\rho(x,y), \quad x, y \in \Omega.$$

Lemma 5. *System f with the closed graph is correct q–shadowing if f is correctly q–normal for δ defined by (13). Properties to be shadowing and correct shadowing are equivalent for multivalued systems f with closed graph $Gr(f)$.*

Proof: The first assertion can be proven analogously to Lemma 4; the second one follows from definitions. □

Appendix B. Proof of Lemma 2

First of all recall a classical result about the structure of *double stochastic matrices.*

A square matrix A with the elements a_{ij}, $i, j = 1, 2, \ldots d$ is said to be double stochastic if for each $k = 1, 2, \ldots d$

$$\sum_{i=1}^{d} a_{ik} = \sum_{j=1}^{d} a_{kj} = 1. \tag{15}$$

(We will treat i as the row number and j as the column number.) A one–to–one mapping σ of the set $I_d = \{1, 2, \ldots, d\}$ onto itself is called *permutation.* To any permutation σ is corresponded the matrix $P(\sigma)$ defined by

$$p_{ij}(\sigma) = \begin{cases} 1 & \text{if} \quad j = \sigma(i) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The matrix (p_{ij}) is called a *permutation matrix.*

Birkhoff Theorem. ([12], Theorem 3.3). *The set $\mathcal{DSM}_d$ of d – square doubly stochastic matrices forms a convex polyhedron with permutations matrices as vertices. In other words, if $A \in \mathcal{DSM}_d$, then $A = \sum_{\kappa=\nu}^{d} \theta_\kappa P_\kappa$, where $P_1, \ldots, P_\nu$ are permutation matrices, $\theta_1, \ldots, \theta_\nu$ are nonnegative, $\sum_{\kappa=1}^{\nu} \theta_\kappa = 1$.*

We will use below a special modification of this Theorem. Denote by $\mathcal{M}_*$ the set of all non–negative square matrices $d \times d$ with elements $a_{i,j}$ satisfying the equalities

$$\sum_{i=1}^{d} a_{ik} = \sum_{j=1}^{d} a_{kj}, \quad k = 1, 2, \ldots, d, \tag{17}$$

instead of (15), and also the equality $\sum_{i,j=1}^{d} a_{ij} = 1$. The d–square matrix P will be called a *semi–permutation matrix* if there exist integers $1 \le i_1 < i_2 < \ldots < i_{d_*} \le d$ and a permutation σ_* of the set $I_* = \{i_1, \ldots, i_{d_*}\}$ such that

$$p_{ij}(\sigma) = \begin{cases} 1/d_* & \text{if} \quad i \in I_*,\ j = \sigma(i), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5. *If $A \in \mathcal{M}_*$, then*

$$A = \sum_{\kappa=\nu}^{d} \theta_\kappa P_\kappa, \tag{18}$$

where $P_1, \ldots, P_\nu$ are semi–permutation matrices and $\theta_1, \ldots, \theta_\nu$ are nonnegative, and $\sum_{\kappa=1}^{\nu} \theta_\kappa = 1$.

Proof: Recall that a nonnegative matrix $A \in \mathcal{M}_{d_0}$ is called *fully decomposable* if for sufficiently large ν all entries of the matrix A^ν are strictly positive.

Lemma 6. *Let be d_0–square fully indecomposable matrix Then there exist a permutation σ_0 of the set $\{1, \ldots, d_0\}$ such that $a_{i\,\sigma(i)} > 0, \quad i = 1, \ldots, d_0$.*

Proof: This lemma follows, for instance from Corollary of Theorem 2.2 and Theorem 3.9 from [12]. □

For a given set I of integers $1 \le i_1 < i_2 < \ldots < i_{d_*} \le d$ and given d–square matrix A denote by $A(I)$ the d_*–square matrix B with entries $b_{l,m} = a_{i_l,i_m}$. Let A be a matrix from $\mathcal{M}_*$. A partition of the set $\{1,\ldots,d\}$ into disjoint subsets $I_k = \{i(1,k),\ldots,i(d_k,k)\}$, $k = 1,\ldots K$ will be called A–*decomposing* if matrices $A(I_k)$, $k = 1,\ldots K$ are fully indecomposable, matrix $A(I_0)$ is a zero matrix (may be empty one), and $a_{i,j} = 0$ if indexes i and j belong to different sets I_k and I_l.

Lemma 7. *For each matrix $A \in \mathcal{M}_*$ there exist a decomposing partition*

Proof: Define I_0 as a set such indexes i satisfying $a_{ij} = a_{ji} = 0, \quad j = 1,\ldots,d$. Consider an auxiliary Markov chain $\mathcal{C}$ with the set of states $I = \{1,\ldots d\} \setminus I_0$ and transition probabilities

$$q_{ij} = \begin{cases} a_{ij} & \text{if} \quad i \ne j \\ 1 - \sum_{k \ne i} a_{ik} & \text{if} \quad i = j. \end{cases}$$

Define subsets $I_k = \{i(1,k),\ldots,i(d_k,k)\}$, $k = 1,\ldots K$ as components of this chain. By virtue of (17) this partition has the necessary properties. □

Let us now complete a proof of Theorem 5. Use induction on $\pi(A)$, the number of positive entries in A. If $\pi(A) = 0$ then we have nothing to prove. Assume that $\pi(A) > 0$ and the theorem holds for all matrices in $\mathcal{M}_*$ with less than $\pi(a)$ positive entries.

Choose a decomposing partition $I_k = \{i(1,k),\ldots,i(d_k,k)\}$, $\quad k = 0,1,\ldots K$, which exists by Lemma 7. If all matrices $A(I_k)$, $k = 1,\ldots,K$ are semi–permutation matrices we again have nothing to prove.

By Lemma 6 there exist a permutation σ_1 of the set $S_1 = \{i(1,1),\ldots,i(d_1,1)\}$, such that $a_{i(l),\sigma(i(l))} > 0, \quad l = 1,\ldots,d_1$. Let $a = \min_l a_{i(l,1),\sigma_1(i(l,1))} = a_{i(l_*,1),\sigma(i(l_*,1))}$ and P be a d-square semi–permutation matrix defined by

$$p_{ij} = \begin{cases} 1/d_1 & \text{if} \qquad i \in I_1,\ j = \sigma_1(i), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $0 < a < 1/d_1$. Also, $A - a\,d_1 P$ is nonnegative. By the construction,

$$B \;=\; (b_{ij}) \;=\; \frac{1}{1 - a\,d_1}(A - a\,d_1 P) \tag{19}$$

belongs to $\mathcal{M}_*$. Now, $\pi(B) \le \pi(A) - 1$, since B has zeros in all positions where A has a zero entry, and in addition $b_{i(l_*,1),\sigma(i(l_*,1))}$. Hence, by the induction hypothesis, $B = \sum_{\kappa=1}^{\nu-1} \lambda_\kappa P_\nu$ where $\tau_1,\ldots,\tau_{\nu-1}$ are nonnegative, $\sum_{\kappa=1}^{\nu-1} \tau_\kappa = 1$. But then, by (19)

$$A \;=\; (1 - ad_1)B + ad_1 P = \left(\sum_{\kappa=1}^{\nu-1}(1 - ad_1)\tau_\kappa P_\kappa\right) + ad_1 P = \sum_{\kappa=1}^{\nu} \theta_\kappa P_\kappa,$$

where $\theta_\kappa = (1 - ad_1)\tau_\kappa$, $\tau_\nu = ad_1$, and $P_\nu = P$. Obviously, the θ_κ are nonnegative. It remains to show that $\sum_{\kappa=1}^{\nu} \tau_\kappa = 1$; it is a simple calculation. □

Now we can prove Lemma 2.

Let μ_* be a given semi–invariant measure of φ. By Theorem 1 from [7] there exists a Markov chain C with transition probabilities $p(\xi,\eta)$, $\xi,\eta \in \mathbb{L}$ for which μ is invariant and $p(\xi,\eta) = 0$ for $\eta \notin \varphi(\xi)$. Denote the cardinality of $\mathbb{L}$ by d. Choose an enumeration $\xi_1,\ldots,\xi_d$ of $\mathbb{L}$. Consider a matrix A with elements $a_{ij} = \mu(\xi_i)p(\xi_i,\xi_j)$. This matrix belongs to $\mathcal{M}_*$ because μ is invariant for the chain C. By Theorem 5 it can be represented in the form (18). Therefore, the measure μ can be represented as $\mu = \sum_{\kappa=1}^{\nu} \theta_\kappa \mu_\kappa$ where μ_k is the uniform measure with the support $\mathrm{Supp}(\mu_k) = \{\xi_i :$ there exist j such that $p_{i,j}(\kappa) \neq 0\}$. Thus, the lemma is proven. □

References

[1] M. Blank, Small perturbations of chaotic dynamical systems, *Russian Math. Soc. Surveys*, **44**, No 6 (1989), 1–33.

[2] A. Boyarsky, Randomness implies order, *J. Math. Anal. Applns.*, **76** (1986), 483–497.

[3] I.P. Cornfeld, S.V. Fomin and Ya.G. Sinai, *Ergodic Theory*, Springer–Verlag, New York, 1982.

[4] P. Diamond and P. Kloeden, Spatial discretization of mappings, *Comp. Math. Applns.*, **25**, No 6, (1993), 85–94.

[5] P. Diamond, P. Kloeden and A. Pokrovskii, 1993 An invariant measure arising in computer simulation of a chaotic dynamical system, *J. Nonlinear Sciences*, in press.

[6] P. Diamond, P. Kloeden and A. Pokrovskii, Weakly chain recurrent points and spatial discretizations of dynamical systems, *Random & Computational Dynamics* in press.

[7] P. Diamond, P. Kloeden and A. Pokrovskii, Interval stochastic matrices and the computation of invariant measures, submitted for publication.

[8] S. Ethier and T. Kurtz, *Markov Processes: Characterization and Convergence,* New York: Wiley, New York, 1986.

[9] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer–Verlag, New York, 1983.

[10] M. Hurley, Attractors: persistence and density of their basins, *Trans. Amer. Math. Soc.*, **269**, No 1 (1982), 247–271.

[11] Y. Kifer, *Random Perturbations of Dynamical Systems*, Birkhauser, Boston, 1988.

[12] H. Minc, *Encyclopedia of Mathematics and Applications.* **V. 6.** *Permaments*, Addison–Wesley Publishing Co., London, 1978.